Effective Field Theories

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There is a high energy scale M where an effective theory breaks down. Its Lagrangian describes light particles ($m_i \ll M$) and their interactions at $p_i \ll M$; physics at distances $\lesssim 1/M$ produces local interactions of these light fields.

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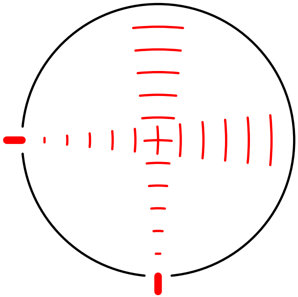
The Lagrangian contains all possible operators (allowed by symmetries). Coefficients of operators of dimension $n+4$ contain $1/M^n$. If M is much larger than energies we are interested in, we can retain only renormalizable terms (dimension 4), and, maybe, a power correction or two.

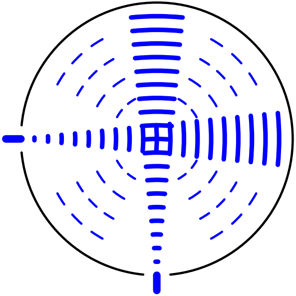
Photonia



Quantum PhotoDynamics (QPD)

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Photonia



Quantum PhotoDynamics (QPD)

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + c_1\left(F_{\mu\nu}F^{\mu\nu}\right)^2 + c_2 F_{\mu\nu}F^{\nu\alpha}F_{\alpha\beta}F^{\beta\mu}$$

Dimension 6

$$F_\lambda{}^\mu F_\mu{}^\nu F_\nu{}^\lambda = 0$$

Dimension 6

$$F_{\mu\nu}\partial^2 F^{\mu\nu} = -2\left(\partial^\mu F_{\nu\mu}\right)\left(\partial_\lambda F^{\nu\lambda}\right)$$

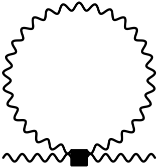
$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$$

Dimension 6

$$(\partial^\mu F_{\lambda\mu})\left(\partial_\nu F^{\lambda\nu}\right) = 0$$

$$\text{Equation of motion } \partial_\nu F^{\lambda\nu} = j^\lambda = 0$$

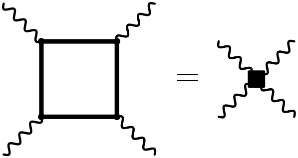
No corrections to the propagator



$= 0$

Qedland

Matching S-matrix elements



Qedland

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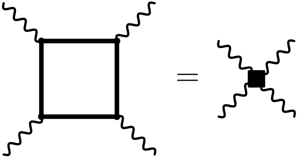
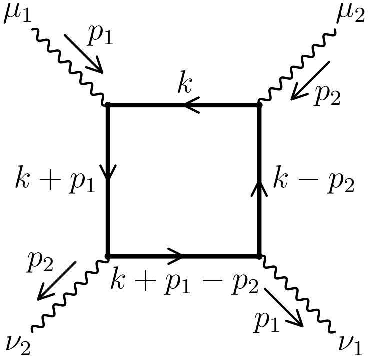
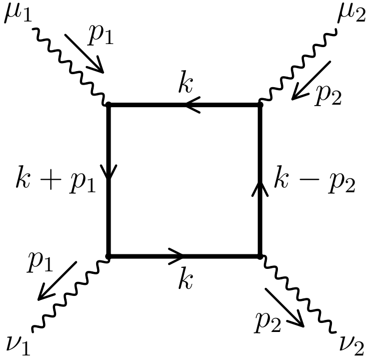
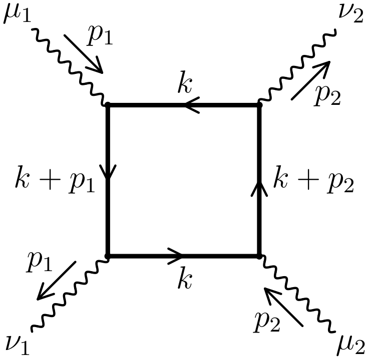


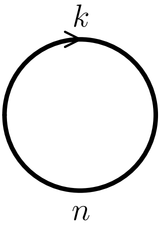
$T^{\mu_1\mu_2\nu_1\nu_2}(p_1, p_2, p'_1, p'_2) = c_1 T_1^{\mu_1\mu_2\nu_1\nu_2} + c_2 T_2^{\mu_1\mu_2\nu_1\nu_2}$
For example $p'_1 = p_1$, $p'_2 = p_2$, $p_1^2 = p_2^2 = 0$,
$(p_1 + p_2)^2 = s \ll M^2$
Compare $T^{\mu_1\mu_2\nu_1\nu_2} g_{\mu_1\nu_1} g_{\mu_2\nu_2}$ and $T^{\mu_1\mu_2\nu_1\nu_2} g_{\mu_1\mu_2} g_{\nu_1\nu_2}$







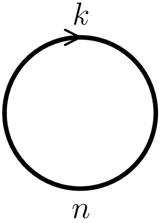
+ (opposite line direction)



$$= \frac{1}{i\pi^{d/2}} \int \frac{d^d k}{D^n} = M^{d-2n} V(n)$$

$$D = M^2 - k^2 - i0$$

$$V(n) = \frac{\Gamma\left(n - \frac{d}{2}\right)}{\Gamma(n)}$$



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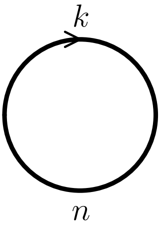
$$T^{\mu_1\mu_2\nu_1\nu_2} = \frac{e_0^4 M^{-4-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \frac{(d-4)(d-6)}{2880} \times \left(-5T_1^{\mu_1\mu_2\nu_1\nu_2} + 14T_2^{\mu_1\mu_2\nu_1\nu_2}\right)$$



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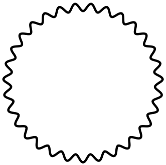
Heisengerg–Euler Lagrangian

$$L_1 = \frac{\pi\alpha^2}{180M^4}\left(-5O_1 + 14O_2\right)$$

Thermal radiation $T \ll m$



$\sim T^4$

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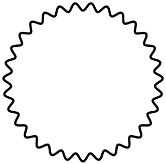
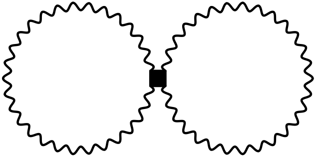
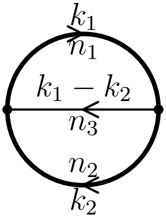


$\sim T^4$



$\sim \frac{\alpha^2}{M^4} T^8$

Two loops



$$\int \frac{d^d k_1 \, d^d k_2}{D_1^{n_1} D_2^{n_2} D_3^{n_3}} = -\pi^d M^{2(d-n_1-n_2-n_3)} V(n_1, n_2, n_3)$$

$$D_1 = M^2 - k_1^2 \quad D_2 = M^2 - k_2^2 \quad D_3 = -(k_1 - k_2)^2$$

$$V(n_1, n_2, n_3) =$$

$$\frac{\Gamma\left(\frac{d}{2} - n_3\right) \Gamma\left(n_1 + n_3 - \frac{d}{2}\right) \Gamma\left(n_2 + n_3 - \frac{d}{2}\right) \Gamma(n_1 + n_2 + n_3 - d)}{\Gamma\left(\frac{d}{2}\right) \Gamma(n_1) \Gamma(n_2) \Gamma(n_1 + n_2 + 2n_3 - d)}$$

A. Vladimirov (1980)

Wilson line

Classical charge

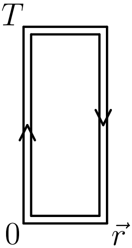
$$S_{\text{int}} = e \int_l dx^\mu A_\mu(x)$$

