# Anomalous dimensions and evolution equations
## of the unintegrated parton densities

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The renormalization-group properties of the transverse-momentum dependent (TMD) parton densities, or distribution functions (PDFs) are addressed, and analysis of the leading-order UV anomalous dimensions is given in the light-cone gauge. A generalized definition of the TMD PDF, based on the renormalization procedure for the Wilson exponentials with obstructions, is discussed and the $Q^2$-evolution equations are proposed.

- **Integrated parton densities:** definition; gauge invariance; RG properties

- **Unintegrated (TMD) densities**: complete gauge invariance, extra divergences, RG properties

- **Generalized definition** and cancelation of extra divergences; UV-evolution

- **Conclusions and outlook**

**Integrated parton densities:** definition; gauge invariance; RG properties

$$q_{i/h}(x) = \frac{1}{2}\int \frac{d\xi^-}{2\pi}\, \mathrm{e}^{-ik^+\xi^-} \langle h(P)|\bar{\psi}_i(\xi^-, \mathbf{0}_\perp)\gamma^+\psi_i(0^-, \mathbf{0}_\perp)|h(P)\rangle \sim$$

$$\sim \langle h(P)|a_i^\dagger a_i(x)|h(P)\rangle$$

Corresponds to the number of partons (probabilistic interpretation)

**Gauge invariance** is saved by the insertion of the **gauge link**:

$$[y, x|\Gamma] = \mathcal{P}\exp\left[-ig\int_{x[\Gamma]}^{y} dz_\mu A_a^\mu(z) t_a\right]$$

so that

$$\hat{q}_{i/h}(x) = \frac{1}{2}\int \frac{d\xi^-}{2\pi}\, \mathrm{e}^{-ik^+\xi^-} \langle h(P)|\bar{\psi}_i(\xi^-, \mathbf{0}_\perp)[\xi^-, 0^-]\gamma^+\psi_i(0^-, \mathbf{0}_\perp)|h(P)\rangle$$

**Renormalization properties** are described by the DGLAP equation:

$$\mu\frac{d}{d\mu}\hat{q}_{i/h}(x,\mu) = \sum_j \int_x^1 \frac{dz}{z} P_{ij}\left(\frac{x}{z}\right) \hat{q}_{j/h}(x,\mu)$$

$P_{ij}\left(\frac{x}{z}\right)$ is the DGLAP integral kernel, which controls the dependence from the UV scale $\mu$—therefore, the (logarithmic) scale $Q^2$-dependence stems from DGLAP:

$$\hat{q}_{i/h}(x,\mu) \rightarrow \hat{q}_{i/h}(x,Q^2)$$

# Unintegrated (TMD) parton densities

"Naive" definition:

$$f_i(x, \boldsymbol{k}_\perp) = \frac{1}{2} \int \frac{d\xi^- d^2\xi_\perp}{2\pi(2\pi)^2} \; \mathrm{e}^{-ik^+\xi^- + ik_\perp \cdot \xi_\perp}.$$

$$\cdot \langle p | \bar{\psi}_i(\xi^-, \xi_\perp) [\xi^-, \boldsymbol{\xi}_\perp; \infty^-, \boldsymbol{\xi}_\perp;]^\dagger \gamma^+ [\infty^-, \mathbf{0}_\perp; 0^-, \mathbf{0}_\perp] \psi_i(0^-, 0_\perp) | p \rangle \; |_{\xi^+=0}$$

Formally:

$$\int d^2k_\perp f_i(x, \boldsymbol{k}_\perp) = q_i(x)$$

$f_i(x, \boldsymbol{k}_\perp)$ accumulates a lot of the phenomenologically important quantities. E.g., $T-$odd functions, responsible for the single-spin asymmetries in SIDIS, Drell-Yan, etc.

**Factorization** of semi-inclusive processes

$$F(x_B, z_h, p_h, Q^2) = \sum_i e_i^2 \cdot$$
$$\cdot H(Q^2, \mu^2) \otimes \mathcal{F}_D(x_B, \boldsymbol{k}_\perp, \mu^2, \eta) \otimes \mathcal{F}_F(z_h, \boldsymbol{q}_\perp, \mu^2, \hat{\eta}) \otimes S(\mu^2)$$

**However:** this definition suffers from several shortcomings.

- **gauge invariance** is not complete: in the light-cone gauge, dependence on the pole prescription in the gluon propagator still takes place
- **extra** (rapidity) **divergences** associated with the features of the light-cone gauge, or the light-like Wilson lines (*in the integrated case, these divergences cancel*)
- **reduction to the integrated case**: formal integration doesn't produce correct result because of additional uncanceled UV divergences

