Reduction of 1-loop Tensor integrals

J.Fleischer

Project in collaboration with T.Diakonides, B.Tausk and T. Riemann

1 Mini-Review

There are well known methods for tensor reduction:

- G.Passarino, M.Veltman: Nucl. Phys. **B160**(1979)151 - Solving a sysem of equations → Gram determinants
- A.I. Davydychev, *Phys.Lett.***B263**(1991)107 and J.F., F.Jegerlehner, O.V.Tarasov *Nucl. Phys.***D54**(1996)6479 - integrals in higher dimensions and recursion relations
- A.Denner, S.Dittmaier :Nucl. Phys. **B734**(2006)62 - based on PV, higher point integrals and higher rank tensors reduced to higher rank boxes
- T.Binoth et al.: *JHEP* **10**(2005)015 - integrals in higher dimensions

For a recent review : S.Weinzierl, *Automated calculations for multileg processes*, **ACAT**(2007)005.

2 Tensor Integrals

Arbitrary n-point functions. Scalar propagators

$c_j = (k - q_j)^2 - m_j^2 + i\varepsilon, \ \ (\mathrm{j} < \mathrm{n})$ and $\mathrm{c_n} = \mathrm{k}^2 - \mathrm{m_n^2} + \mathrm{i}\varepsilon.$ $\mathrm{q_j}$ are the "**chords**"; kinematics wih $\mathrm{q_n} = 0.$

We follow **A.I. Davydychev** and **J.F., F.Jegerlehner, O.V.Tarasov**

$$
\begin{aligned}
I_n^{\mu} &= \int^d k^{\mu} \prod_{r=1}^{n} c_r^{-1} = - \sum_{i=1}^{n} q_i^{\mu} \, I_{n,i}^{[d+]} \\
I_n^{\mu\,\nu} &= \int^d k^{\mu}\, k^{\nu} \prod_{r=1}^{n} c_r^{-1} = \sum_{i,j=1}^{n} q_i^{\mu}\, q_j^{\nu}\, n_{ij}\, I_{n,ij}^{[d+]^2} - \frac{1}{2} g^{\mu\nu}\, I_n^{[d+]} \,, \\
I_n^{\mu\,\nu\,\lambda} &= \int^d k^{\mu}\, k^{\nu}\, k^{\lambda} \prod_{r=1}^{n} c_r^{-1} = - \sum_{i,j,k=1}^{n-1} q_i^{\mu}\, q_j^{\nu}\, q_k^{\lambda}\, n_{ijk}\, I_{n,ijk}^{[d+]^3} \\
&\quad + \frac{1}{2} \sum_{i=1}^{n} (g^{\mu\nu}\, q_i^{\lambda} + g^{\mu\lambda}\, q_i^{\nu} + g^{\nu\lambda}\, q_i^{\mu}) I_{n,i}^{[d+]^2} \,, \\
I_n^{\mu\,\nu\,\lambda\,\rho} &= \int^d k^{\mu}\, k^{\nu}\, k^{\lambda}\, k^{\rho} \prod_{r=1}^{n} c_r^{-1} = \sum_{i,j,k,l=1}^{n} q_i^{\mu}\, q_j^{\nu}\, q_k^{\lambda}\, q_l^{\rho}\, n_{ijkl}\, I_{n,ijkl}^{[d+]^4} \\
&\quad - \frac{1}{2} \sum_{i,j=1}^{n} (g^{\mu\nu}\, q_i^{\lambda}\, q_j^{\rho} + g^{\mu\lambda}\, q_i^{\nu}\, q_j^{\rho} + g^{\nu\lambda}\, q_i^{\mu}\, q_j^{\rho} + g^{\mu\rho}\, q_i^{\nu}\, q_j^{\lambda} + g^{\nu\rho}\, q_i^{\mu}\, q_j^{\lambda} + g^{\lambda\rho}\, q_i^{\mu}\, q_j^{\nu})\, n_{ij} I_{n,ij}^{[d+]^3} \\
&\quad + \frac{1}{4} \left(g^{\mu\nu}\, g^{\lambda\rho} + g^{\mu\lambda}\, g^{\nu\rho} + g^{\mu\rho}\, g^{\nu\lambda} \right) I_n^{[d+]^2} \,,
\end{aligned}
$$

$$I_{p,\,i\,j\,k\cdots}^{[d+]^l,stu\cdots} = \int^{[d+]^l} \prod_{r=1}^{n} \frac{1}{c_r^{1+\delta_{ri}+\delta_{rj}+\delta_{rk}+\cdots-\delta_{rs}-\delta_{rt}-\delta_{ru}-\cdots}}, \qquad \int^d \equiv \int \frac{d^d k}{\pi^{d/2}},$$

$[d+]^l = 4 + 2l - 2\varepsilon$ (p is the number of scalar propagators of the "p-point function"; equal lower and upper indices "cancel").
n_{ij}, n_{ijk} and n_{ijkl} : for the successive application of recurrence relations to reduce higher dimensional integrals:
$n_{ij} = \nu_{ij}, n_{ijk} = \nu_{ij}\nu_{ijk}$ and $n_{ijkl} = \nu_{ij}\nu_{ijk}\nu_{ijkl}$ with $\nu_{ij} = 1 + \delta_{ij}, \nu_{ijk} = 1 + \delta_{ik} + \delta_{jk}$ and $\nu_{ijkl} = 1 + \delta_{il} + \delta_{jl} + \delta_{kl}$.
Gram determinants:

$$Y_{ij} = -(q_i - q_j)^2 + m_i^2 + m_j^2,$$

"modified Cayley determinant" of the diagram with internal lines $1 \ldots n$:

$$()_n \equiv \begin{vmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & Y_{11} & Y_{12} & \ldots & Y_{1n} \\ 1 & Y_{12} & Y_{22} & \ldots & Y_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & Y_{1n} & Y_{2n} & \ldots & Y_{nn} \end{vmatrix},$$

$()_n = -|G_{ik}|, \;\; G_{ik} = 2q_i \cdot q_k; \;\; i,k = 1,\ldots n-1$. "Signed minors:"

$$\begin{pmatrix} j_1 & j_2 & \cdots j_m \\ k_1 & k_2 & \cdots k_m \end{pmatrix}_n,$$

labeled by rows $j_1, j_2, \cdots j_m$ and columns $k_1, k_2, \cdots k_m$ excluded from $()_n$. Sign: $(-1)^{j_1+j_2+\cdots+j_m+k_1+k_2+\cdots+k_m} \times$ Signature$[j_1, j_2, \cdots j_m] \times$ Signature$[k_1, k_2, \cdots k_m]$, "Signature" = sign of permutations to place the indices in increasing order.

