The one-loop pentagon to higher orders in epsilon

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Outline

- The BDS ansatz & high-energy limits
- The negative dimension approach in a nutshell
- The scalar massless pentagon in the high-energy limit

The BDS ansatz

- Bern, Dixon and Smirnov conjectured that MHV amplitudes MSYM can be written as:

$$M_n(\epsilon) = 1 + \sum_{l=1}^{\infty} a^l \, M_n^{(l)}(\epsilon) = \exp \sum_{l=0}^{\infty} a^l \left[f^{(l)}(\epsilon) \, M_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon) \right],$$

- The BDS ansatz reproduces correctly the infrared poles of the amplitude.

The BDS ansatz

- In practice, the BDS ansatz implies a tower of iteration formulæ in the number of loops, e.g. for two loops

$$M_n^{(2)}(\epsilon) = \frac{1}{2}\left(M_n^{(1)}(\epsilon)\right)^2 + f^{(2)}(\epsilon)\, M_n^{(1)}(2\epsilon) + C^{(2)} + \mathcal{O}(\epsilon),$$

The BDS ansatz

- In practice, the BDS ansatz implies a tower of iteration formulæ in the number of loops, e.g. for two loops

$$M_n^{(2)}(\epsilon) = \frac{1}{2}\left(M_n^{(1)}(\epsilon)\right)^2 + f^{(2)}(\epsilon)\, M_n^{(1)}(2\epsilon) + C^{(2)} + \mathcal{O}(\epsilon),$$

Requires the knowledge of the one-loop amplitude to higher orders.

The BDS ansatz

	n=4	n=5	n=6
l=2			
l=3			

The BDS ansatz

	n=4	n=5	n=6
l=2	✓		
l=3	✓		

The BDS ansatz

	n=4	n=5	n=6
l=2	✓	✓ (num.)	
l=3	✓		

The BDS ansatz

	n=4	n=5	n=6
l=2	✓	✓ (num.)	● (num.)
l=3	✓		

The BDS remainder function

- Modified BDS ansatz, including an arbitrary function of conformal cross-ratios, e.g. for n=6,

$$M_6^{(2)}(\epsilon) = \frac{1}{2}\left(M_6^{(1)}(\epsilon)\right)^2 + f^{(2)}(\epsilon)\, M_6^{(1)}(2\epsilon) + C^{(2)}$$

$$+\mathcal{R}_6^{(2)}(u_1, u_2, u_3)$$

BDS remainder function

$$+\mathcal{O}(\epsilon)$$

The BDS remainder function

- How can we get a handle on the BDS remainder function?
- Solution I:

  Direct analytic computation of the one and two - loop six-point amplitudes.

  ➡ Needs the analytic evaluation of the one and two-loop scalar hexagon integrals.

  ➡ Completely out of reach for the moment!

The BDS remainder function

- How can we get a handle on the BDS remainder function?
- Solution II:

  Analytic evaluation of the one and two-loop six-point amplitude in some simplified kinematics.

  ➡ Collinear limit: This limit is verified ,by construction'.

  ➡ High-energy limit.

- We want to explore what higher point amplitudes looks like, so we start form the simplest non-trivial case, the 5-point amplitude in the high-energy limit.