Feynman path integral: $\exp(iS)$ contains

$$W_l = \exp\left(ie \int_l dx^\mu A_\mu(x) \right)$$

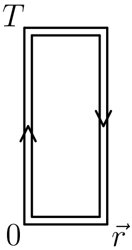
The vacuum-to-vacuum transition amplitude is the vacuum average of the Wilson lines

Potential



$$= e^{-iU(\vec{r})T}$$

Potential

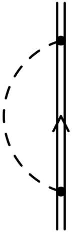


$$= e^{-iU(\vec{r})T}$$

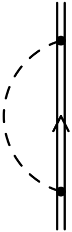
Coulomb gauge

$$D^{00}(q) = -\frac{1}{\vec{q}^2}$$

$$D^{ij}(q) = \frac{1}{q^2 + i0}\left(\delta^{ij} - \frac{q^i q^j}{\vec{q}^2}\right)$$



$$= 0$$



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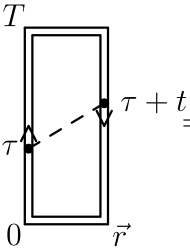

$$= -i\, e^2\, T \int D^{00}(t, \vec{r})\, dt$$

$$= -i\, e^2\, T \int \frac{d^{d-1}\vec{q}}{(2\pi)^{d-1}}\, D^{00}(0, \vec{q})\, e^{i\, \vec{q}\cdot\vec{r}}$$

Coulomb potential

$$U(\vec{q}) = e^2 D^{00}(0, \vec{q}) = -\frac{e^2}{\vec{q}^2}$$

$$U(\vec{r}) = -\frac{\alpha}{r}$$

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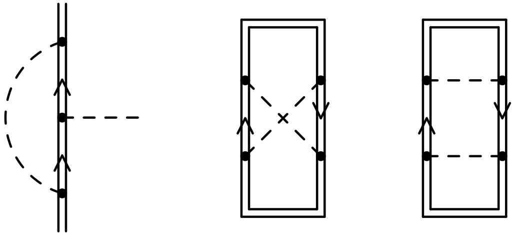

No corrections

Contact interaction

In the presence of sources

$$L_c = c\left(\partial^{\mu} F_{\lambda\mu}\right)\left(\partial_{\nu} F^{\lambda\nu}\right)$$

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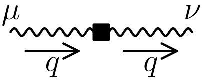

$$= 2icq^2\left(q^2 g_{\mu\nu} - q_\mu q_\nu\right)$$

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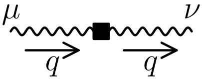

$$= 2icq^2\left(q^2 g_{\mu\nu} - q_\mu q_\nu\right)$$

$$U_c(\vec{r}) = 2c\delta(\vec{r})$$

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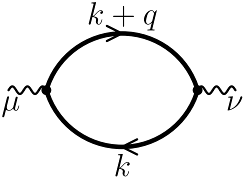
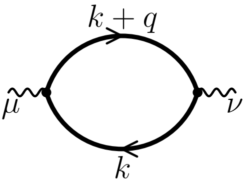


$$= i\left(q^2 g_{\mu\nu} - q_\mu q_\nu\right)\Pi(q^2)$$

Qedland



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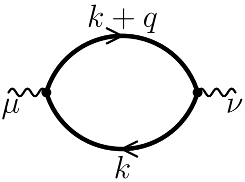
$$\Pi(q^2) = \frac{4ie_0^2}{(d-1)q^2}\int \frac{d^dk}{(2\pi)^d}\frac{N}{D_1 D_2}$$

$$D_1 = M_0^2 - (k+q)^2 \qquad D_2 = M_0^2 - k^2$$

$$N = \frac{1}{4}\,\mathrm{Tr}\,\gamma_\mu(\not{k} + \not{q} + M_0)\gamma^\mu(\not{k} + M_0)$$

$$= \frac{d-2}{2}(D_1 + D_2 + q^2) + 2M_0^2$$

Qedland



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$$\Pi(q^2) = -\frac{4}{3}\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\left(1 - \frac{d-4}{10}\frac{q^2}{M_0^2} + \cdots\right)$$

Electron charge

$$e'^2 = e_0^2 \left[1 - \frac{4}{3} \frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \right]$$

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$\overline{\mathrm{MS}}$

$$e_0^2 = e^2(\mu) Z_\alpha(\alpha(\mu)) \qquad Z_\alpha(\alpha) = 1 - \beta_0 \frac{\alpha}{4\pi\varepsilon} + \cdots$$

$$\frac{\alpha(\mu)}{4\pi} = \frac{e^2(\mu)\, \mu^{-2\varepsilon}}{(4\pi)^{d/2}} e^{-\gamma_E \varepsilon}$$

Electron charge

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$$\frac{\alpha(\mu)}{4\pi} = \frac{e^2(\mu)\, \mu^{-2\varepsilon}}{(4\pi)^{d/2}} e^{-\gamma_E \varepsilon}$$

At 1 loop

$$\frac{\alpha(\mu)}{4\pi} = \frac{e'^2 \mu^{-2\varepsilon}}{(4\pi)^{d/2}} e^{-\gamma_E \varepsilon} \times \left[1 + \frac{e'^2 M^{-2\varepsilon}}{(4\pi)^{d/2}} \left(\frac{\beta_0}{\varepsilon} \left(\frac{M}{\mu} \right)^{2\varepsilon} + \frac{4}{3} e^{\gamma_E \varepsilon} \Gamma(\varepsilon) \right) + \cdots \right]$$

Decoupling: electron charge

$$\alpha(\mu) = \alpha' \left(1 - \beta_0 \frac{\alpha'}{4\pi} \log \frac{\mu^2}{M^2} + \cdots \right)$$

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RG equation

$$\frac{d \, \log \alpha(\mu)}{d \, \log \mu} = -2\beta(\alpha(\mu)) \qquad \beta(\alpha) = \beta_0 \frac{\alpha}{4\pi} + \cdots$$

Initial condition

$$\alpha(M) = \alpha'$$

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Contact interaction

$$c = -\frac{2}{15} \frac{\alpha}{4\pi} \frac{1}{M^2}$$

$$U_c(\vec{q}) = -\frac{4}{15} \frac{\alpha^2}{M^2} \qquad U_c(\vec{r}) = -\frac{4}{15} \frac{\alpha^2}{M^2} \delta(\vec{r})$$

The full theory and the effective theory

QPD

$$L' = -\frac{1}{4} F'_{0\mu\nu} F'^{\mu\nu}_{0} - \frac{1}{2a'_0} \left(\partial_\mu A'^{\mu}_{0}\right)^2$$

$$A'_0 = A'(\mu) = A'_{\rm os} \qquad a'_0 = a'(\mu) = a'_{\rm os}$$

The full theory and the effective theory

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QED

$$\psi_0 = Z_\psi^{1/2}(\alpha(\mu))\psi(\mu) \qquad A_0 = Z_A^{1/2}(\alpha(\mu))A(\mu)$$

$$a_0 = Z_A(\alpha(\mu))a(\mu)$$

$$e_0 = Z_\alpha^{1/2}(\alpha(\mu))e(\mu) \qquad M_0 = Z_m(\alpha(\mu))M(\mu)$$

$\overline{\text{MS}}$

$$Z_i(\alpha) = 1 + \frac{z_1}{\varepsilon}\frac{\alpha}{4\pi} + \left(\frac{z_{22}}{\varepsilon^2} + \frac{z_{21}}{\varepsilon}\right)\left(\frac{\alpha}{4\pi}\right)^2 + \cdots$$