Example

$$\Delta_n = \begin{vmatrix} Y_{11} & Y_{12} & \ldots & Y_{1n} \\ Y_{12} & Y_{22} & \ldots & Y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \ldots & Y_{nn} \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}_n .$$

Recursion Relations:

The integrals $I^{[d+]^l, stu\cdots}_{p, i\,j\,k\cdots}$ occurring in the tensor decomposition are supposed to be reduced to integrals in generic dimension and index 1 (scalar n-point functions). For the reduction of indices the "integration by parts" method (K.Chetyrkin and T.Tkachov, *Nucl.Phys.* **B192**(1981)159) is usefull, reducing dimensions: O.Tarasov *Phys.Rev.***D54**(1996)6478.

Without derivation: 2 relevant relations (reducing an index and the dimension, respectively dimension only)

$$\left(\right)_n \nu_j \mathbf{j}^+ I_n^{(d+2)} = \left[-\binom{j}{0}_n + \sum_{k=1}^n \binom{j}{k}_n \mathbf{k}^- \right] I_n^{(d)} \quad \text{and} \quad \left(d - \sum_{i=1}^n \nu_i + 1 \right) \left(\right)_n I_n^{(d+2)} = \left[\binom{0}{0}_n - \sum_{k=1}^n \binom{0}{k}_n \mathbf{k}^- \right] I_n^{(d)}$$

$\mathbf{j}^\pm$ etc. shift the indices $\nu_j \to \nu_j \pm 1$; Examples:

$$\nu_{ij} I_{5,ij}^{[d+]^3} = -\frac{\binom{j}{0}_5}{\left(\right)_5} I_{5,i}^{[d+]^2} + \sum_{k=1,k\neq i}^{5} \frac{\binom{j}{k}_5}{\left(\right)_5} I_{4,i}^{[d+]^2,k} + \frac{\binom{i}{j}_5}{\left(\right)_5} I_5^{[d+]^2}, \quad \text{and} \quad I_4^{[d+],s} = \left[\frac{\binom{0s}{0s}_5}{\binom{s}{s}_5} I_4^s - \sum_{k=1,t\neq s}^{5} \frac{\binom{0s}{ks}_5}{\binom{s}{s}_5} I_3^{sk} \right] \frac{1}{d-3}$$

Applications:

Scalar 5-point function ($n = 5$)

$$(d-4)\,()_5\, I_5^{[d+]} = \binom{0}{0}_5 I_5 - \sum_{s=1}^{5} \binom{0}{s}_5 I_4^s$$

With $I_5^{[d+]}$ finite for $d = 4$ we have in this limit

$$E \equiv I_5 = \frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \binom{0}{s}_5 I_4^s.$$

Similarly for the **vector** we obtain

$$I_5^\mu = \sum_{i=1}^{4} q_i^\mu E_i,$$

$$E_i \equiv -I_{5,i}^{[d+]} = (d-4)\frac{\binom{0}{i}_5}{\binom{0}{0}_5} I_5^{[d+]} - \frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \binom{0i}{0s}_5 I_4^s,$$

where again in the limit $d \to 4$ the $I_5^{[d+]}$ disappears (see Z.Bern,L.J.Dixon, and D.A.Kosower *Nucl. Phys.* **B412**(1994)751.
These two cases are simple and lead to a direct reduction to scalar integrals, without the Gram determinant $()_5$.

Applying the first recursion relation to $I_{n,i}^{[d+]}$ we can write the vector n-point function for arbitrary $n \le 5$:

$$I_n^\mu \;=\; I_n \sum_{i=1}^{n} q_i^\mu \frac{\binom{0}{i}_n}{()_n} - \sum_{s=1}^{n} I_{n-1}^s \sum_{i=1}^{n} q_i^\mu \frac{\binom{s}{i}_n}{()_n} \equiv I_n Q_0^\mu - \sum_{s=1}^{n} I_{n-1}^s Q_s^\mu,$$

where we introduced the **universal** vectors

$$Q_s^\mu = \sum_{i=1}^{n} q_i^\mu \frac{\binom{s}{i}_n}{()_n} \quad , \quad s = 0 \dots n.$$

For higher tensors a complication arises due to the appearence of the $g^{\mu\nu}$ tensor, which we have to express in terms of the chords

$$g^{\mu\nu} = 2 \sum_{i,j=1}^{5} q_i^\mu q_j^\nu \frac{\binom{i}{j}_5}{()_5}$$

for which we need 4 independent vectors. Therefore this representation applies for $n = 5$ only.

Disregarding this (technical) problem for the time beeing, the observation is that we might obtain acceptable results from the Davydychev formula if $g^{\mu\nu}$ is eliminated such that considerable simplifications (cancellations) can occur among the various terms when appying the recursion relations.

Thus a representation of $g^{\mu\nu}$ as above is essenial: we express all tensors in terms of independent chords - and moreover we speak the same language for all terms, which is the algebra of the signed minors!

In the next steps we will investigate $n = 5$ and cancel the Gram determinant $()_5$; later we find results for arbitrary $n \le 6$!