The high-energy limit

- Multi-Regge kinematics are defined by

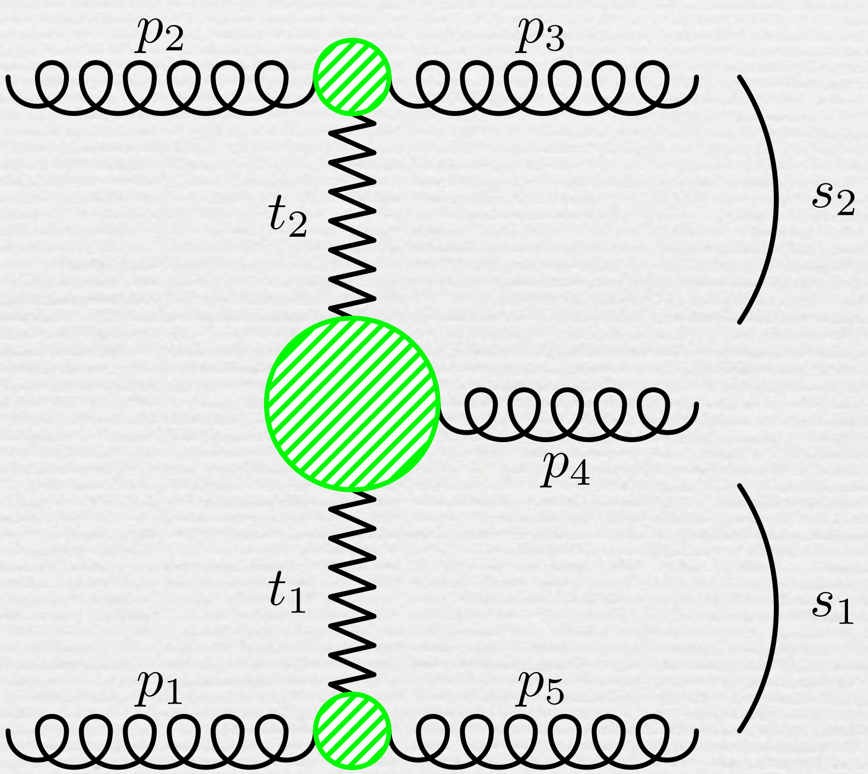
$$y_3 \gg y_4 \gg y_5$$

$$|p_{3\perp}| \simeq |p_{4\perp}| \simeq |\, p_{5\perp}|$$

- This implies a hierarchy of scales:

$$s \gg s_1, s_2 \gg -t_1, -t_2$$

$$st_1 \simeq st_2 \simeq s_1 s_2$$



[See Del Duca's talk last week]

The scalar massless pentagon

- According to the BDS ansatz, it is enough to know the 5-point amplitude to higher orders in ϵ .

$$M_n^{(2)}(\epsilon) = \frac{1}{2}\left(M_n^{(1)}(\epsilon)\right)^2 + f^{(2)}(\epsilon)\, M_n^{(1)}(2\epsilon) + C^{(2)} + \mathcal{O}(\epsilon),$$

$$M_n^{(1)}(\epsilon) \;= -\frac{1}{2}\, G(\epsilon) \sum_{\text{cyclic}} s_{12}s_{23}I_4^{1m}(1,2,3,45,\epsilon) - \epsilon\, G(\epsilon)\, \epsilon_{1234} I_5^{6-2\epsilon}(\epsilon)$$

[Bern, Dixon, Dunbar, Kosower]

The scalar massless pentagon

- According to the BDS ansatz, it is enough to know the 5-point amplitude to higher orders in ϵ .

$$M_n^{(1)}(\epsilon) \; = -\frac{1}{2}\,G(\epsilon)\sum_{\text{cyclic}} s_{12}s_{23}I_4^{1m}(1,2,3,45,\epsilon) - \epsilon\,G(\epsilon)\,\epsilon_{1234}I_5^{6-2\epsilon}(\epsilon)$$

- We performed the computation in two different ways:

➡ using the Negative Dimension approach (NDIM).

[Halliday, Ricotta]

➡ using the Mellin-Barnes approach (MB).

[See Riemann's and Smirnov's lectures]

NDIM in a nutshell

- We start from the Schwinger parametrization.

$$I_n^D\Big(\{\nu_i\};\{Q_i^2\};\{M_i\}\Big) = \int \mathcal{D}\alpha \int \frac{\mathrm{d}^D k}{i\pi^{D/2}} \exp\Big(\sum_{i=1}^{n} \alpha_i\, D_i\Big)$$

$$= \int \mathcal{D}\alpha\, \frac{1}{\mathcal{P}^{D/2}} \exp(\mathcal{Q}/\mathcal{P})\, \exp(-\mathcal{M})$$

with

$$\int \mathcal{D}\alpha = e^{\gamma_E \epsilon} \prod_{i=1}^{n} \frac{(-1)^{\nu_i}}{\Gamma(\nu_i)} \int_0^\infty \mathrm{d}\alpha_i\, \alpha_i^{\nu_i - 1}$$

- We can now expand the exponentials

$$\int \mathcal{D}\alpha \sum_{n_1,\ldots,n_n=0}^{\infty} \int \frac{d^D k}{i\pi^{D/2}} \prod_{i=1}^{n} \frac{(\alpha_i D_i)^{n_i}}{n_i!} = \int \mathcal{D}\alpha \sum_{n=0}^{\infty} \frac{\mathcal{Q}^n \mathcal{P}^{-n-\frac{D}{2}}}{n!} \sum_{m=0}^{\infty} \frac{(-\mathcal{M})^m}{m!},$$