On-shell renormalization scheme

$$A_0 = (Z_A^{\text{os}})^{1/2} A_{\text{os}}$$

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Photon propagator

$$D_\perp(p^2) = Z_A^{\text{os}} D_\perp^{\text{os}}(p^2)$$

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Near the mass shell

$$D_\perp(p^2) = \frac{1}{1 - \Pi(p^2)} \frac{1}{p^2} = \frac{1}{1 - \Pi(0)} \frac{1}{p^2} + \cdots$$

On-shell renormalization scheme

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Photon propagator

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Near the mass shell

$$D_\perp(p^2) = \frac{1}{1 - \Pi(p^2)} \frac{1}{p^2} = \frac{1}{1 - \Pi(0)} \frac{1}{p^2} + \cdots$$

By definition $D_\perp^{\text{os}}(p^2) \to 1/p^2$

$$Z_A^{\text{os}} = \frac{1}{1 - \Pi(0)}$$

Decoupling: photon field

The propagators of both $A_{\rm os}$ and $A'_{\rm os}$ at $p^2 \to 0$ are equal to the free propagator

$$A_{\rm os} = A'_{\rm os}$$

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Decoupling (bare)

$$A_0 = \left(\zeta_A^0\right)^{1/2} A'_0$$

up to corrections suppressed by powers of $1/M$

$$\zeta_A^0 = Z_A^{\rm os}$$

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up to corrections suppressed by powers of $1/M$

$$\zeta_A^0 = Z_A^{\rm os}$$

Decoupling ($\overline{\rm MS}$)

$$A(\mu) = \zeta_A^{1/2}(\mu) A'(\mu)$$

$$\zeta_A(\mu) = \frac{\zeta_A^0}{Z_A} = \frac{Z_A^{\rm os}}{Z_A}$$

1 loop

$$\left(\zeta_A^0\right)^{-1} = \left(Z_A^{\text{os}}\right)^{-1} = 1 - \Pi(0) = 1 + \frac{4}{3}\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon) + \cdots$$

1 loop

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Via renormalized quantities

$$\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon) = e^{L\varepsilon} e^{\gamma\varepsilon}\Gamma(1+\varepsilon)\frac{\alpha(\mu)}{4\pi\varepsilon} Z_\alpha Z_m^{-2\varepsilon}$$

$$L = 2\log\frac{\mu}{M(\mu)}$$

$\zeta_A^{-1} = Z_A (\zeta_A^0)^{-1} = Z_A / Z_A^{\text{os}}$ must be finite at $\varepsilon \to 0$
Substitute $Z_A = 1 + z_1 \alpha(\mu)/(4\pi\varepsilon)$ with an unknown z_1

$$Z_A = 1 - \frac{4}{3}\frac{\alpha(\mu)}{4\pi\varepsilon} + \cdots$$

1 loop

$$\left(\zeta_A^0\right)^{-1} = (Z_A^{\text{os}})^{-1} = 1 - \Pi(0) = 1 + \frac{4}{3}\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon) + \cdots$$

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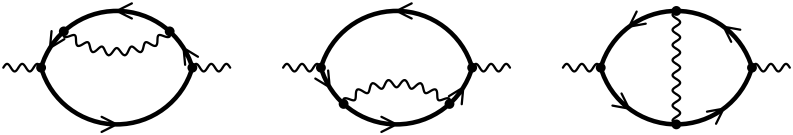
2 loops



$$\Pi(0) = -\frac{4}{3}\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon) - \frac{2}{3}\frac{(d-4)(5d^2-33d+34)}{d(d-5)}\left(\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\right)^2 + \cdots$$

Via renormalized quantities

$$\left(\zeta_A^0\right)^{-1} = (Z_A^{\rm os})^{-1} = 1 - \Pi(0) = 1 + \frac{4}{3} e^{L\varepsilon} \frac{\alpha(\mu)}{4\pi\varepsilon} Z_\alpha Z_m^{-2\varepsilon} - \varepsilon\left(6 - \frac{13}{3}\varepsilon + \cdots\right) e^{2L\varepsilon} \left(\frac{\alpha(\mu)}{4\pi\varepsilon}\right)^2 + \cdots$$

$$Z_\alpha = Z_A^{-1} = 1 + \frac{4}{3}\frac{\alpha(\mu)}{4\pi\varepsilon} + \cdots \qquad Z_m = 1 - 3\frac{\alpha(\mu)}{4\pi\varepsilon} + \cdots$$

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Electron charge

QED Ward identities

$$Z_\alpha = Z_A^{-1} \qquad Z_\alpha^{\mathrm{os}} = \left(Z_A^{\mathrm{os}}\right)^{-1}$$

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$\overline{\text{MS}}$ decoupling

$$\alpha(\mu) = \zeta_\alpha(\mu)\alpha'(\mu)$$

$$\zeta_\alpha(\mu) = \frac{Z_\alpha^{\text{os}}}{Z_\alpha} = \zeta_A^{-1}(\mu)$$

Where to match?

$$\mu_0 = M(\mu_0) \qquad (L = 0)$$

$$\zeta_\alpha(\mu_0) = 1 + \frac{13}{3}\left(\frac{\alpha(\mu_0)}{4\pi}\right)^2 + \cdots$$

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$$\mu = M_{\text{os}} \qquad \frac{M(\mu)}{M_{\text{os}}} = 1 - 6\left(\log\frac{\mu}{M_{\text{os}}} + \frac{2}{3}\right)\frac{\alpha}{4\pi} + \cdots$$

$$\zeta_\alpha(M_{\text{os}}) = 1 + 15\left(\frac{\alpha(M_{\text{os}})}{4\pi}\right)^2 + \cdots$$

Qedland

Two ways to search for new physics:

- increase the energy of e^+e^- colliders to produce pairs of new particles
- performing high-precision experiments at low energies

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Massless electron. Dimension 5 operator

$$\bar{\psi} F_{\mu\nu} \sigma^{\mu\nu} \psi$$

violates the helicity conservation

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Dimension 6 operators

$$O_n = (\bar{\psi}\gamma_{(n)}\psi)(\bar{\psi}\gamma_{(n)}\psi) \qquad \gamma_{(n)} = \gamma^{[\mu_1} \ldots \gamma^{\mu_n]}$$

conserve helicity at odd n

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conserve helicity at odd n

$$(\partial_\mu F^{\lambda\mu})(\partial^\nu F_{\lambda\nu}) = \bar{\psi} \partial_\nu F^{\mu\nu} \gamma_\mu \psi = O_1$$

equations of motion

$$\bar{\psi} \partial_\lambda F_{\mu\nu} \gamma^{[\lambda} \gamma^\mu \gamma^{\nu]} \psi = 0$$

Power counting

$$\lambda \sim \frac{p_i}{M}$$

$p \sim \lambda$, $x \sim 1/\lambda$, $\partial \sim \lambda$

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Soft photon

$$<0|T\left\{A_\mu(x)A_\nu(0)\right\}|0> \sim \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x}\frac{1}{p^2}\left[g_{\mu\nu} - (1-a)\frac{p_\mu p_\nu}{p^2}\right]$$

$A \sim \lambda$, $D_\mu \sim \lambda$

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$\psi \sim \lambda^{3/2}$

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$p \sim \lambda$, $x \sim 1/\lambda$, $\partial \sim \lambda$

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Lagrangian: $F_{\mu\nu}F^{\mu\nu} \sim \lambda^4$, $\bar{\psi} i \not{D} \psi \sim \lambda^4$

Action: ~ 1

Corrections to the Lagrangian $\sim \lambda^6$, to the action $\sim \lambda^2$

We can add higher-dimensional contributions to the Lagrangian, with further unknown coefficients. To any finite order in $1/M$, the number of such coefficients is finite, and the theory has predictive power.