3 The algebra of signed minors

Signed minors are subdeterminants and it is surprising how many useful relations between them can be derived. In fact these were specified in the article of Melrose. : *Nuovo Cim.* **40**(1965)181. Important general (arbitrary n) relations are

$$\binom{il}{jk}_n ()_n = \binom{i}{j}_n \binom{l}{k}_n - \binom{i}{k}_n \binom{l}{j}_n ; \quad i,j,k,l = 0,\ldots,n \ ,$$

$$\sum_{i=1}^{n} \binom{0}{i}_n = ()_n \quad \text{and} \quad \sum_{i=1}^{n} \binom{j}{i}_n = 0, \ (j \neq 0).$$

Further, “extensionals“ are needed, i.e. relations valid for $()_n$ can be extended to any minor of $()_n$; extensionals of the last two relations are

$$\sum_{i=1}^{n} \binom{0k}{il}_n = \binom{k}{l}_n \quad \text{and} \quad \sum_{i=1}^{n} \binom{jk}{il}_n = 0, \ (j \neq 0).$$

For the reduction of 5-point tensors to scalars we need

$$-\binom{0}{j}_5 \binom{ts}{0i}_5 \binom{s}{s}_5 - \binom{s}{j}_5 \binom{ts}{is}_5 \binom{0}{0}_5 + \binom{0}{s}_5 \binom{ts}{0s}_5 \binom{i}{j}_5 = ()_5 X_{ij}^{st} \quad \texttt{with} \quad X_{ij}^{st} = -\binom{0s}{0j}_5 \binom{ts}{is}_5 + \binom{0i}{sj}_5 \binom{ts}{0s}_5 .$$

The proof is obained by direcly evaluating the r.h.s., $X_{ij}^{st}()_5$, resulting in

$$\binom{s}{0}_5 \binom{0}{j}_5 \binom{ts}{is} - \binom{i}{s}_5 \binom{0}{j}_5 \binom{ts}{0s} = -\binom{0}{j}_5 \binom{ts}{0i} \binom{s}{s}_5 , \tag{3.1}$$

which in turn is proven by multiplying again with $()_5$. Further relations will be shown when needed.

4 The tensor integral of rank 2

The tensor integral of **rank 2** can be written without a $g_{\mu\nu}$-term:

$$I_5^{\mu\,\nu} = \sum_{i,j=1}^{4} q_i^{\mu}\, q_j^{\nu} I_{5,ij}.$$

Dropping $I_5^{[d+]}$ (as above it drops out) we have

$$I_{5,ij} = \nu_{ij} I_{5,ij}^{[d+]^2} = -\frac{\binom{0}{j}_5}{()_5} I_{5,i}^{[d+]} + \sum_{s=1,s\neq i}^{5} \frac{\binom{s}{j}_5}{()_5} I_{4,i}^{[d+],s} = \frac{\binom{0}{j}_5}{()_5} E_i + \sum_{s=1,s\neq i}^{5} \frac{\binom{s}{j}_5}{()_5} I_{4,i}^{[d+],s} \qquad \texttt{and with}$$

$$I_{4,i}^{[d+],s} = -\frac{\binom{0s}{is}_5}{\binom{s}{s}_5} I_4^s + \sum_{t=1,t\neq s}^{5} \frac{\binom{ts}{is}_5}{\binom{s}{s}_5} I_3^{st} \qquad \texttt{we have}$$

$$\begin{aligned}
I_{5,ij} = \; & \frac{1}{\binom{0}{0}_5 ()_5} \sum_{s=1,s\neq i}^{5} \frac{1}{\binom{s}{s}_5} \left\{ -\binom{0}{j}_5 \binom{0s}{0i}_5 \binom{s}{s}_5 - \binom{s}{j}_5 \binom{0s}{is}_5 \binom{0}{0}_5 + \binom{s}{0}_5 \binom{0s}{0s}_5 \binom{i}{j}_5 \right\} I_4^s - \\
& \frac{\binom{i}{j}_5}{\binom{0}{0}_5 ()_5} \sum_{s=1,s\neq i}^{5} \frac{1}{\binom{s}{s}_5} \binom{s}{0}_5 \binom{0s}{0s}_5 I_4^s - \\
& \frac{1}{\binom{0}{0}_5 ()_5} \sum_{s,t=1,s\neq i,t}^{5} \frac{1}{\binom{s}{s}_5} \left\{ -\binom{0}{j}_5 \binom{ts}{0i}_5 \binom{s}{s}_5 - \binom{s}{j}_5 \binom{ts}{is}_5 \binom{0}{0}_5 + \binom{s}{0}_5 \binom{ts}{0s}_5 \binom{i}{j}_5 \right\} I_3^{st} + \\
& \frac{\binom{i}{j}_5}{\binom{0}{0}_5 ()_5} \sum_{s,t=1,s\neq i,t}^{5} \frac{1}{\binom{s}{s}_5} \binom{s}{0}_5 \binom{ts}{0s}_5 I_3^{st}.
\end{aligned}$$

With the notation (recombining $g^{\mu\nu}$)

$$I_5^{\mu\,\nu} = \sum_{i,j=1}^{4} q_i^{\mu}\, q_j^{\nu} E_{ij} + g^{\mu\nu} E_{00}, \quad E_{ij} = \sum_{s=1}^{5} S_{ij}^{4,s} I_4^s + \sum_{s,t=1}^{5} S_{ij}^{3,st} I_3^{st},$$

we have

$$S_{ij}^{4,s} = \frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \frac{1}{\binom{s}{s}_5} X_{ij}^{s0}, \quad S_{ij}^{3,st} = -\frac{1}{\binom{0}{0}_5} \sum_{s,t=1}^{5} \frac{1}{\binom{s}{s}_5} X_{ij}^{st},$$

and taking into account the expression for $I_4^{[d+]}$

$$E_{00} = -\frac{1}{2} \frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \frac{\binom{s}{0}_5}{\binom{s}{s}_5} \left[\begin{pmatrix} 0s \\ 0s \end{pmatrix}_5 I_4^s - \sum_{t=1}^{5} \begin{pmatrix} ts \\ 0s \end{pmatrix}_5 I_3^{st} \right] = -\frac{1}{2} \frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \begin{pmatrix} s \\ 0 \end{pmatrix}_5 I_4^{[d+],s}.$$

In this way we have cancelled the Gram determinant $()_5$ for the tensor of degree 2 and represented the tensor in relatively simple form.