## Looking for the solution:

- **gauge invariance** is completely restored by means of the additional transverse Wilson line at light-cone infinity (`Belitsky, Ji, Yuan`). This gauge link contributes only in the light-cone gauge and cancels the pole-prescription dependence

- **extra divergences** can be avoided by using the non-light-like gauge connectors in covariant gauges, or an axial gauge off the light cone (`Collins, Soper`). This entails the introduction of an additional rapidity parameter $\zeta = (p \cdot n)^2/n^2$ (with $n^2 \neq 0$) to encode the deviation from the light cone; the calculations become more complicated; problems with factorization could arise.

- **generalized renormalization** procedure for the light-like Wilson lines (or a subtractive method) (`Collins, Hautmann`): extra divergences cancel by the additional "soft" factor, defined by the vacuum average of particular Wilson lines (demonstrated explicitly in the covariant gauge, in the 1-loop order)

## Towards the "completely correct" definition:

- calculate the **anomalous dimension** of the TMD PDF in the light-cone gauge and identify extra divergences in terms of the *defect* of anomalous dimension

- perform the **generalized renormalization** of TMD PDF, in analogue to the renormalization of the Wilson contours with cusps or self-intersection

- propose the **modified definition** of the TMD PDF, free of the above mentioned disadvantages

In the **tree** approximation, the TMD quark distribution reads

$$f^{(0)}(x, \boldsymbol{k}_\perp) =$$

$$= \frac{1}{2} \int \frac{d\xi^- d^2\xi_\perp}{2\pi(2\pi)^2} \mathrm{e}^{-ik^+\xi^- + ik_\perp \cdot \xi_\perp} \langle p| \bar{\psi}(\xi^-, \xi_\perp)\gamma^+ \psi(0^-, 0_\perp)|p\rangle =$$

$$= \delta(1-x)\delta^{(2)}(\boldsymbol{k}_\perp)$$

The one-gluon exchanges, contributing to the UV-divergences, are described by the diagrams:

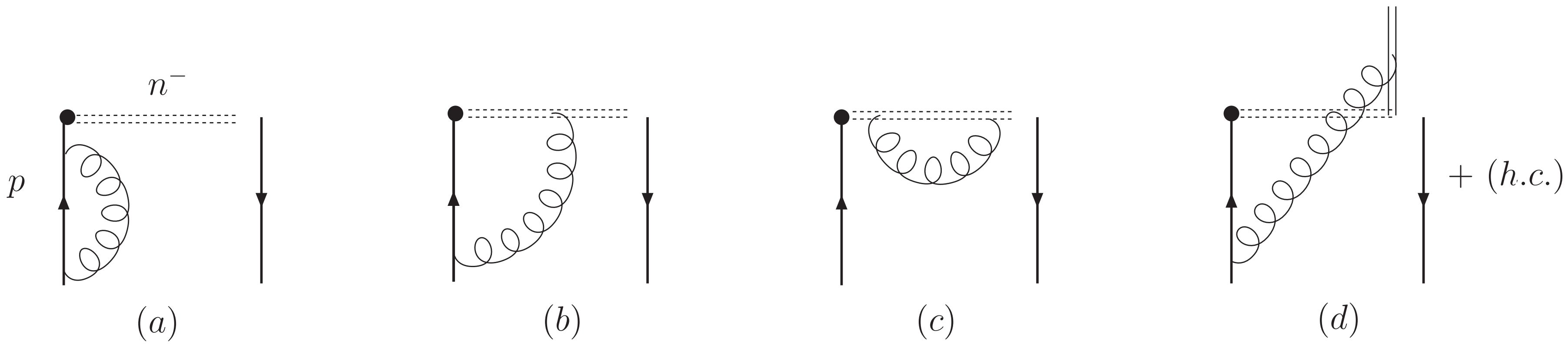


Figure 1: One-gluon exchanges for the TMD PDF: diagrams producing UV divergences. Only $(a)$ and $(d)$ contribute in the light-cone gauge

**Source of the uncertainties and extra divergences:** pole in the gluon propagator

$$D^{\mu\nu}_{\mathrm{LC}}(q) = \frac{1}{q^2}\left[g^{\mu\nu} - \frac{q^\mu n^{-\nu}}{[q^+]} - \frac{q^\nu n^{-\mu}}{[q^+]}\right]$$