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For example, if we want to work at the order $1/M^4$, then either a single $1/M^4$ (dimension 8) vertex or two $1/M^2$ ones (dimension 6) can occur in a diagram. UV divergences which appear in diagrams with two dimension 6 vertices are compensated by dimension 8 counterterms. So, the theory can be renormalized.

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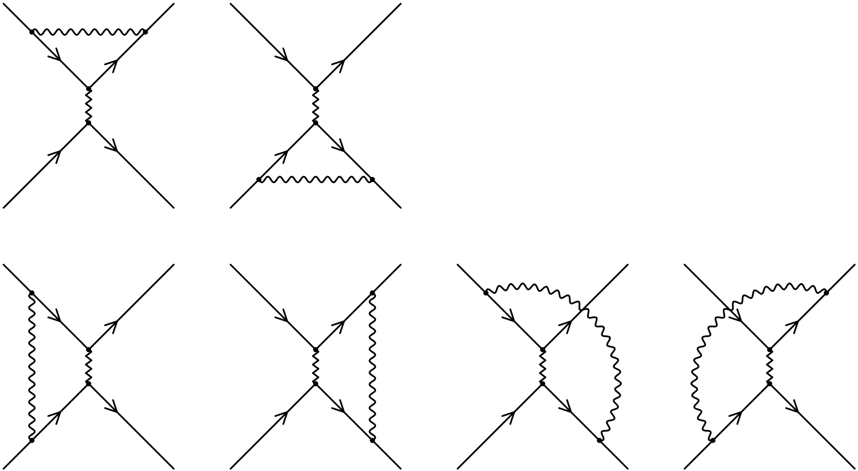
The usual arguments about non-renormalizability are based on considering diagrams with arbitrarily many vertices of nonrenormalizable interactions (operators of dimensions > 4); this leads to infinitely many free parameters in the theory.

Renormalization of 4-fermion operatprs

$$O_n^0 = (\bar{\psi}_{10}\gamma_{(n)}\psi_{20})(\bar{\psi}_{30}\gamma_{(n)}\psi_{40})$$



$$
\begin{aligned}
&2\left[\frac{1}{d}\gamma_\mu\gamma_\nu\gamma_{(n)}\gamma^\nu\gamma^\mu\otimes\gamma_{(n)}-(1-a)\gamma_{(n)}\otimes\gamma_{(n)}\right]\frac{\alpha}{4\pi\varepsilon}\\
-&2\left[\frac{1}{d}\gamma_\mu\gamma_\nu\gamma_{(n)}\otimes\gamma^\mu\gamma^\nu\gamma_{(n)}-(1-a)\gamma_{(n)}\otimes\gamma_{(n)}\right]\frac{\alpha}{4\pi\varepsilon}\\
+&2\left[\frac{1}{d}\gamma_{(n)}\gamma_\nu\gamma_\mu\otimes\gamma^\mu\gamma^\nu\gamma_{(n)}-(1-a)\gamma_{(n)}\otimes\gamma_{(n)}\right]\frac{\alpha}{4\pi\varepsilon}\,.
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&+2\left[\frac{1}{d}\gamma_{(n)}\gamma_\nu\gamma_\mu\otimes\gamma^\mu\gamma^\nu\gamma_{(n)}-(1-a)\gamma_{(n)}\otimes\gamma_{(n)}\right]\frac{\alpha}{4\pi\varepsilon}\,.
\end{aligned}
$$

$$
\begin{aligned}
&\gamma^\mu\gamma_{(n)}\gamma_\mu=(-1)^n(d-2n)\gamma_{(n)}\\
&\gamma_\mu\gamma_{(n)}\otimes\gamma^\mu\gamma_{(n)}=\gamma_{(n+1)}\otimes\gamma_{(n+1)}+n(d-n+1)\gamma_{(n-1)}\otimes\gamma_{(n-1)}\\
&\gamma_{(n)}\gamma_\mu\otimes\gamma^\mu\gamma_{(n)}=(-1)^n\\
&\qquad\times\left[\gamma_{(n+1)}\otimes\gamma_{(n+1)}-n(d-n+1)\gamma_{(n-1)}\otimes\gamma_{(n-1)}\right]
\end{aligned}
$$

When $n=5$ (Dirac structure vanishing in 4 dimensions), the contribution $\gamma_{(n-1)}\otimes\gamma_{(n-1)}$ (Dirac structure not vanishing in 4 dimensions) comes with the factor $\sim\varepsilon$.

$$
<O_n^0> = \left[1 + 2(n-1)(n-3)\frac{\alpha}{4\pi\varepsilon}\right]\gamma_{(n)} \otimes \gamma_{(n)}
$$

$$
- \left[\gamma_{(n+2)} \otimes \gamma_{(n+2)} + n(n-1)(d-n+1)(d-n+2)\gamma_{(n-2)} \otimes \gamma_{(n-2)}\right]\frac{\alpha}{4\pi\varepsilon}
$$

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$$\left(\begin{array}{c} <O_1^0> \\ <O_3^0> \\ \hline <O_5^0> \\ \vdots \end{array}\right) = \left[1 + \left(\begin{array}{cc|cc} 0 & -1 & & \\ -36 & 0 & -1 & \\ \hline & 40\varepsilon & 16 & -1 \\ & & \vdots & \ddots \end{array}\right)\frac{\alpha}{4\pi\varepsilon}\right]\left(\begin{array}{c} \gamma_{(1)} \otimes \gamma_{(1)} \\ \gamma_{(3)} \otimes \gamma_{(3)} \\ \hline \gamma_{(5)} \otimes \gamma_{(5)} \\ \vdots \end{array}\right)$$

Renormalization

Renormalization

$$O^0 = Z(\alpha(\mu))O(\mu) \qquad O(\mu) = Z^{-1}(\alpha(\mu))O^0$$

RG equations

$$\frac{dO(\mu)}{d\log\mu} + \gamma(\alpha(\mu))O(\mu) = 0$$

Anomalous dimension matrix

$$\gamma = Z^{-1}\frac{dZ}{d\log\mu} = -\frac{dZ^{-1}}{d\log\mu}Z$$

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$$\gamma = Z^{-1}\frac{dZ}{d\log\mu} = -\frac{dZ^{-1}}{d\log\mu}Z$$

Renormalized evanescent operators vanish

$$O(\mu) = \begin{pmatrix} O_1(\mu) \\ O_3(\mu) \\ \hline 0 \\ \vdots \end{pmatrix}$$

We have to include the $\mathcal{O}(1)$ term in Z

$$Z(\alpha) = 1 + \left(\begin{array}{cc|cc} 0 & -1 & & \\ -36 & 0 & -1 & \\ \hline & 40\varepsilon & 16 & -1 \\ & & \vdots & \ddots \end{array}\right) \frac{\alpha}{4\pi\varepsilon}$$

Then

$$O_5(\mu) = O_5^0 - 40\frac{\alpha(\mu)}{4\pi} O_3^0 + \cdots$$

and the $\gamma_{(3)} \otimes \gamma_{(3)}$ contributions of O_5^0 and O_3^0 cancel in matrix elements of $O_5(\mu) = 0$.