NDIM in a nutshell

$$\int \mathcal{D}\alpha \sum_{n_1,\ldots,n_n=0}^{\infty} \int \frac{d^D k}{i\pi^{D/2}} \prod_{i=1}^{n} \frac{(\alpha_i D_i)^{n_i}}{n_i!} \quad = \int \mathcal{D}\alpha \sum_{n=0}^{\infty} \frac{\mathcal{Q}^n \mathcal{P}^{-n-\frac{D}{2}}}{n!} \sum_{m=0}^{\infty} \frac{(-\mathcal{M})^m}{m!};$$

- For negative D, we could use the binomial theorem for $\mathcal{P}^{-n-D/2}$.

$$\int \mathcal{D}\alpha \sum_{n_1,\ldots,n_n=0}^{\infty} I_n^D\Big(-n_1,\ldots,-n_n;\{Q_i^2\},\{M_i^2\}\Big) \prod_{i=1}^{n} \frac{\alpha_i^{n_i}}{n_i!} \quad =$$

$$\int \mathcal{D}\alpha \sum_{\substack{p_1,\ldots,p_n=0\\ q_1,\ldots,q_q=0\\ m_1,\ldots,m_n=0}}^{\infty} \frac{\mathcal{Q}_1^{q_1}\ldots\mathcal{Q}_q^{q_q}}{q_1!\ldots q_q!} \frac{\alpha_1^{p_1}\ldots\alpha_n^{p_n}}{p_1!\ldots p_n!} \frac{(-\alpha_1 M_1^2)^{m_1}}{m_1!} \ldots \frac{(-\alpha_n M_n^2)^{m_n}}{m_n!} (p_1+\ldots+p_n)!,$$

- We can now match the powers of the Schwinger parameters...

NDIM in a nutshell

- ... and obtain a series representation of the Feynman integral

$$I_n^D\left(\{\nu_i\};\{Q_i^2\};\{M_i\}\right) \equiv e^{\gamma_E \epsilon} \sum_{\substack{p_1,\ldots,p_n=0\\ q_1,\ldots,q_q=0\\ m_1,\ldots,m_n=0}}^{\infty} (Q_1^2)^{q_1}\ldots(Q_q^2)^{q_q}\,(-M_1^2)^{m_1}\ldots(-M_n^2)^{m_n}$$

$$\times\left(\prod_{i=1}^{n}\frac{\Gamma(1-\nu_i)}{\Gamma(1+m_i)\Gamma(1+p_i)}\right)\left(\prod_{i=1}^{q}\frac{1}{\Gamma(1+q_i)}\right)\Gamma\left(1+\sum_{k=1}^{n}p_k\right),$$

- In general, more than one solution might be obtained, and the Feynman integral is a combination of hypergeometric series.

NDIM in a nutshell

- Caveat: Some of the series are only convergent in a given region of phase space!

  E.g., in the four point case

$$\sum_{n=0}^{\infty} \frac{(a)_n}{n!} \left(\frac{s}{t}\right)^n = {}_2F_1(a,b,b;s/t), \quad \text{if } s < t$$

$$\sum_{n=0}^{\infty} \frac{(a)_n}{n!} \left(\frac{t}{s}\right)^n = {}_2F_1(a,b,b;t/s), \quad \text{if } t < s$$

- Recipe: Only the convergent series contirbute to a given region.
- The different regions are linked by analytic continuation:

$${}_2F_1(a,b,b;t/s) = \left(-\frac{s}{t}\right)^a {}_2F_1(a,b,b;s/t)$$

The pentagon from NDIM

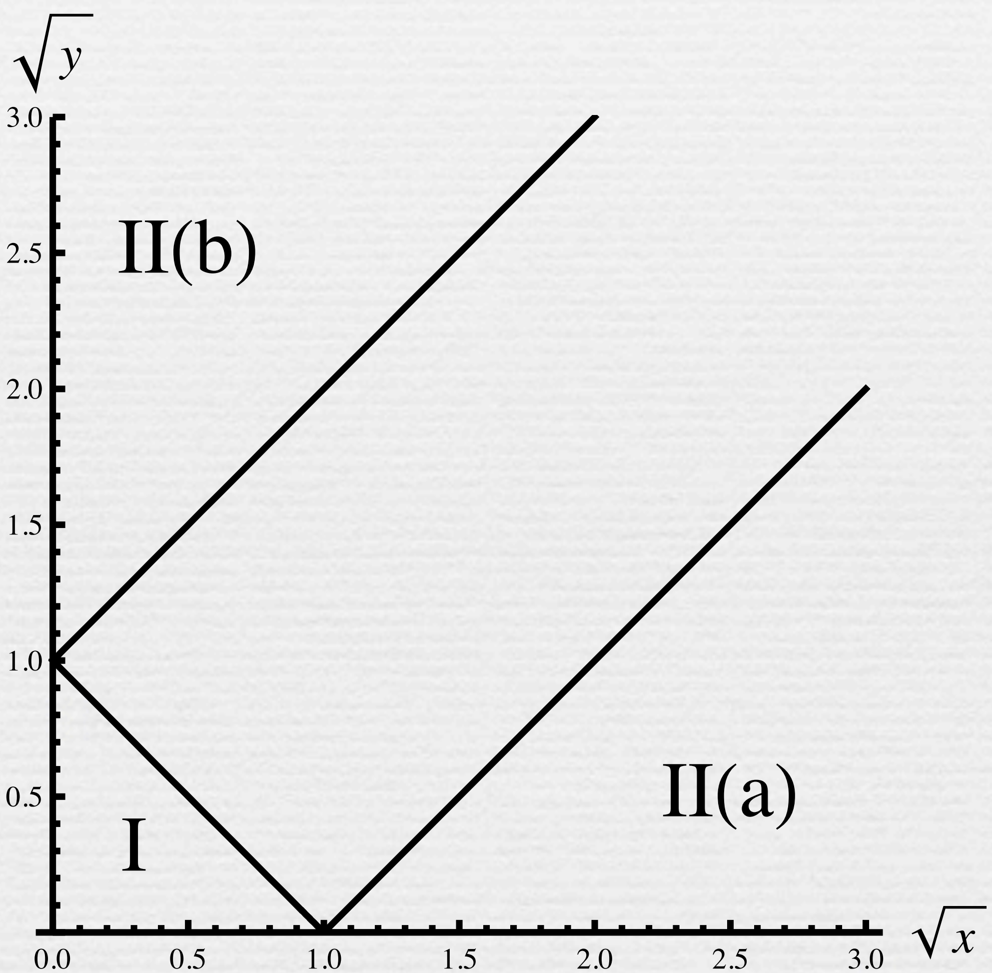
- For the pentagon in general kinematics, we find 125 4-fold hypergeometric sums:

$$F(a,b,c,d,e,f;x_1,x_2,x_3,x_4) = \sum_{n_1,n_2,n_3,n_4=0}^{\infty} \frac{(a)_{n_1+n_2+n_3+n_4}\,(b)_{n_1+n_2+n_3}\,(c)_{n_2+n_3+n_4}}{(d)_{n_1+n_2}\,(e)_{n_2+n_3}\,(f)_{n_3+n_4}} \frac{x_1^{n_1}}{n_1!}\frac{x_2^{n_2}}{n_2!}\frac{x_3^{n_3}}{n_3!}\frac{x_4^{n_4}}{n_4!}$$

- After imposing multi-Regge kinematics, the sums reduce to double sums.

The pentagon from NDIM

- For the pentagon there are three regions of convergence:



$$x_1 = \frac{st_1}{s_1 s_2} \qquad x_2 = \frac{st_2}{s_1 s_2}$$

- ➡ Region I: $\sqrt{x_1} + \sqrt{x_2} < 1$
- ➡ Region II(a): $-\sqrt{x_1} + \sqrt{x_2} > 1$
- ➡ Region II(b): $\sqrt{x_1} - \sqrt{x_2} > 1$

- The three regions are connected by analytic continuation.

The pentagon from NDIM

$$
\begin{aligned}
&\mathcal{I}_{\mathrm{ND}}^{(IIa)}(s, s_1, s_2, t_1, t_2) \\
&\quad = -\frac{1}{\epsilon^3}\, y_2^{-\epsilon}\, \Gamma(1-2\epsilon)\, \Gamma(1+\epsilon)^2\, F_4\Big(1-2\epsilon, 1-\epsilon, 1-\epsilon, 1-\epsilon; -y_1, y_2\Big) \\
&\quad + \frac{1}{\epsilon^3}\, \Gamma(1+\epsilon)\, \Gamma(1-\epsilon)\, F_4\Big(1, 1-\epsilon, 1-\epsilon, 1+\epsilon; -y_1, y_2\Big) \\
&\quad - \frac{1}{\epsilon^2}\, y_1^{\epsilon}\, y_2^{-\epsilon} \Bigg\{ \big[\ln y_1 + \psi(1-\epsilon) - \psi(-\epsilon)\big]\, F_4\big(1, 1-\epsilon, 1+\epsilon, 1-\epsilon; -y_1, y_2\big) \\
&\qquad\qquad + \frac{\partial}{\partial\delta} F_{0,2}^{2,1} \left(\begin{matrix} 1+\delta & 1+\delta-\epsilon \\ - & - \end{matrix} \middle| \begin{matrix} 1 & - & - & - \\ 1+\delta & 1-\epsilon & 1+\epsilon+\delta & - \end{matrix} \middle| -y_1, y_2 \right)_{|\delta=0} \Bigg\} \\
&\quad + \frac{1}{\epsilon^2}\, y_1^{\epsilon} \Bigg\{ \big[\ln y_1 + \psi(1+\epsilon) - \psi(-\epsilon)\big]\, F_4\big(1, 1+\epsilon, 1+\epsilon, 1+\epsilon; -y_1, y_2\big) \\
&\qquad\qquad + \frac{\partial}{\partial\delta} F_{0,2}^{2,1} \left(\begin{matrix} 1+\delta & 1+\delta+\epsilon \\ - & - \end{matrix} \middle| \begin{matrix} 1 & - & - & - \\ 1+\delta & 1+\epsilon & 1+\epsilon+\delta & - \end{matrix} \middle| -y_1, y_2 \right)_{|\delta=0} \Bigg\}
\end{aligned}
$$