5 Reduction of the rank 3 tensor integral

The tensor integral of **rank 3** can be written as

$$I_5^{\mu\,\nu\,\lambda} = \sum_{i,j,k=1}^{4} q_i^{\mu}\, q_j^{\nu}\, q_k^{\lambda} I_{5,ijk}.$$

As before, substituting $g^{\mu\nu}$

$$I_{5,ijk} = -\nu_{ij}\nu_{ijk} I_{5,ijk}^{[d+]^3} + \left[\frac{\binom{j}{k}_5}{\binom{}{}_5} I_{5,i}^{[d+]^2} + \frac{\binom{i}{k}_5}{\binom{}{}_5} I_{5,j}^{[d+]^2} + \frac{\binom{i}{j}_5}{\binom{}{}_5} I_{5,k}^{[d+]^2} \right].$$

Inserting recursion relations and keeping in mind to drop $I_5^{[d+]}$, we have

$$I_{5,ijk} = \frac{\binom{0}{k}_5}{\binom{}{}_5} I_{5,ij} + \frac{\binom{i}{j}_5}{\binom{}{}_5} \sum_{s=1}^{5} \frac{\binom{s}{k}_5}{\binom{}{}_5} I_4^{[d+],s} - \sum_{s=1,s\neq i,j}^{5} \frac{\binom{s}{k}_5}{\binom{}{}_5} \nu_{ij} I_{4,ij}^{[d+]^2,s} = \frac{\binom{0}{k}_5}{\binom{}{}_5} \left\{ E_{ij} + 2\frac{\binom{i}{j}_5}{\binom{}{}_5} E_{00} \right\} + \cdots,$$

from which we obtain for the $\binom{i}{j}_5$ terms

$$\frac{\binom{i}{j}_5}{\binom{0}{0}_5} \frac{1}{\binom{}{}_5^2} \sum_{s=1}^{5} \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}_5 \begin{pmatrix} s \\ k \end{pmatrix}_5 - \begin{pmatrix} 0 \\ k \end{pmatrix}_5 \begin{pmatrix} s \\ 0 \end{pmatrix}_5 \right] I_4^{[d+],s} = \frac{\binom{i}{j}_5}{\binom{0}{0}_5} \frac{1}{\binom{}{}_5} \sum_{s=1}^{5} \begin{pmatrix} 0s \\ 0k \end{pmatrix}_5 I_4^{[d+],s},$$ yielding a $g^{\mu\nu}$! Need:

$$\nu_{ij} I_{4,ij}^{[d+]^2,s} = -\frac{\binom{0s}{js}_5}{\binom{s}{s}_5} I_{4,i}^{[d+],s} + \sum_{t=1,t\neq s,i}^{5} \frac{\binom{ts}{js}_5}{\binom{s}{s}_5} I_{3,i}^{[d+],st} + \frac{\binom{is}{js}_5}{\binom{s}{s}_5} I_4^{[d+],s}$$ and $$I_{4,i}^{[d+],s} = -\frac{\binom{0s}{is}_5}{\binom{s}{s}_5} I_4^s + \sum_{t=1,t\neq s}^{5} \frac{\binom{ts}{is}_5}{\binom{s}{s}_5} I_3^{st}$$

Disregarding the $g^{\mu\nu}$- term and the 3-point term $I_{3,i}^{[d+],st}$ for the time being, we have after $\binom{0}{0}_5 \binom{s}{k}_5 = \binom{0s}{0k}_5 ()_5 + \binom{s}{0}_5 \binom{0}{k}_5$

$$
\begin{aligned}
3\binom{0}{0}_5 I_{5,ijk} = \cdots \; & - \; \frac{1}{()_5 \binom{s}{s}_5^2}\binom{0}{k}_5 \cdot \\
& \left\{ \underline{\binom{s}{s}_5} \left[\underline{\binom{0s}{0i}_5} \binom{0s}{js}_5 - \binom{0j}{si}_5 \binom{0s}{0s}_5 \right] + \underline{\binom{s}{0}_5} \left[\binom{0s}{js}_5 \underline{\binom{0s}{is}_5} + \binom{is}{js}_5 \binom{0s}{0s}_5 \right] \right\} I_4^s \\
& + \; \frac{1}{()_5 \binom{s}{s}_5^2}\binom{0}{k}_5 \cdot \\
& \left\{ \binom{s}{s}_5 \left[\binom{0s}{0j}_5 \binom{ts}{is}_5 - \binom{0i}{sj}_5 \binom{ts}{0s}_5 \right] + \binom{s}{0}_5 \left[\binom{0s}{js}_5 \binom{ts}{is}_5 + \binom{js}{is}_5 \binom{ts}{0s}_5 \right] \right\} I_3^{st} \\
& + \; (i \leftrightarrow k) + (j \leftrightarrow k).
\end{aligned}
$$

Here the following **"master formula"** is of great help

$$
\binom{s}{i}_5 \binom{\tau s}{0s}_5 = \binom{s}{0}_5 \binom{\tau s}{is}_5 + \binom{s}{s}_5 \binom{\tau s}{0i}_5, \quad \tau = 0, 1, \ldots 5,
$$

which yields explicitely

$$
\underline{\binom{s}{s}_5 \binom{0s}{0i}_5 + \binom{s}{0}_5 \binom{0s}{is}_5} = \binom{s}{i}_5 \binom{0s}{0s}_5 \quad \text{and} \quad \binom{s}{s}_5 \binom{0j}{si}_5 - \binom{s}{0}_5 \binom{is}{js}_5 = -\binom{s}{j}_5 \binom{0s}{is}_5,
$$

so that the above curly brackets $\{\}$ can be combined to give

$$
-\frac{1}{()_5 \binom{s}{s}_5^2} \binom{0}{k}_5 \cdot \left\{ \left[\binom{s}{i}_5 \binom{0s}{js}_5 + \binom{s}{j}_5 \binom{0s}{is}_5 \right] \binom{0s}{0s}_5 I_4^s - \left[\binom{s}{j}_5 \binom{0s}{0s}_5 \binom{ts}{is}_5 + \binom{s}{i}_5 \binom{0s}{js}_5 \binom{ts}{0s}_5 \right] I_3^{st} \right\}.
$$

As next we use

$$\binom{0}{k}_5\binom{s}{i}_5 = -\binom{0i}{sk}_5 ()_5 + \binom{i}{k}_5\binom{s}{0}_5 \text{ and not } \binom{0}{k}_5\binom{s}{i}_5 = \binom{0s}{ki}_5 ()_5 + \binom{0}{i}_5\binom{s}{k}_5 !$$

In this way one factor $()_5$ is cancelled and the other yields a further contribution to the metric tensor!