**$q^-$-independent pole prescriptions:**

$$d^{\mu\nu}_{\mathrm{PV}}(q) = -(q^\mu n^{-\nu} + q^\nu n^{-\mu})\frac{1}{2}\left(\frac{1}{q^+ + i\eta} + \frac{1}{q^+ - i\eta}\right)$$

$$d^{\mu\nu}_{\mathrm{Adv/Ret}}(q) = -(q^\mu n^{-\nu} + q^\nu n^{-\mu})\frac{1}{q^+ \mp i\eta}$$

**Mandelstam-Leibbrandt pole prescriptions:**

$$\frac{1}{[q^+]_{\mathrm{ML}}} = \begin{cases} \frac{1}{q^+ + i0q^-} \\ \frac{q^-}{q^+q^- + i0} \end{cases}$$

The **UV divergent** part read:

$$\Sigma^{UV}_{\rm left}(p, \alpha_s; \epsilon) =$$

$$= -\frac{\alpha_s}{\pi} C_{\rm F} \frac{1}{\epsilon} \left[ -\frac{3}{4} - \ln \frac{\eta}{p^+} + \frac{i\pi}{2} + i\pi \, C_\infty \right] + \alpha_s C_{\rm F} \; \frac{1}{\epsilon} [i C_\infty] =$$

$$= -\frac{\alpha_s}{\pi} C_{\rm F} \frac{1}{\epsilon} \left[ -\frac{3}{4} - \ln \frac{\eta}{p^+} + \frac{i\pi}{2} \right]$$

Imaginary part of AD — gluons in the Glauber regime (Idilbi, Majumder)

**prescription dependence** is canceled

$$C_\infty = \begin{cases} 0 \; , \; \text{Advanced} : \frac{1}{[q^+]} = \frac{1}{q^+ - i\eta} \\ -1 \; , \; \text{Retarded} : \frac{1}{[q^+]} = \frac{1}{q^+ + i\eta} \\ -\frac{1}{2} \; , \; \text{PrincipalValue} : \; \frac{1}{[q^+]} = \frac{1}{2} \left( \frac{1}{q^+ - i\eta} + \frac{1}{q^+ + i\eta} \right) \end{cases}$$

Taking into account $(h.c.)$ contributions, one gets total real UV divergent part:

$$\Sigma_{\rm tot}(p, \alpha_s(\mu); \epsilon) = \Sigma_{\rm left} + \Sigma_{\rm right} =$$

$$= -\frac{\alpha_s}{4\pi} C_{\rm F} \frac{2}{\epsilon} \left( -3 - 4 \ln \frac{\eta}{p^+} \right)$$

Dependence on $\eta$ remains:

- standard renormalization procedure is **not sufficient**
- gauge invariance is **not complete**
- AD does **not coincide** with $\mathrm{AD}_{2q}$

One-loop **anomalous dimension** is defined via the renormalization constant

$$\gamma = \frac{1}{2} \frac{1}{Z^{(1)}} \mu \frac{\partial \alpha_s(\mu)}{\partial \mu} \frac{\partial Z^{(1)}(\mu, \alpha_s(\mu); \epsilon)}{\partial \alpha_s}$$

and reads

$$\gamma_{\rm LC} = \gamma_{\rm smooth} - \delta\gamma \quad , \quad \gamma_{\rm smooth} = \frac{3}{4}\frac{\alpha_s}{\pi} C_{\rm F} + O(\alpha_s^2)$$

Defect of anomalous dimension

$$\delta\gamma = -\frac{\alpha_s}{\pi} C_{\rm F} \ \ln \frac{\eta}{p^+}$$

contains undesirable $p^+$-dependent term which should be removed by a consistent procedure.

Note, that $\delta\gamma$ is nothing else, but the cusp anomalous dimension:

$$p^+ = (p \cdot n^-) \sim \cosh \chi$$

defines an angle $\chi$ between the direction of the quark momentum $p_\mu$ and the light-like vector $n^-$. In the large $\chi$ limit:

$$\ln p^+ \to \chi, \; \chi \to \infty$$

Renormalization of the Wilson operators with obstructions (cusps, self-intersections) requires additional renormalization factor depending on the cusp angle (Korchemsky, Radyushkin)

$$Z_\chi = \left[ \langle 0 | \mathcal{P} \exp \left[ ig \int_\chi d\zeta^\mu \ \hat{A}^a_\mu(\zeta) \right] |0\rangle \right]^{-1}$$