The pentagon from NDIM

- Appell function:

$$F_4(a,b,c,d;x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}\,(b)_{m+n}}{(c)_m (d)_n} \frac{x^m}{m!} \frac{y^n}{n!}$$

- Kampé de Fériet function:

$$F^{p,q}_{p',q'}\left(\begin{matrix} \alpha_i \\ \alpha'_k \end{matrix}\middle|\begin{matrix} \beta_j & \gamma_j \\ \beta'_\ell & \gamma'_\ell \end{matrix}\middle| x,y\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\prod_i (\alpha_i)_{m+n} \prod_j (\beta_j)_m (\gamma_j)_n}{\prod_k (\alpha'_k)_{m+n} \prod_\ell (\beta'_\ell)_m (\gamma'_\ell)_n} \frac{x^m}{m!} \frac{y^n}{n!}$$

The pentagon from NDIM

- After expansion in ϵ, we find an expression for the pentagon in terms of new transcendental functions:

$$\mathcal{M}(\vec{\imath},\vec{\jmath},\vec{k};x_1,x_2) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\binom{n_1+n_2}{n_1}^2 S_{\vec{\imath}}(n_1)\,S_{\vec{\jmath}}(n_2)\,S_{\vec{k}}(n_1+n_2)\,x_1^{n_1}\,x_2^{n_2}$$

- In simple cases we could resum these series, but we could not do it in general.

The pentagon from NDIM

- We can easily perform the analytic continuation to the physical region, because the hypergeometric functions stay real.
  - ➡ The convergence criterion is $\sqrt{|y_1|} + \sqrt{|y_2|} < 1$
  - ➡ y1 and y2 change sign, but not in absolute value.
  - ➡ So the only imaginary parts come form the coefficients.

$$\mathcal{I}_{\mathrm{ND}}^{(IIa)}(s, s_1, s_2, t_1, t_2)$$
$$= -\frac{1}{\epsilon^3}\, y_2^{-\epsilon}\, \Gamma(1-2\epsilon)\, \Gamma(1+\epsilon)^2\, F_4\Big(1-2\epsilon, 1-\epsilon, 1-\epsilon, 1-\epsilon; -y_1, y_2\Big)$$
$$+\frac{1}{\epsilon^3}\, \Gamma(1+\epsilon)\, \Gamma(1-\epsilon)\, F_4\Big(1, 1-\epsilon, 1-\epsilon, 1+\epsilon; -y_1, y_2\Big)$$

The pentagon from MB

- The pentagon in general kinematics can be written as a four-fold MB integral.

[Bern et al.]

- After imposing multi-Regge kinematics, the 4-fold integral reduces to a double integral, e.g. in Region II(a):

$$\mathcal{I}_{\mathrm{MB}}^{(IIa)}(\kappa, t_1, t_2) = \frac{-y_1^{\epsilon}}{\Gamma(1+\epsilon)\,\Gamma(1-\epsilon)^2}\,\frac{1}{(2\pi i)^2}\int_{-i\infty}^{+i\infty} \mathrm{d}z_1\,\mathrm{d}z_2\, y_1^{z_1} y_2^{z_2}\,\Gamma\left(-\epsilon - z_1\right)\Gamma\left(-z_1\right)^2$$
$$\times\,\Gamma\left(z_1+1\right)\Gamma\left(-\epsilon - z_2\right)\Gamma\left(-z_2\right)\Gamma\left(z_1+z_2+1\right)\Gamma\left(\epsilon + z_1 + z_2 + 1\right),$$

- Closing the contours, and taking residues, we exactly reproduce the NDIM result.