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The one-loop anomalous dimension matrix is

$$\gamma(\alpha) = -2 \left(\begin{array}{cc|cc} 0 & -1 & & \\ -36 & 0 & -1 & \\ \hline & & 16 & -1 \\ & & \vdots & \ddots \end{array}\right) \frac{\alpha}{4\pi}$$

2 loops

The non-minimal renormalization matrix

$$Z(\alpha) = 1 + \left(Z_{10} + \frac{Z_{11}}{\varepsilon}\right)\frac{\alpha}{4\pi} + \left(Z_{20} + \frac{Z_{21}}{\varepsilon} + \frac{Z_{22}}{\varepsilon^2}\right)\left(\frac{\alpha}{4\pi}\right)^2$$

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The anomalous dimension matrix must be finite at $\varepsilon \to 0$

$$Z_{22} = \frac{1}{2} Z_{11} (Z_{11} - \beta_0)$$

$$\gamma(\alpha) = -2 Z_{11} \frac{\alpha}{4\pi} - 2(2Z_{21} - Z_{10} Z_{11} - Z_{11} Z_{10} + \beta_0 Z_{10}) \left(\frac{\alpha}{4\pi} \right)^2$$

The 1-loop $\mathcal{O}(\varepsilon^0)$ term

$$Z_{10} = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$$

$1/\varepsilon$ divergences of 1-loop integrals
(momentum-independent) times ε from γ-matrix algebra

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$1/\varepsilon$ divergences of 1-loop integrals
(momentum-independent) times ε from γ-matrix algebra

$$Z_{11} = \begin{pmatrix} b & c \\ 0 & d \end{pmatrix}$$

$$Z_{21} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

g — $1/\varepsilon^2$ divergences of 2-loop integrals
(momentum-independent) times ε from γ-matrix algebra
The lower left corner of γ must vanish

$$g = \frac{1}{2}(ab + da - \beta_0 a)$$

Evolution of the physical operators — the upper left corner of γ. At 2 loops

$$-2(2e + ca)\left(\frac{\alpha}{4\pi}\right)^2$$

e — the $1/\varepsilon$ part of 2-loop diagrams with the insertion of a physical operator

Effective low-energy QED

$$L = L_0 + L_1 \qquad L_1 = c_1^0 O_1^0 + c_3^0 O_3^0 = c_1(\mu) O_1(\mu) + c_3(\mu) O_3(\mu)$$

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Matrix form

$$L_1 = c_0^T O_0 = c^T(\mu) O(\mu)$$
$$O_0 = Z(\alpha(\mu)) O(\mu) \qquad c(\mu) = Z^T(\alpha(\mu)) c_0$$

RG equations

$$\frac{dc(\mu)}{d \log \mu} = \gamma^T(\alpha(\mu)) c(\mu)$$

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RG equations

$$\frac{dc(\mu)}{d\log\mu} = \gamma^T(\alpha(\mu)) c(\mu)$$

Evolution of coefficients of physical operators doesn't depend on coefficients of evanescent operators.

Initial conditions

$c_i(\mu_0)$ are determined by matching — equating some S-matrix elements in the full theory (expanded in p_i/M) and in the effective theory. It is most convenient to use $\mu_0 \sim M$; then $c_i(\mu_0)$ are given by perturbative series in $\alpha(\mu_0)$ containing no large logarithms. They contain all the information about physics at the scale M which is important for low-energy processes.

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The Wilson coefficients $c_i(\mu)$ at low normalization scales μ are obtained by solving the RG equations. The effective theory knows nothing about M; the only information about it is contained in $c_i(\mu)$. When the effective Lagrangian is applied to some physical process with small momenta $p_i \ll M$, it is most convenient to use μ of the order of the characteristic momenta: then the results will contain no large logarithms. This solution of the RG equation sums large logarithmic terms in perturbation series.

Contact interactions

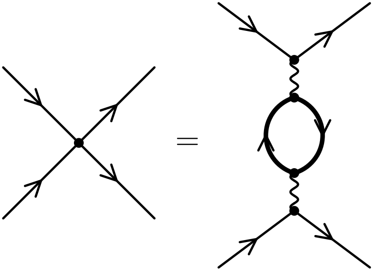
The q^2 term in the muon loop

$$\Delta L = cO \qquad c = -\frac{2}{15}\frac{\alpha}{4\pi}\frac{1}{M^2} + \mathcal{O}(\alpha^2) \qquad O = (\partial^\mu F_{\lambda\mu})(\partial_\nu F^{\lambda\nu})$$

EOM $O = e^2 O_1$

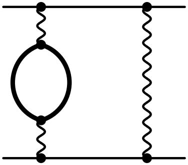
$$c_1(M) = -\frac{2}{15}\frac{\alpha^2(M) + \mathcal{O}(\alpha^3)}{M^2}$$

Matching



$$2ic_1^0 = i\frac{e_0^2}{q^2}\frac{4}{3}\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\frac{d-4}{10}\frac{q^2}{M_0^2} \Rightarrow -\frac{4}{15}i\frac{e_0^4}{(4\pi)^{d/2}}\frac{1}{M_0^{2+2\varepsilon}}$$

$\gamma_{(3)} \otimes \gamma_{(3)}$



$$c_3(M) = \frac{\mathcal{O}(\alpha^3(M))}{M^2}$$

Solving RG equations

$$\frac{dc}{d\log\alpha} = -\frac{\gamma^T(\alpha)}{2\beta(\alpha)}c$$

$$\beta(\alpha) = \beta_0\frac{\alpha}{4\pi} + \cdots \qquad \gamma^T(\alpha) = \gamma_0^T\frac{\alpha}{4\pi} + \cdots$$

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1 loop — solution with a matrix exponent

$$c(\mu) = \left(\frac{\alpha(\mu)}{\alpha(M)} \right)^{-\gamma_0^T/(2\beta_0)} c(M)$$

Solving RG equations

$$\frac{dc}{d\log\alpha} = -\frac{\gamma^T(\alpha)}{2\beta(\alpha)}c$$

$$\beta(\alpha) = \beta_0\frac{\alpha}{4\pi} + \cdots \qquad \gamma^T(\alpha) = \gamma_0^T\frac{\alpha}{4\pi} + \cdots$$

1 loop — solution with a matrix exponent

$$c(\mu) = \left(\frac{\alpha(\mu)}{\alpha(M)}\right)^{-\gamma_0^T/(2\beta_0)} c(M)$$

If eigenvectors v_i of γ_0^T ($\gamma_0^T v_i = \lambda_i v_i$) form a full basis, then

$$c(\mu) = \sum A_i \left(\frac{\alpha(\mu)}{\alpha(M)}\right)^{-\lambda_i/(2\beta_0)} v_i$$

where $c(M) = \sum A_i v_i$.

The full theory

The full theory — QED with massless electrons and muons having mass M. When we consider processes with characteristic momenta $p_i \ll M$, the existence of muons is not important. Everything can be described by an effective low-energy theory, in which there are no muons. In other words, muons only exist in loops of size $\sim 1/M$; if we are interested in processes having characteristic distances much larger than $1/M$, such loops can be replaced by local interactions of electrons and photons.

The low-energy effective theory

The effective low-energy theory contains only the light fields — electrons and photons. The Lagrangian of this theory, describing interactions of these fields at low energies, contains all operators constructed from these fields which are allowed by the symmetries. Operators with dimensionalities $d_i > 4$ are multiplied by coefficients having negative dimensionalities; these coefficients contain $1/M^{d_i-4}$. Therefore, this Lagrangian can be viewed as an expansion in $1/M$. The coefficients in this Lagrangian are fixed by matching — equating S-matrix elements up to some power of p_i/M.