Remains the $I_{3,i}^{[d+],st}$ term. After having applied again $\binom{0}{0}_5\binom{s}{k}_5 = \binom{0s}{0k}_5 ()_5 + \binom{s}{0}_5\binom{0}{k}_5$ we use

$$\binom{s}{0}_5\binom{ts}{js}_5 = \binom{s}{j}_5\binom{ts}{0s}_5 - \binom{s}{s}_5\binom{ts}{0j}_5 .$$

The $\binom{s}{s}_5$ in the second term cancels and the remaining factor is antisymmetric in s and t, i.e. this term drops out after summation over s, t. Using again $\binom{0}{k}_5\binom{s}{j}_5 = -\binom{0j}{sk}_5 ()_5 + \binom{j}{k}_5\binom{s}{0}_5$ we can collect all contributions, using the notation

$$I_5^{\mu\,\nu\,\lambda} = \sum_{i,j,k=1}^{4} q_i^{\mu}\, q_j^{\nu}\, q_k^{\lambda} E_{ijk} + \sum_{k=1}^{4} g^{[\mu\nu} q_k^{\lambda]} E_{00k}, \qquad E_{ijk} = \sum_{s=1}^{5} S_{ijk}^{4,s} I_4^{s} + \sum_{s,t=1}^{5} S_{ijk}^{3,st} I_3^{st} + \sum_{s,t,u=1}^{5} S_{ijk}^{2,stu} I_2^{stu} .$$

$$S^{4,s}_{ijk} = \frac{1}{3\binom{0}{0}_5\binom{s}{s}_5^2}$$

$$\left\{-\binom{0s}{0k}_5\left[\binom{0s}{is}_5\binom{0s}{js}_5+\binom{is}{js}_5\binom{0s}{0s}_5\right]+\left[\binom{0i}{sk}_5\binom{0s}{js}_5+\binom{0j}{sk}_5\binom{0s}{is}_5\right]\binom{0s}{0s}_5 +(i\leftrightarrow k)+(j\leftrightarrow k)\right\},$$

$$S^{3,st}_{ijk} = \frac{1}{3\binom{0}{0}_5\binom{s}{s}_5^2}\left\{\binom{0s}{0k}_5\left[\binom{ts}{is}_5\binom{0s}{js}_5+\binom{is}{js}_5\binom{ts}{0s}_5+\frac{\binom{s}{s}_5\binom{0st}{ist}_5}{\binom{st}{st}_5}\binom{ts}{js}_5\right]\right.$$

$$-\left[\binom{0i}{sk}_5\binom{0s}{js}_5+\binom{0j}{sk}_5\binom{0s}{is}_5\right]\binom{ts}{0s}_5-\left[\binom{0i}{sk}_5\binom{ts}{js}_5+\binom{0j}{sk}_5\binom{ts}{is}_5\right]\frac{\binom{s}{s}_5\binom{0st}{0st}_5}{2\binom{st}{st}_5}$$

$$\left. +(i\leftrightarrow k)+(j\leftrightarrow k)\right\} \quad \text{and}$$

$$S^{2,stu}_{ijk} = -\frac{1}{3\binom{0}{0}_5\binom{s}{s}_5\binom{st}{st}_5}$$

$$\left\{\binom{0s}{0k}_5\binom{ts}{js}_5\binom{ust}{ist}_5-\frac{1}{2}\left[\binom{0j}{sk}_5\binom{ust}{ist}_5+\binom{0i}{sk}_5\binom{ust}{jst}_5\right]\binom{ts}{0s}_5+(i\leftrightarrow k)+(j\leftrightarrow k)\right\}$$

The following relation (proven by multiplying with $\binom{s}{s}_5$)

$$\binom{s}{0}_5 \binom{\mu st}{jst}_5 = \binom{s}{j}_5 \binom{\mu st}{0st}_5 - \binom{\mu s}{0j}_5 \binom{st}{st}_5 + \binom{ts}{0j}_5 \binom{ts}{\mu s}_5, \quad \mu = 0, 1, \cdots, 4$$

turns out to be useful for the simplification of the coefficients of I_3^{st} ($\mu = 0$) and I_2^{stu} ($\mu = u$) of E_{00j}. For I_2^{stu} we consider the sum over all permutations of any fixed set of values of s, t and u; I_2^{stu} being symmetric in s, t and u:

$$\sum_{permutations} \frac{1}{\binom{s}{s}_5 \binom{st}{st}_5} \binom{ts}{0s}_5 \left[\binom{us}{0j}_5 \binom{st}{st}_5 - \binom{ts}{0j}_5 \binom{ts}{us}_5 \right] = 0$$

(explicitely calculated with *Mathematica*). Finally we have

$$\begin{aligned}
3\binom{0}{0}_5 E_{00j} = & -\frac{1}{2} \sum_{s=1}^{5} \frac{1}{\binom{s}{s}_5^2} \left[3\binom{s}{0}_5 \binom{0s}{js}_5 - \binom{s}{j}_5 \binom{0s}{0s}_5 \right] \binom{0s}{0s}_5 I_4^s \\
& + \frac{1}{2} \sum_{s,t=1}^{5} \frac{1}{\binom{s}{s}_5^2} \left[3\binom{s}{0}_5 \binom{0s}{js}_5 - \binom{s}{j}_5 \frac{\binom{ts}{0s}_5^2}{\binom{st}{st}_5} \right] \binom{ts}{0s}_5 I_3^{st} \\
& - \frac{1}{2} \sum_{s,t,u=1}^{5} \frac{1}{\binom{s}{s}_5^2} \binom{s}{j}_5 \frac{\binom{s}{s}_5 \binom{ust}{0st}_5}{\binom{st}{st}_5} \binom{ts}{0s}_5 I_2^{stu}.
\end{aligned}$$

Ref.: T.Diakonidis, J.Fleischer, J.Gluza, K.Kaida, T.Riemann, B.Tausk,*Nucl.Phys.Proc.Suppl.***183**(2008)109; *arXiv:0812.2134*; *Phys.Rev.* **in print**