The pentagon from MB

- In this case we can do more!

- We can exchange one MB integral for an Euler integral:

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \mathrm{d}z_1\, \Gamma(-z_1)\,\Gamma(c-z_1)\,\Gamma(b+z_1)\,\Gamma(a+z_1)\,X^{z_1}$$
$$= \Gamma(a)\,\Gamma(b+c)\,\int_0^1 \mathrm{d}v\, v^{b-1}\,(1-v)^{a+c-1}\,(1-(1-X)v)^{-a}$$

- The remaining MB integral now becomes trivial, and we obtain an Euler integral representation for the pentagon.

The pentagon from MB

- The Euler integral can be completely solved in terms of Goncharov's multiple polylogarithm

$$M(a_1, a_2, \ldots; z) = \int_0^z \frac{dx}{x - a_1} M(a_2, \ldots; x) = \int_0^z \frac{dx}{x - a_1} \int_0^x \frac{dy}{y - a_2} \ldots$$

- A small sample:

$$\frac{1}{\sqrt{\lambda_K}} \Big\{ \Big(\tfrac{1}{2}\ln^2 x_2 + \tfrac{\pi^2}{2}\Big) M(\lambda_1) + \Big(- \tfrac{1}{2}\ln^2 x_2 - \tfrac{\pi^2}{2}\Big) M(\lambda_2) - \ln x_2 M(\lambda_1, 0) -$$
$$\ln x_2 M(\lambda_1, 1) + \ln x_2 M(\lambda_1, \lambda_3) + \ln x_2 M(\lambda_2, 0) + \ln x_2 M(\lambda_2, 1) - \ln x_2 M(\lambda_2, \lambda_3) +$$
$$M(\lambda_1, 0, 0) + M(\lambda_1, 0, 1) - M(\lambda_1, 0, \lambda_3) + M(\lambda_1, 1, 0) + M(\lambda_1, 1, 1) - M(\lambda_1, 1, \lambda_3) -$$

$$\lambda_1 = \frac{1}{2}\left(1 + x_1 - x_2 - \sqrt{\lambda_K}\right)$$
$$\lambda_2 = \frac{1}{2}\left(1 + x_1 - x_2 + \sqrt{\lambda_K}\right)$$

$$\lambda_K = \lambda(x_1, x_2, -1)$$

What did we learn?

- We used different representations to get different kinds of information

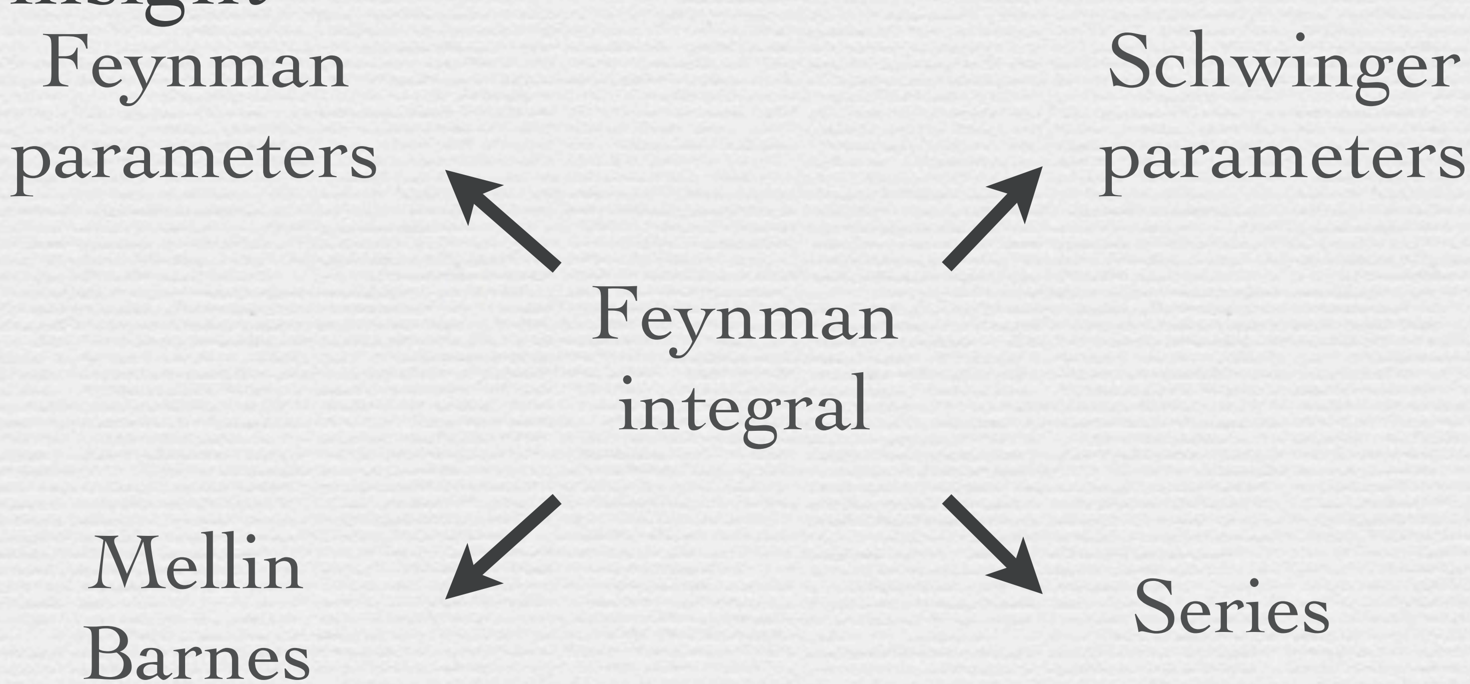
  - ➡ NDIM: Hypergeometric functions, analytic continuation.
  - ➡ MB: Closed form in terms of generalized polylog's.