Operators and parameters

Operators of the full theory are also expansions in $1/M$, in terms of all operators of the effective theory with appropriate quantum numbers. In particular, the bare electron and the photon fields of the full theory are, up to $1/M^2$ corrections,

$$\psi_0 = \left(\zeta_\psi^0\right)^{1/2} \psi_0' \qquad A_0 = \left(\zeta_A^0\right)^{1/2} A_0'$$

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$$\psi_0 = \left(\zeta_\psi^0\right)^{1/2} \psi'_0 \qquad A_0 = \left(\zeta_A^0\right)^{1/2} A'_0$$

The bare parameters in the Lagrangians of the two theories are related by

$$e_0 = \left(\zeta_\alpha^0\right)^{1/2} e'_0 \qquad a_0 = \zeta_A^0 a'_0$$

$\overline{\text{MS}}$ renormalized fields and parameters

$$\psi(\mu) = \zeta_\psi^{1/2}(\mu)\psi'(\mu) \qquad A(\mu) = \zeta_A^{1/2}(\mu)A'(\mu)$$

$$\alpha(\mu) = \zeta_\alpha(\mu)\alpha'(\mu) \qquad a(\mu) = \zeta_A(\mu)a'(\mu)$$

where

$$\zeta_\psi(\mu) = \frac{Z'_\psi(\alpha'(\mu), a'(\mu))}{Z_\psi(\alpha(\mu), a(\mu))}\zeta_\psi^0$$

$$\zeta_A(\mu) = \frac{Z'_A(\alpha'(\mu))}{Z_A(\alpha(\mu))}\zeta_A^0 \qquad \zeta_\alpha(\mu) = \frac{Z'_\alpha(\alpha'(\mu))}{Z_\alpha(\alpha(\mu))}\zeta_\alpha^0$$

Decoupling: photon field

The photon propagators in the two theories are related by

$$
\begin{aligned}
&D_\perp(p^2)\left(g_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)+a_0\frac{p_\mu p_\nu}{(p^2)^2}\\
&=\zeta_A^0\left[D'_\perp(p^2)\left(g_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)+a'_0\frac{p_\mu p_\nu}{(p^2)^2}\right]+\mathcal{O}\left(\frac{1}{M^2}\right)
\end{aligned}
$$

Matching at $p^2 \to 0$ — power-suppressed terms play no role

Bare decoupling coefficient

The full-theory propagator near the mass shell is

$$D_\perp(p^2) = \frac{Z_A^{\text{os}}}{p^2} \qquad Z_A^{\text{os}} = \frac{1}{1 - \Pi(0)}$$

Only diagrams with muon loops contribute to $\Pi(0)$, all the other diagrams contain no scale.

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In the effective theory

$$D'_\perp(p^2) = \frac{Z_A'^{\text{os}}}{p^2} \qquad Z_A'^{\text{os}} = \frac{1}{1 - \Pi'(0)} = 1$$

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$$\zeta_A^0 = \frac{Z_A^{\mathrm{os}}}{Z_A'^{\mathrm{os}}} = \frac{1}{1-\Pi(0)}$$

1 loop

n_l light lepton "flavours" in the effective theory and
$n_f = n_l + 1$ "flavours" in the full one

$$Z_A = 1 - \frac{4}{3} n_f \frac{\alpha(\mu)}{4\pi\varepsilon} + \cdots \qquad Z'_A = 1 - \frac{4}{3} n_l \frac{\alpha'(\mu)}{4\pi\varepsilon} + \cdots$$

With this accuracy $\alpha'(\mu) = \alpha(\mu)$

$$\zeta_A^{-1}(\mu) = 1 + \frac{4}{3} L \frac{\alpha(\mu)}{4\pi} + \cdots$$

Photon field renormalization with n_f

$$\left(Z_A^{\rm os}\right)^{-1} = 1 - \Pi(0) = 1 + \frac{4}{3} n_f \frac{\alpha(\mu_0)}{4\pi\varepsilon} + \left(\frac{16}{9} n_f + 2\varepsilon\right) \left(\frac{\alpha(\mu_0)}{4\pi\varepsilon}\right)^2$$

$Z_A^{\rm os}$ should be equal to the minimal renormalization constant $Z_A(\alpha(\mu_0))$ times an expression finite at $\varepsilon \to 0$

$$Z_A(\alpha) = 1 - \frac{4}{3} n_f \frac{\alpha}{4\pi\varepsilon} - 2\varepsilon n_f \left(\frac{\alpha}{4\pi\varepsilon}\right)^2$$

Z'_A: $n_f \to n_l$

Decoupling at 2 loops

$$\left(\zeta_A^0\right)^{-1} = \left(Z_A^{\text{os}}\right)^{-1} = 1 - \Pi(0) = 1 + \frac{4}{3} e^{L\varepsilon} \frac{\alpha(\mu)}{4\pi\varepsilon} Z_\alpha Z_m^{-2\varepsilon} - \varepsilon \left(6 - \frac{13}{3}\varepsilon + \cdots\right) e^{2L\varepsilon} \left(\frac{\alpha(\mu)}{4\pi\varepsilon}\right)^2 + \cdots$$

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does not depend on n_l (this dependence appears at 3 loops)

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RG equation

$$\frac{d \log \zeta_A(\mu)}{d \log \mu} + \gamma_A(\alpha(\mu)) - \gamma_A'(\alpha'(\mu)) = 0$$

Decoupling: electron field

The electron propagators in the two theories are related by

$$\not{p} S(p) = \zeta_\psi^0 \not{p} S'(p) + \mathcal{O}\left(\frac{p^2}{M^2}\right)$$

Matching at $p \to 0$ — power-suppressed terms play no role.

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where $\Sigma(p) = \Sigma_V(p^2)\not{p}$. Only diagrams with muon loops contribute to $\Sigma_V(0)$.

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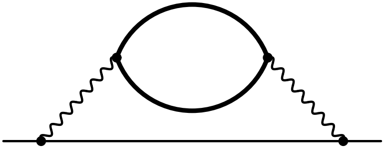
2 loops

The first diagram contributing to $\Sigma_V(0)$ appears at 2 loops



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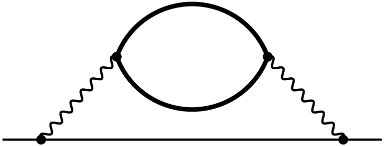

$$-i\not{p}\Sigma_V(p^2) = \int \frac{d^dk}{(2\pi)^d} ie_0\gamma^\mu i\frac{\not{k}+\not{p}}{(k+p)^2} ie_0\gamma^\nu \left(\frac{-i}{k^2}\right)^2 \\ i(k^2 g_{\mu\nu} - k_\mu k_\nu)\Pi(k^2)$$

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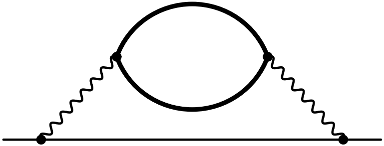

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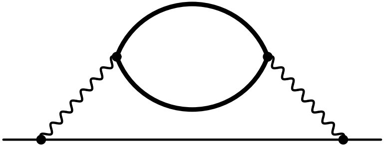

$$-i\not{p}\Sigma_V(p^2) = \int \frac{d^d k}{(2\pi)^d} ie_0\gamma^\mu i \frac{\not{k}+\not{p}}{(k+p)^2} ie_0\gamma^\nu \left(\frac{-i}{k^2}\right)^2 \\ i(k^2 g_{\mu\nu} - k_\mu k_\nu)\Pi(k^2)$$

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$$\zeta_\psi^0 = Z_\psi^{os} = 1 + \frac{e_0^4 M_0^{-4\varepsilon}}{(4\pi)^d} \Gamma^2(\varepsilon) \frac{2(d-1)(d-4)(d-6)}{d(d-2)(d-5)(d-7)}$$