6 Reduction of the rank 4 tensor integral

The tensor integral of **rank 4** can be written like the one of rank 3:

$$
\begin{aligned}
I_5^{\mu\,\nu\,\lambda\,\rho} &= \sum_{i,j,k,l=1}^{4} q_i^{\mu}\, q_j^{\nu}\, q_k^{\lambda}\, q_l^{\rho} I_{5,ijkl} \qquad \texttt{with} \\
I_{5,ijkl} &= \nu_{ij}\nu_{ijk}\left[-\frac{{0\choose l}_5}{()_5} I_{5,ijk}^{[d+]^3} + \sum_{s=1,s\neq i,j,k}^{5} \frac{{s\choose l}_5}{()_5} I_{4,ijk}^{[d+]^3,s} + \frac{{i\choose l}_5}{()_5} I_{5,jk}^{[d+]^3} + \frac{{j\choose l}_5}{()_5} I_{5,ik}^{[d+]^3} + \frac{{k\choose l}_5}{()_5} I_{5,ij}^{[d+]^3}\right] \\
&- \left[\nu_{kl}\frac{{i\choose j}_5}{()_5} I_{5,kl}^{[d+]^3} + \nu_{jl}\frac{{i\choose k}_5}{()_5} I_{5,jl}^{[d+]^3} + \nu_{il}\frac{{j\choose k}_5}{()_5} I_{5,il}^{[d+]^3} + \nu_{jk}\frac{{i\choose l}_5}{()_5} I_{5,jk}^{[d+]^3} + \nu_{ik}\frac{{j\choose l}_5}{()_5} I_{5,ik}^{[d+]^3} + \nu_{ij}\frac{{k\choose l}_5}{()_5} I_{5,ij}^{[d+]^3}\right] \\
&+ \frac{1}{()_5^2}\left[{i\choose j}_5 {k\choose l}_5 + {i\choose k}_5 {j\choose l}_5 + {j\choose k}_5 {i\choose l}_5\right] I_5^{[d+]^2} \qquad \texttt{and} \\
\nu_{ij} I_{5,ij}^{[d+]^3} &= -\frac{{0\choose j}_5}{()_5} I_{5,i}^{[d+]^2} + \sum_{s=1,s\neq i}^{5} \frac{{s\choose j}_5}{()_5} I_{4,i}^{[d+]^2,s} + \frac{{i\choose j}_5}{()_5} I_5^{[d+]^2} \qquad \texttt{cancel and so does } I_5^{[d+]} \texttt{such that} \\
4{0\choose 0}_5 I_{5,ijkl} &= \frac{{0\choose 0}_5 {0\choose l}_5}{()_5} I_{5,ijk} - \left[\frac{{i\choose j}_5}{()_5} \sum_{s=1,s\neq k}^{5} \frac{{0\choose 0}_5 {s\choose l}_5}{()_5} I_{4,k}^{[d+]^2,s} + (i\leftrightarrow k) + (j\leftrightarrow k)\right] + \\
& \frac{1}{3}\left[\nu_{ij}\nu_{ijk} \sum_{s=1,\neq i,j,k}^{5} \frac{{0\choose 0}_5 {s\choose l}_5}{()_5} I_{4,ijk}^{[d+]^3,s} + (i\leftrightarrow k) + (j\leftrightarrow k)\right] + (i\leftrightarrow l) + (j\leftrightarrow l) + (k\leftrightarrow l),
\end{aligned}
$$

The second term is reckognized as $I_{4,ijk}$, i.e. we have

$$I_5^{\mu\nu\lambda\rho} = I_5^{\mu\nu\lambda} \sum_{i=1}^{5} q_i^\rho \frac{\binom{0}{i}_5}{()_5} - \sum_{s=1}^{5} I_4^{\mu\nu\lambda,s} \sum_{i=1}^{5} q_i^\rho \frac{\binom{s}{i}_5}{()_5} \equiv I_5^{\mu\nu\lambda} Q_0^\rho - \sum_{s=1}^{5} I_4^{\mu\nu\lambda,s} Q_s^\rho,$$

where we introduced the vectors (see 5-point function of rank R=1 !)

$$Q_s^\mu = \sum_{i=1}^{n} q_i^\mu \frac{\binom{s}{i}_n}{()_n} \quad , \quad s = 0 \ldots n.$$

This opens a completely new view on the tensor reduction: we expext that for any tensor we have the scheme

$$Tensor\,\{n,R\} = Tensor\,\{n,R-1\}\,Q_0^{\mu_R} - \sum_{s}^{n} Tensor^s\,\{n-1,R-1\}\,Q_s^{\mu_R} + \cdots .$$

This means: a tremendeous simplification of the tensor structure; e.g. for the 5-point tensor of rank 4(5) we would have

$$I_5^{\mu_1\mu_2\mu_3(\mu_4)\mu} = I_5^{\mu_1\mu_2\mu_3(\mu_4)} Q_0^\mu - \sum_{s=1}^{5} I_4^{\mu_1\mu_2\mu_3(\mu_4),s} Q_s^\mu.$$

These representations have been checked numerically! Moreover: every expression you have calculated will be "recicled"! Apart from that the formalism is quite elegant and simple (in comparison, e.g. with Denner, Dittmaier).

Of course we need a similar representation also for the 4-point functions etc., such that we obtain a complete recursion for the tensors! This causes a problem: for n-point functions with $n \leq 4$ we have not enough independent 4-vectors to build the $g^{\mu\nu}$! Therefore, for these cases, we have first of all to investigate the metric tensor.