- New mathematical structures can appear beyond $\mathcal{O}(\epsilon^0)$

  - ➡ Kampé de Fériet functions
  - ➡ Goncharov's polylogarithms

What did we learn?

- We used different representations to get different kinds of information
  - ➡ NDIM: Hypergeometric functions, analytic continuation.
  - ➡ MB: Closed form in terms of generalized polylog's.
- Switching between different representations can give a valuable insight



The five-point amplitude

- We have now all the ingredients to build the five-point two-loop amplitude (in MRK).

➡ The BDS iteration

$$M_n^{(2)}(\epsilon) = \frac{1}{2}\left(M_n^{(1)}(\epsilon)\right)^2 + f^{(2)}(\epsilon)\, M_n^{(1)}(2\epsilon) + C^{(2)} + \mathcal{O}(\epsilon),$$

➡ The one-loop amplitude to higher orders in ϵ

$$M_n^{(1)}(\epsilon) \;= -\frac{1}{2}G(\epsilon) \sum_{\text{cyclic}} s_{12}s_{23}I_4^{1m}(1,2,3,45,\epsilon) - \epsilon\, G(\epsilon)\, \epsilon_{1234} I_5^{6-2\epsilon}(\epsilon)$$

Conclusion

- We studied for the first time the analytic structure of 5-point one-loop amplitudes to higher orders in ϵ.

- New mathematical structures appear beyond the finite part:
  - ➡ Kampé de Fériet functions
  - ➡ Goncharov's polylogarithms

- As a byproduct, we computed the two-loop 5-point amplitude in MSYM in simplified kinematics.

The BDS remainder function

- The remainder function vanishes for n < 6:

$$\mathcal{R}_4 = \mathcal{R}_5 = 0.$$

- The remainder function is a function of conformal cross-ratios only.
- The remainder function is a symmetric function in the conformal cross-ratios.

- Collinear limits:

$$\mathcal{R}_n \longrightarrow \mathcal{R}_{n-1}$$

In particular, this implies

$$\mathcal{R}_6 \longrightarrow 0$$

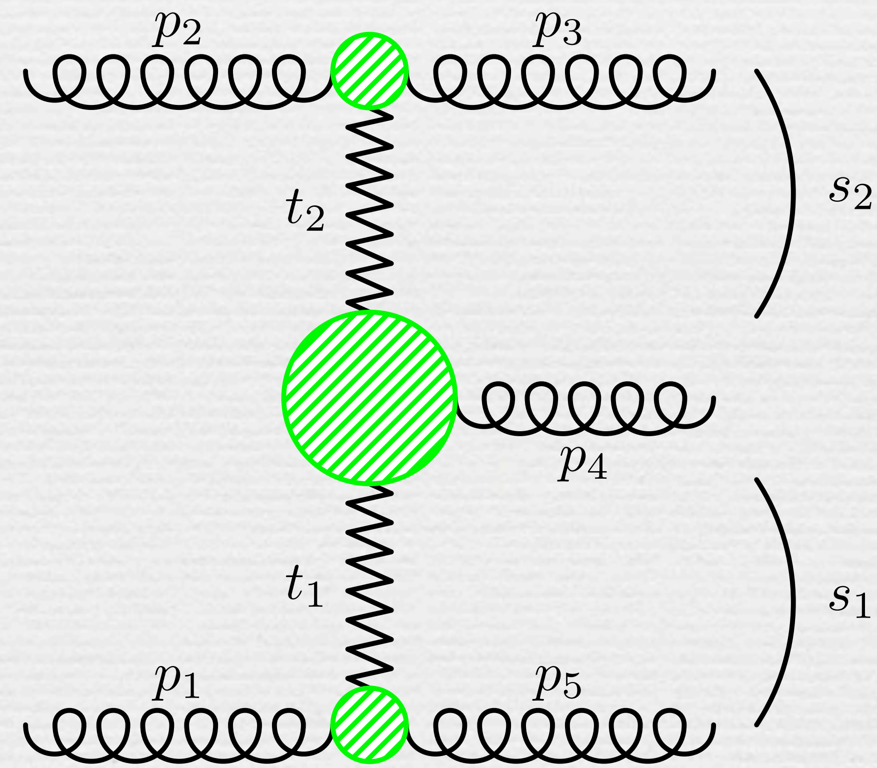
The high-energy limit

- In the high-energy limit, the amplitude is conjectured to factorize,

$$s\,C(p_2,p_3;\epsilon,\tau)\,\frac{1}{t_2}\left(\frac{-s_2}{\tau}\right)^{\alpha(t_2)}$$

$$V(q_2;p_4;q_1;\epsilon,\tau)$$

$$\frac{1}{t_1}\left(\frac{-s_1}{\tau}\right)^{\alpha(t_1)}\qquad C(p_5,p_5;\epsilon,\tau)$$



Determined by the four-point amplitude

We can now extract the Lipatov vertex

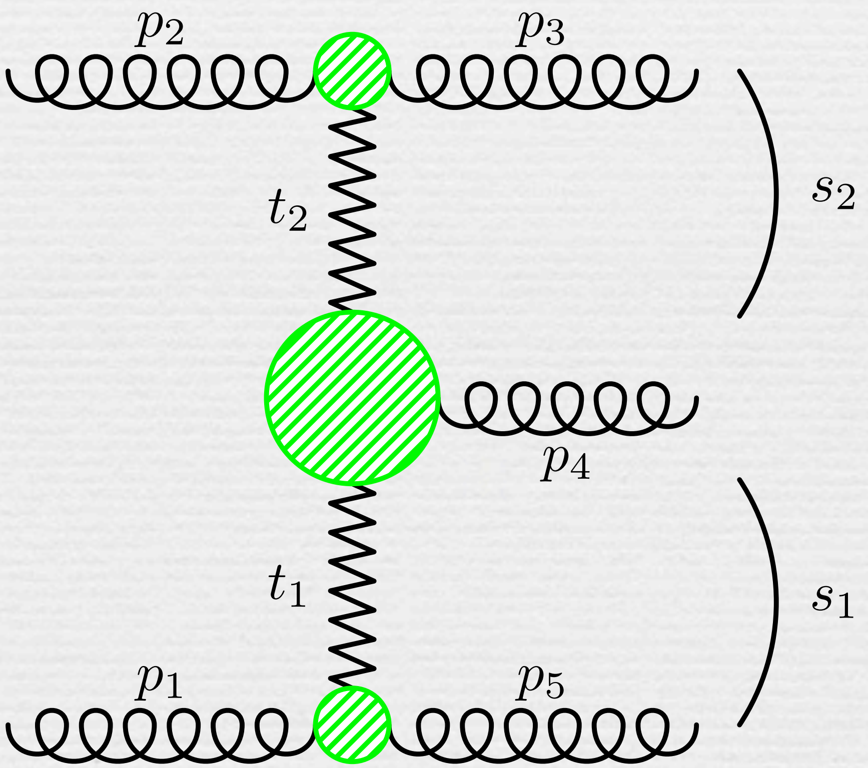
The high-energy limit

- In the high-energy limit, the amplitude is conjectured to factorize,

$$s\, C(p_2, p_3; \epsilon, \tau)\, \frac{1}{t_2} \left(\frac{-s_2}{\tau}\right)^{\alpha(t_2)}$$

$$V(q_2; p_4; q_1; \epsilon, \tau)$$

$$\frac{1}{t_1} \left(\frac{-s_1}{\tau}\right)^{\alpha(t_1)} \quad C(p_5, p_5; \epsilon, \tau)$$



  - ➡ The impact factor C
  - ➡ The Regge trajectory
  - ➡ The Lipatov vertex V

[See Del Duca's talk last week]