Renormalized decoupling coefficient

$$\zeta_\psi(\mu) = \zeta_\psi^0 \frac{Z'_\psi(\alpha'(\mu), a'(\mu))}{Z_\psi(\alpha(\mu), a(\mu))}$$

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$$\gamma_\psi(\alpha, a) = 2a\frac{\alpha}{4\pi} - (4n_f + 3)\left(\frac{\alpha}{4\pi}\right)^2 + \cdots$$

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$$\frac{d\log\zeta_\psi(\mu)}{d\log\mu} + \gamma_\psi(\alpha(\mu), a(\mu)) - \gamma'_\psi(\alpha'(\mu), a'(\mu)) = 0$$

Decoupling: electron charge

$$e_0 \Gamma S S D = \zeta_\psi^0 \left(\zeta_A^0\right)^{1/2} e_0' \Gamma' S' S' D'$$

$S = \zeta_\psi^0 S'$, $D = \zeta_A^0 D'$

$$e_0 \Gamma^\mu = \left(\zeta_\psi^0\right)^{-1} \left(\zeta_A^0\right)^{-1/2} e_0' \Gamma'^\mu$$

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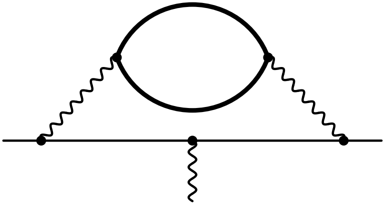

Bare decoupling

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Ward identity

$$\Gamma^\mu(p,p) = \frac{\partial S^{-1}(p)}{\partial p_\mu}$$

Near the mass shell

$$S(p) = \frac{Z_\psi^{\text{os}}}{\not{p}}$$

and therefore

$$Z_\Gamma^{\text{os}} Z_\psi^{\text{os}} = 1$$

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Effective theory: $Z_\Gamma'^{\rm os} Z_\psi'^{\rm os} = 1$

$$\zeta_\Gamma^0 \zeta_\psi^0 = 1 \qquad \zeta_\alpha^0 = \left(\zeta_A^0\right)^{-1}$$

$\overline{\text{MS}}$ renormalization

$S(p) = Z_\psi S_r(p)$, $\Gamma^\mu = Z_\Gamma \Gamma^\mu_r$
Ward identity: $Z_\Gamma Z_\psi$ is finite at $\varepsilon \to 0$

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$$\alpha(\mu_0) = \zeta_\alpha(\mu_0) \alpha'(\mu_0) \qquad \zeta_\alpha(\mu_0) = 1 + \frac{13}{3} \left(\frac{\alpha(\mu_0)}{4\pi} \right)^2 + \cdots$$

Decoupling: bilinear electron currents

$$j_{n0} = \bar{\psi}_0 \gamma_{(n)} \tau \psi_0 \qquad \gamma_{(n)} = \gamma^{[\mu_1} \ldots \gamma^{\mu_n]}$$

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The on-shell matrix element of $j_n(\mu)$ is $M_n(p, p'; \mu) = Z_q^{\text{os}} Z_{jn}^{-1}(\alpha(\mu)) \Gamma_n(p, p')$. It should be equal to $\zeta_{jn}(\mu) M'_n(p, p'; \mu)$, where $M'_n(p, p'; \mu) = Z_q'^{\text{os}} Z_{jn}'^{-1}(\alpha'(\mu)) \Gamma'_n(p, p')$. Both matrix elements are UV-finite; their IR divergences coincide, because both theories are identical in the IR region. Any on-shell momenta p, p' can be used; it is easiest to set $p = p' = 0$, thus excluding power-suppressed terms.

$\Gamma_n(p,p') = \gamma_{(n)} + \Lambda_n(p,p')$ at $p = p' = 0$ has the structure $\Gamma_n(0,0) = \gamma_{(n)}\Gamma_n$, where $\Gamma_n = 1 + \Lambda_n$ is scalar.

$$\zeta_{jn}(\mu) = \frac{Z'_{jn}(\alpha'(\mu))}{Z_{jn}(\alpha(\mu))}\zeta^0_{jn} \qquad \zeta^0_{jn} = Z_q^{\mathrm{os}}\Gamma_n$$

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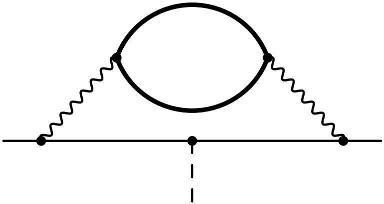

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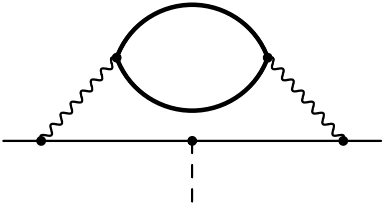

$$\Lambda_n(0,0) = -ie_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{\gamma^\mu \not{k} \gamma_{(n)} \not{k} \gamma^\nu (k^2 g_{\mu\nu} - k_\mu k_\nu)\Pi(k^2)}{(k^2)^4}$$

$$\zeta_{jn}^{0} = 1 - \frac{e_0^4 M_0^{-4\varepsilon}}{(4\pi)^d} \Gamma^2(\varepsilon) \frac{8(d-6)(n-1)(n-d+1)}{d(d-2)(d-5)(d-7)}$$

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$$\begin{aligned} \gamma_{jn} = & -2(n-1)(n-3)\frac{\alpha}{4\pi}\left[1 + \frac{1}{2}\bigl(5(n-2)^2 - 19\bigr)\frac{\alpha}{4\pi}\right] \\ & - \frac{1}{3}(n-1)(n-15)\beta_0\left(\frac{\alpha}{4\pi}\right)^2 + \cdots \end{aligned}$$

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$$\zeta_{jn}(M) = 1 + \frac{1}{54}(n-1)(85n-267)\left(\frac{\alpha(M)}{4\pi}\right)^2+\cdots$$

Vector current

Vector current: $\zeta_{j1} = 1$ to all orders. Diagonal τ — weighted integers (differences of the numbers of leptons and antileptons).

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Multiplying the Ward identity

$$\Gamma_1^\mu(0,0) = \gamma^\mu - \left.\frac{\partial \Sigma(p)}{\partial p_\mu}\right|_{p=0} = \gamma^\mu \left(1 - \Sigma_V(0)\right)$$

by $Z_\psi^{\rm os} = \left[1 - \Sigma_V(0)\right]^{-1}$, we obtain just $\Gamma_1 = 1$. Taking account of the fact that the vector current does not renormalize ($Z_{j1} = 1$, $Z'_{j1} = 1$) yields $\zeta_{j1}(\mu) = 1$.