7 The metric tensor

In $d = 4$ dimensions, one may eliminate $g^{\mu\nu}$ by expressing it in terms of the n different chords of the integral:

$$
\begin{aligned}
g^{\mu\nu} &= 2 \sum_{i,j=1}^{6} q_i^{\mu} q_j^{\nu} \frac{\binom{0i}{0j}_6}{\binom{0}{0}_6}, \\
g^{\mu\nu} &= 2 \sum_{i,j=1}^{5} q_i^{\mu} q_j^{\nu} \frac{\binom{i}{j}_5}{()_5}, \\
g^{\mu\nu} &= 2 \sum_{i,j=1}^{4} q_i^{\mu} q_j^{\nu} \frac{\binom{i}{j}_4}{()_4} + \frac{8 v^{\mu} v^{\nu}}{()_4}, \\
g^{\mu\nu} &= 2 \sum_{i,j=1}^{3} q_i^{\mu} q_j^{\nu} \frac{\binom{i}{j}_3}{()_3} + \frac{4 v^{\mu\lambda} {v^{\nu}}_{\lambda}}{()_3}.
\end{aligned}
$$

For $n = 6, 5$ see [J.F., F.Jegerlehner, O.V.Tarasov *Nucl. Phys.***D54**(1996)6479].
For $n < 5$, we have to introduce extra terms [G.J.Oldenborgh and J.A.M. Vermaseren, *Z.Phys.* **C46** (1990) 425]:

$$
v^{\mu} = \varepsilon(\mu, q_1 - q_4, q_2 - q_4, q_3 - q_4), \quad v^{\mu\lambda} = \varepsilon(\mu, \lambda, q_1 - q_3, q_2 - q_3),
$$

where

$$
v^{\mu\lambda} {v^{\nu}}_{\lambda} = q_1^2 q_2^{\mu} q_2^{\nu} + q_2^2 q_1^{\mu} q_1^{\nu} - q_1 \cdot q_2 (q_1^{\mu} q_2^{\nu} + q_1^{\nu} q_2^{\mu}) - \left[q_1^2 q_2^2 - (q_1 \cdot q_2)^2 \right] g^{\mu\nu},
$$

q_1, q_2 for the two 4-vectors $q_i - q_3, i = 1, 2$; the sums in the above expresssions annihilate the extra terms (orhogonality).

8 The 6-point function

Let us now first look at $(6,R)$-integrals. Representing the $g^{\mu\nu}$-tensor as above, as generalisation of the 5-point tensor one expects:

$$I_6^{\mu_1\cdots\mu_{R-1}\rho} = I_6^{\mu_1\cdots\mu_{R-1}}\bar{Q}_0^\rho - \sum_{s=1}^{6} I_5^{\mu_1\cdots\mu_{R-1},s}\bar{Q}_s^\rho. \tag{8.1}$$

The integral $I_5^{\mu_1\cdots\mu_{R-1},s}$ is derived from $I_6^{\mu_1\cdots\mu_{R-1}\mu}$ by scratching line s, and the auxiliary vectors $\bar{Q}$ read, due to the particular form of the metric tensor

$$\bar{Q}_0^\mu = \sum_{i=1}^{6} q_i^\mu \frac{\binom{00}{0i}_6}{\binom{0}{0}_6} \equiv 0, \tag{8.2}$$

$$\bar{Q}_s^\mu = \sum_{i=1}^{6} q_i^\mu \frac{\binom{0s}{0i}_6}{\binom{0}{0}_6} \quad , \quad s = 1\ldots 6. \tag{8.3}$$

Since $\binom{00}{0i}_6 = 0$ only the second term contributes and this is what has been shown already before **Denner et al., Diakonidis et al.**.

In this respect, also the 6-point function fits into the general scheme of representing the tensors.

9 The (n=5, R=3) recursions

The ”master formula“ for $(5,3)$-integrals has on the r.h.s $(5,2)$- and $(4,2)$-integrals:

$$I_5^{\mu_1\mu_2\mu} = I_5^{\mu_1\mu_2} Q_0^{\mu} - \sum_{s=1}^{5} I_4^{\mu_1\mu_2,s} Q_s^{\mu}.$$

Prodeeding recursively, the main discussion in the following concerns the evaluation of $n < 5$ integrals:
The $(5,2)$-integrals have been discussed before and it remains to investigate the (n=4, R=2) tensor. From

$$\nu_{ij} I_{4,ij}^{[d+]^2,s} = -\frac{\binom{0s}{js}_5}{\binom{s}{s}_5} I_{4,i}^{[d+],s} + \sum_{t=1,\neq i}^{5} \frac{\binom{ts}{js}_5}{\binom{s}{s}_5} I_{3,i}^{[d+],st} + \frac{\binom{is}{js}_5}{\binom{s}{s}_5} I_4^{[d+],s},$$

it immediately follows

$$I_4^{\mu\nu,s} = I_4^{\mu,s} Q_0^{s,\nu} - \sum_{t=1}^{4} I_3^{\mu,st} Q_t^{s,\nu} - \frac{4 \upsilon^{s,\mu} \upsilon^{s,\nu}}{\binom{s}{s}_5} I_4^{[d+],s},$$

where the vectors are known from the above, i.e.

$$I_4^{\mu,s} = I_4^{s} Q_0^{s,\mu} - \sum_{u=1}^{5} I_3^{st} Q_u^{s,\mu} \quad \text{and} \quad I_3^{\mu,st} = I_3^{st} Q_0^{st,\mu} - \sum_{u=1}^{5} I_2^{ust} Q_u^{st,\mu},$$

s and t as upper indices standing for scratched lines.

10 The (n=5, R=4) recursions

$$I_5^{\mu_1\mu_2\mu_3\mu} = I_5^{\mu_1\mu_2\mu_3} Q_0^\mu - \sum_{s=1}^{5} I_4^{\mu_1\mu_2\mu_3,s} Q_s^\mu .$$

Here we need $I_4^{\mu_1\mu_2\mu_3,s}$ - but not only here! Investigating the case $n = 4$, we drop scratches! Recursion yields

$$I_4^{\mu\nu\lambda} = I_4^{\mu\nu} Q_0^\lambda - \sum_{t=1}^{4} I_3^{\mu\nu,t} Q_t^\lambda + G^{\mu\lambda} V^\nu + G^{\nu\lambda} V^\mu ,$$

with the ”extra term“

$$G^{\mu\lambda} = \frac{1}{2} g^{\mu\lambda} - \sum_{i,j=1}^{4} q_i^\mu q_i^\lambda \frac{\binom{i}{j}_4}{()_4} = \frac{4 v^\mu v^\nu}{()_4}, \quad \text{and} \quad V^\mu = -I_4^{[d+]} Q_0^\mu + \sum_{t=1}^{4} I_3^{[d+],t} Q_t^\mu .$$

In V^μ enters besides $I_4^{[d+]}$ (known from above) also $I_3^{[d+],t}$, which may be reduced as follows

$$I_3^{[d+],t} = \left[\frac{\binom{0t}{0t}_4}{\binom{t}{t}_4} I_3^t - \sum_{u=1}^{4} \frac{\binom{ut}{0t}_4}{\binom{t}{t}_4} I_2^{tu} \right] \frac{1}{d-2} .$$