Generalized renormalization:

$$\mathcal{O}_{\mathrm{ren}}(\chi, ...) = Z_\chi Z_{\mathrm{R}} \, \mathcal{O}(\chi, ...)$$

Choose the integration path as follows

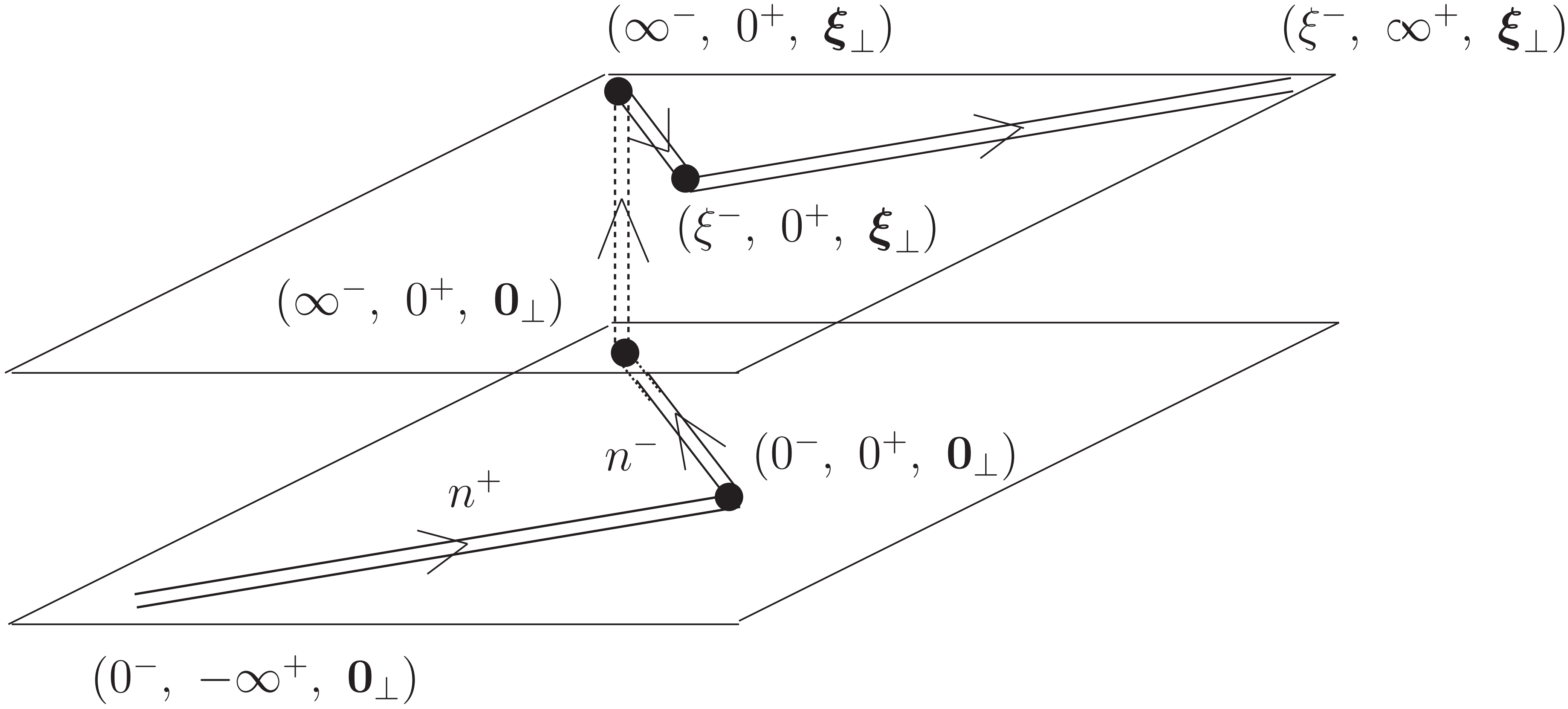


Figure 2: Space-time picture: integration trajectory for the additional cusp-dependent renormalization factor