't Hooft–Veltman and anticommuting γ_5

$j_{\mathrm{AC}}(\mu) = Z_{j0}^{-1}(\alpha(\mu))\bar{\psi}_0\gamma_5^{\mathrm{AC}}\tau\psi_0$ and
$j_{\mathrm{HV}}(\mu) = Z_{j4}^{-1}(\alpha(\mu))\bar{\psi}_0\gamma_5^{\mathrm{HV}}\tau\psi_0$ are related by

$$j_{\mathrm{AC}}(\mu) = Z_P(\alpha(\mu))j_{\mathrm{HV}}(\mu) \qquad Z_P(\alpha) = 1 + z_{P1}\frac{\alpha}{4\pi} + z_{P2}\left(\frac{\alpha}{4\pi}\right)^2 + \cdots$$

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$$j_{\mathrm{AC}}^{\mu}(\mu) = Z_A(\alpha(\mu))j_{\mathrm{HV}}^{\mu}(\mu)$$

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$$j^\mu_{\mathrm{AC}}(\mu) = Z_A(\alpha(\mu))j^\mu_{\mathrm{HV}}(\mu)$$

RG equations

$$\frac{d\log Z_P(\alpha)}{d\log\alpha} = \frac{\gamma_{j0}(\alpha) - \gamma_{j4}(\alpha)}{2\beta(\alpha)}$$

$$\frac{d\log Z_A(\alpha)}{d\log\alpha} = \frac{\gamma_{j1}(\alpha) - \gamma_{j3}(\alpha)}{2\beta(\alpha)} \quad \text{where } \gamma_{j1} = 0$$

$$Z_P(\alpha) = 1 - 8\frac{\alpha}{4\pi} + \frac{8}{9}n_f\left(\frac{\alpha}{4\pi}\right)^2 + \cdots$$
$$Z_A(\alpha) = 1 - 4\frac{\alpha}{4\pi} + \frac{2}{9}(99 + 2n_f)\left(\frac{\alpha}{4\pi}\right)^2 + \cdots$$

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The currents j_4 and j_3 differ from j_0 and j_1 by insertion of γ_5^{HV}. Inserting γ_5^{AC} does not change the decoupling coefficient.

$$\zeta_{j4} = \zeta_{j0}\frac{Z'_P}{Z_P} \qquad \zeta_{j3} = \zeta_{j1}\frac{Z'_A}{Z_A}$$

Decoupling: electron mass

Now let's take a small electron mass m into account

$$S(p) = \frac{1}{\not{p} - m_0 - \Sigma(p)} = \frac{1}{1 - \Sigma_V(p^2)} \, \frac{1}{\not{p} - \dfrac{1 + \Sigma_S(p^2)}{1 - \Sigma_V(p^2)} m_0}$$

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$$\frac{1}{1 - \Sigma_V(p^2)} \frac{1}{\not{p} - \dfrac{1 + \Sigma_S(p^2)}{1 - \Sigma_V(p^2)} m_0} = \zeta_\psi^0 \frac{1}{1 - \Sigma'_V(p^2)} \frac{1}{\not{p} - \dfrac{1 + \Sigma'_S(p^2)}{1 - \Sigma'_V(p^2)} m'_0}$$

On-shell mass is the same

$$\frac{1 + \Sigma_S(p^2)}{1 - \Sigma_V(p^2)} m_0 = \frac{1 + \Sigma'_S(p^2)}{1 - \Sigma'_V(p^2)} m'_0$$

$$\zeta_m^0 = \frac{m_0}{m_0'} = \left(\zeta_\psi^0\right)^{-1} \frac{1 + \Sigma_S'(p^2)}{1 + \Sigma_S(p^2)}$$

This equation should hold for all $m \ll M$, $p \ll M$. It is easiest to set $m = 0$, and use $1 + \Sigma_S(p^2)|_{m_0=0} = \Gamma_0(p,p)$. Setting $p = 0$:

$$\zeta_m^0 = \left(\zeta_{j0}^0\right)^{-1}$$

$Z_m = Z_{j0}^{-1}$, $Z_m' = Z_{j0}'^{-1}$

$$\zeta_m(\mu) = \frac{Z_m'}{Z_m}\zeta_m^0 = \zeta_{j0}^{-1}(\mu)$$

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$$\zeta_m(\mu) = \frac{Z_m'}{Z_m}\zeta_m^0 = \zeta_{j0}^{-1}(\mu)$$

$$\zeta_m(M) = 1 - \frac{89}{18}\left(\frac{\alpha(M)}{4\pi}\right)^2 + \cdots$$

RG equation

$$\frac{d\log\zeta_m(\mu)}{d\log\mu} + \gamma_m(\alpha(\mu)) - \gamma_m'(\alpha'(\mu)) = 0$$

QCD

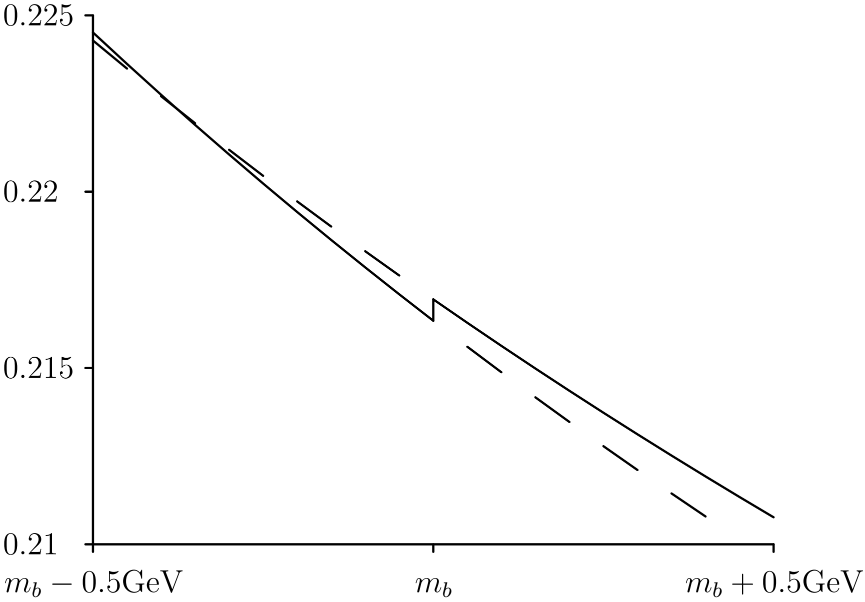
$$\alpha_s^{(n_l+1)}(\mu) = \zeta_\alpha(\mu)\alpha_s^{(n_l)}(\mu)$$

$$\zeta_\alpha(\mu_0) = 1 + \left(\frac{13}{3}C_F - \frac{32}{9}C_A\right) T_F \left(\frac{\alpha_s(\mu_0)}{4\pi}\right)^2 + \cdots$$

RG equation

$$\frac{d\log\zeta_\alpha(\mu)}{d\log\mu} + 2\beta^{(n_l+1)}(\alpha_s^{(n_l+1)}(\mu)) - 2\beta^{(n_l)}(\alpha_s^{(n_l)}(\mu)) = 0$$

QCD



In the past

Only renormalizable theories were considered well-defined: they contain a finite number of parameters, which can be extracted from a finite number of experimental results and used to predict an infinite number of other potential measurements. Non-renormalizable theories were rejected because their renormalization at all orders in non-renormalizable interactions involve infinitely many parameters, so that such a theory has no predictive power. This principle is absolutely correct, if we are impudent enough to pretend that our theory describes the Nature up to arbitrarily high energies (or arbitrarily small distances).

At present

We accept the fact that our theories only describe the Nature at sufficiently low energies (or sufficiently large distances). They are effective low-energy theories. Such theories contain all operators (allowed by the relevant symmetries) in their Lagrangians. They are necessarily non-renormalizable. This does not prevent us from obtaining definite predictions at any fixed order in the expansion in E/M, where E is the characteristic energy and M is the scale of new physics. Only if we are lucky and M is many orders of magnitude larger than the energies we are interested in, we can neglect higher-dimensional operators in the Lagrangian and work with a renormalizable theory.

Conclusion

Practically all physicists believe that the Standard Model is also a low-energy effective theory. But we don't know what is a more fundamental theory whose low-energy approximation is the Standard Model. Maybe, it is some supersymmetric theory (with broken supersymmetry); maybe, it is not a field theory, but a theory of extended objects (superstrings, branes); maybe, this more fundamental theory lives in a higher-dimensional space, with some dimensions compactified; or maybe it is something we cannot imagine at present. The future will tell.