Finally, our representation of $I_4^{\mu\nu\lambda}$ contains the integral $I_3^{\mu\nu,t}$ and we have to reduce this also:

$$I_3^{\mu\nu,t} = I_3^{\mu,t} Q_0^{t,\nu} - \sum_{u=1}^{4} I_2^{\mu,tu} Q_u^{t,\nu} - I_3^{[d+],t} \cdot \frac{2 v^{t,\mu\lambda} v_{\lambda}^{t,\nu}}{\binom{t}{t}_4}.$$

In the Q-vector we have to introduce now again scratches t

$$Q_u^{t,\nu} = \sum_{i=1}^{4} q_i^{\nu} \frac{\binom{ut}{it}_4}{\binom{t}{t}_4}, \quad u = 0, \ldots, 4$$

and the extra term reads

$$G^{t,\mu\nu} = \frac{1}{2} g^{\mu\nu} - \sum_{i,j=1}^{4} q_i^{\mu} q_i^{\nu} \frac{\binom{it}{jt}_4}{\binom{t}{t}_4} = \frac{2 v^{t,\mu\lambda} v^{t,\nu}{}_{\lambda}}{\binom{t}{t}_4}.$$

11 The (n=5, R=5) recursions

Now the procedure is ”standard“:

$$I_5^{\mu_1\mu_2\mu_3\mu_4\mu} = I_5^{\mu_1\mu_2\mu_3\mu_4} Q_0^{\mu} - \sum_{s=1}^{5} I_4^{\mu_1\mu_2\mu_3\mu_4,s} Q_s^{\mu}.$$

Recursion yields:

$$I_4^{\mu\nu\lambda\rho} = I_4^{\mu\nu\lambda} Q_0^{\rho} - \sum_{t=1}^{4} I_3^{\mu\nu\lambda,t} Q_t^{\rho} - G^{\mu\rho} T^{\nu\lambda} - G^{\nu\rho} T^{\mu\lambda} - G^{\lambda\rho} T^{\mu\nu}$$

with

$$T^{\mu\nu} = -V^{\mu} Q_0^{\nu} + \sum_{t=1}^{4} W^{t,\mu}\, Q_t^{\nu} \; - G^{\mu\nu} I_4^{[d+]^2},$$

with the new term

$$W^{t,\mu} = \sum_{i=1}^{4} q_i^{\mu} I_{3,i}^{[d+]^2,t} = -I_3^{[d+],t} Q_0^{t,\mu} + \sum_{u=1}^{4} I_2^{[d+],tu} Q_u^{t,\mu}. \quad \text{and}$$

$$I_3^{\mu\nu\lambda,t} = I_3^{\mu\nu,t} Q_0^{t,\lambda} - \sum_{u=1}^{4} I_2^{\mu\nu,tu} Q_u^{t,\lambda} + G^{t,\mu\lambda} W^{t,\nu} + G^{t,\nu\lambda} W^{t,\mu}.$$

As an exercise we have a look at

$$I_2^{\mu\nu,tu} = I_2^{\mu,tu} Q_0^{tu,\nu} - \sum_{v=1}^{4} I_1^{\mu,tuv} Q_v^{tu,\nu} - G^{tu,\mu\nu} I_2^{[d+],tu}.$$

With $\binom{tu}{tu}_4 = -2(q_i - q_{i'})^2 \equiv -2q^2$ ($i < i'$ and $i, i' \neq t, u$), we get simple explicit expressions:

$$Q_0^{tu,\nu} = \frac{1}{2}(q_i + q_{i'})^\nu - \frac{1}{2q^2}(m_i^2 - m_{i'}^2)(q_i - q_{i'})^\nu,$$

$$Q_i^{tu,\nu} = 2q^2(q_i - q_{i'})^\nu,$$

$$G^{tu,\mu\nu} = \frac{1}{2}\left(g^{\mu\nu} - \frac{(q_i - q_{i'})^\mu (q_i - q_{i'})^\nu}{q^2}\right) \quad \texttt{and}$$

$$I_2^{\mu,tu} == \frac{1}{2} I_2^{tu}(q_i + q_{i'})^\mu - \frac{1}{2}(m_i^2 - m_{i'}^2)\frac{I_2^{tu}(q^2) - I_2^{tu}(0)}{q^2}(q_i - q_{i'})^\mu. \quad \texttt{Further:}$$

$$I_1^{\mu,tuv} = -\sum_{i=1}^{4} q_i^\mu I_{1,i}^{[d+],tuv} = q_i I_1^{tuv}, i \neq t, u, v. \;\; \texttt{due to} \;\; I_{1,i}^{[d+],tuv} = -\frac{\binom{0tuv}{ituv}_4}{\binom{tuv}{tuv}_4} I_1^{tuv} = -I_1^{tuv}.$$

12 Comment on Q_s

Assume we have in the numerator of the original integral a scalar product of the integration momentum and a chord.
Standard: the scalar product $q_i \cdot k$ is expressed in terms of the difference of two propagators, which can then be cancelled.
This simplification has to be handled as very first step, considering the original diagram ($q_n = 0$) .
Our approach offers an alternative:

$$q_i Q_0 = \sum_{j=1}^{n-1} q_i q_j \frac{\binom{0}{j}_n}{()_n} \;=\; -\frac{1}{2}\left(Y_{in} - Y_{nn}\right),\; i = 1, \cdots, n-1,$$

$$q_i Q_s = \sum_{j=1}^{n-1} q_i q_j \frac{\binom{s}{j}_n}{()_n} \;=\; \frac{1}{2}\left(\delta_{is} - \delta_{ns}\right),\; i = 1, \ldots, n-1, \qquad s = 1, \cdots, n,$$

with

$$Y_{jk} = -(q_j - q_k)^2 + m_j^2 + m_k^2.$$

Further we have:

$$Q_0^2 = \frac{1}{2}\frac{\binom{0}{0}_n}{()_n} + \frac{1}{2}Y_{nn},$$

$$Q_s^2 = \frac{1}{2}\frac{\binom{s}{s}_n}{()_n}, \qquad s = 1, \ldots, n.$$