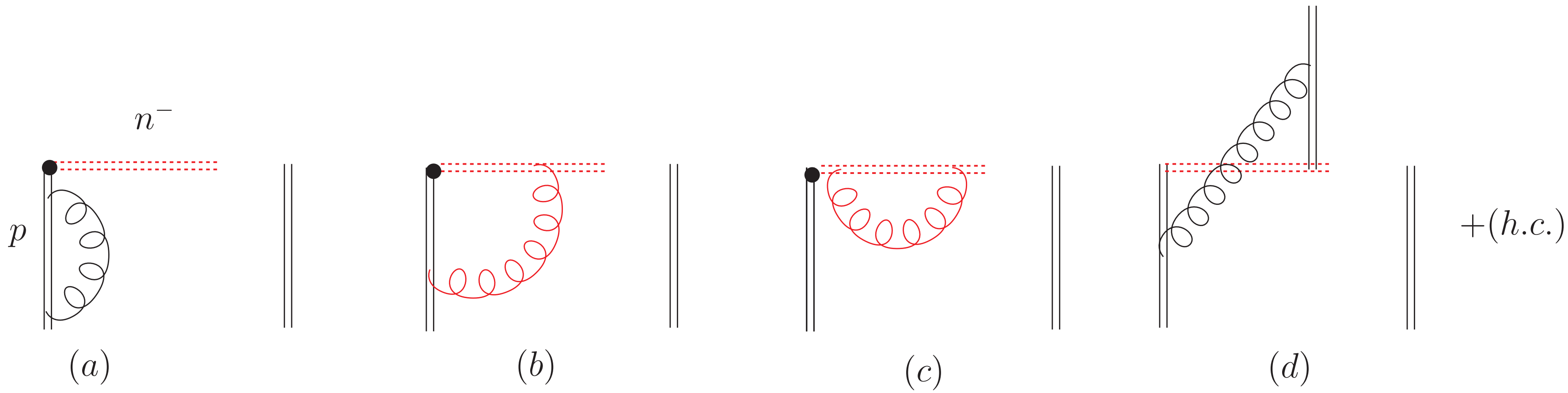


Figure 3: One-gluon exchanges for the generalized multiplicative renormalization factor

The generalized **renormalization constant** reads

$$\hat{Z}_{\rm mod} = 1 + \frac{\alpha_s}{4\pi} C_{\rm F} \frac{2}{\epsilon} \left( -3 - 4 \ln \frac{\eta}{p^+} + 4 \ln \frac{\eta}{p^+} \right) = 1 - \frac{3\alpha_s}{4\pi} C_{\rm F} \frac{2}{\epsilon}$$

so that

$$\frac{1}{2} \mu \frac{d}{d\mu} \ln \hat{Z}_{\rm mod}(\mu, \alpha_s, p^+) = \frac{3}{4} \frac{\alpha_s}{\pi} C_{\rm F} + O(\alpha_s^2)$$

i.e., equal to the anomalous dimension of the corresponding operator with the **smooth gauge connector**, according to the anomalous dimensions sum rule.

ADSR can be formulated in the following form

$$\mathbf{AD}\ \frac{1}{2}\int \frac{d\xi^- d^2\xi_\perp}{2\pi(2\pi)^2} \mathrm{e}^{-ik^+\xi^- + ik_\perp\cdot\xi_\perp}\ \langle p|\bar{\psi}(\xi)\gamma^+[\xi,0]_{\text{direct link}}\psi(0)|p\rangle =$$

$$= \mathbf{AD}\ \frac{1}{2}\int \frac{d\xi^- d^2\xi_\perp}{2\pi(2\pi)^2} \mathrm{e}^{-ik^+\xi^- + ik_\perp\cdot\xi_\perp}$$

$$\cdot\langle p|\bar{\Psi}(\xi|\infty)\gamma^+\Psi(0|\infty)|p\rangle \cdot \Phi(p^+, n^-|0^-, \mathbf{0}_\perp)\Phi^\dagger(p^+, n^-|\xi^-, \boldsymbol{\xi}_\perp)$$

Generalized definition of TMD PDF:

$$\mathcal{F}\left(x, \boldsymbol{k}_\perp; \mu, \eta\right) =$$

$$\frac{1}{2}\int \frac{d\xi^- d^2\boldsymbol{\xi}_\perp}{2\pi(2\pi)^2} \mathrm{e}^{-ik^+\xi^- + ik_\perp\cdot\xi_\perp} \left\langle q(p)|\bar{\psi}(\xi^-, \boldsymbol{k}_\perp)[\xi^-, \boldsymbol{\xi}_\perp; \infty^-, \boldsymbol{\xi}_\perp]^\dagger\right.$$

$$\times[\infty^-, \boldsymbol{\xi}_\perp; \infty^-, \boldsymbol{\infty}_\perp]^\dagger \gamma^+ [\infty^-, \boldsymbol{\infty}_\perp; \infty^-, \mathbf{0}_\perp][\infty^-, \mathbf{0}_\perp; 0^-, \mathbf{0}_\perp]$$

$$\left.\times\psi(0^-, \mathbf{0}_\perp)|q(p)\right\rangle \left[\Phi(p^+, n^-|0^-, \mathbf{0}_\perp)\Phi^\dagger(p^+, n^-|\xi^-, \boldsymbol{\xi}_\perp)\right]^{-1}$$

Soft factor:

$$\Phi(p^+, n^-|0) = \left\langle 0\left|\mathcal{P}\exp\left[ig\int_{\mathcal{C}_{\mathrm{cusp}}} d\zeta^\mu\, t^a A^a_\mu(\zeta)\right]\right|0\right\rangle$$

$$\Phi^\dagger(p^+, n^-|\xi) = \left\langle 0\left|\mathcal{P}\exp\left[-ig\int_{\mathcal{C'}_{\mathrm{cusp}}} d\zeta^\mu\, t^a A^a_\mu(\xi+\zeta)\right]\right|0\right\rangle$$

Dependence on the dimensional regularization scale $\mu$ of the re-defined TMD PDF (**UV-evolution**):

$$\frac{1}{2}\mu\frac{d}{d\mu}\,\mathcal{F}(x,\boldsymbol{k}_\perp;\mu) = \int d^2\boldsymbol{q}_\perp \int_x^1 \frac{dz}{z}\, P_\perp\left(\frac{x}{z},\boldsymbol{q}_\perp,\alpha_s\right)\, \mathcal{F}(z,\boldsymbol{q}_\perp,\mu)$$

$$P_\perp\,(y,\boldsymbol{q}_\perp,\alpha_s) = \gamma_{\rm mod}\,\delta(1-y)\,\delta^{(2)}(\boldsymbol{k}_\perp-\boldsymbol{q}_\perp) + O(\alpha_s^2)\ ,$$

$$\gamma_{\rm mod} = \gamma_{2q} = -\frac{1}{2}\mu\frac{d}{d\mu}\ln\Sigma_{\rm mod}(\alpha_s,\epsilon) = \frac{3}{4}\frac{\alpha_s}{\pi}C_{\rm F} + O(\alpha_s^2)\ .$$

Consistency equation:

$$\mu\frac{d}{d\mu}\left[\eta\frac{d}{d\eta}\mathcal{F}(x,\boldsymbol{k}_\perp;\mu,\eta)\right] = 0$$

## **Evolution equations** for TMD PDFs

- **UV-evolution**

$$\mu\frac{d}{d\mu}\,\mathcal{F}(x,\boldsymbol{k}_\perp,\mu,\eta) = \mathcal{K}_{\rm UV}\otimes\mathcal{F}(x,\boldsymbol{k}_\perp,\mu,\eta)$$

- **rapidity evolution** (Collins-Soper equation)

$$\eta\frac{d}{d\eta}\,\mathcal{F}(x,\boldsymbol{k}_\perp,\mu,\eta) = \mathcal{K}_{\rm CS}\otimes\mathcal{F}(x,\boldsymbol{k}_\perp,\mu,\eta)$$

- **BFKL evolution**

$$x\frac{d}{dx}\,\mathcal{F}(x,\boldsymbol{k}_\perp,\mu,\eta) = \mathcal{K}_{\rm BFKL}\otimes\mathcal{F}(x,\boldsymbol{k}_\perp,\mu,\eta)$$

Reduction to the integrated PDF

$$\int d^{\omega-2}\boldsymbol{k}_\perp \mathcal{F}(x, \boldsymbol{k}_\perp; \mu, \eta) \to \hat{q}(x, \mu)$$

restores the DGLAP evolution

## Probabilistic interpretation

The distribution functions cannot be calculated from first principles, but their **evolution** can. The RG properties define the necessary condition for the unintegrated PDF to be a number density.

The requirement that the off-the-light-cone two-quark matrix element should have an anomalous dimension equal to that of the corresponding quantity with the **smooth gauge connector** in order to respect the probabilistic interpretation, is tantamount to the **anomalous dimensions sum rule**. Therefore, the RG properties can serve to define the necessary condition for the PDF to be a number density.

The generalized TMD PDF obeys this condition.

# Conclusions

- The anomalous dimension of the TMD quark distribution in the **light-cone gauge** with different pole prescriptions is calculated in 1-loop order;

- The generalized completely gauge invariant **definition of TMD PDF** is proposed;

- The direct connection to the **integrated PDF** is established;

- The **UV-evolution** of the modified TMD PDF is discussed.


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