Differential reduction of generalized hypergeometric functions in application to Feynman diagrams

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Evaluation of Multi-Loop Feynman Diagram (FD)

- Reduction to Master integrals :

Integration by part relations (IBP), Lorentz-invariance identities (they are the consequence of IBP relations, Roman Lee)

Key feature: differentiation over loop momenta and obtaining recurrence relations.

$$F(\alpha)=\int\frac{\mathrm{d}^d k}{(k^2-m^2)^\alpha},\quad \int \mathrm{d}^d k\frac{\partial}{\partial k_\mu}k_\mu\frac{1}{(k^2-m^2)^\alpha}=0,$$

$$(d-2\alpha)F(\alpha)-2\alpha m^2 F(\alpha+1)=0$$

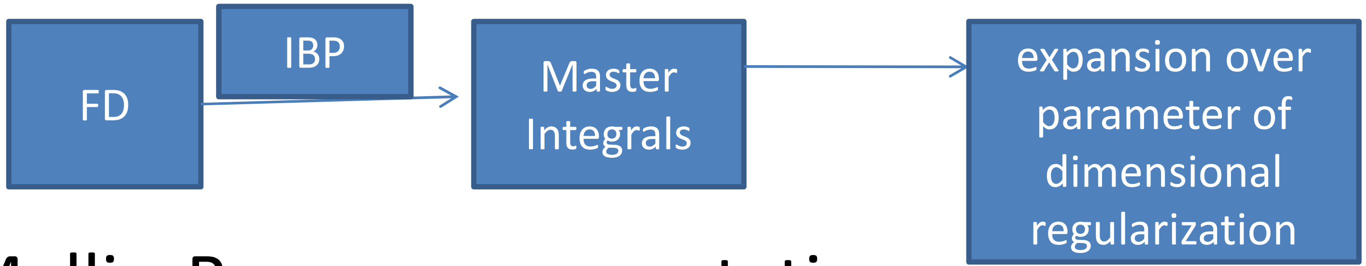
(Implementation: Baikov's method etc.)

method of shifting dimensions (O. Tarasov)

Disadvantages: for example, for double box FD with one off-shell leg it was necessary to solve linear systems of thousands of equations.

For exhaustive review see book of V. A. Smirnov "Evaluating Feynman Integrals"

Master Integral Evaluation



- Mellin-Barnes representation

$$\frac{1}{(m^2-k^2)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}\int \mathrm{d}z \frac{(m^2)^z}{(-k^2)^{\lambda+z}}\Gamma(\lambda+z)\Gamma(-z)$$

- Differential equations

(to take derivatives of integral with respect to kinematical invariants and masses and try to solve this system)

Hypergeometric function approach of multi-loop FD evaluation

- Applying to FD multiple Mellin-Barnes integral representation,

$$\Phi(n) \sim \int_{-i\infty}^{+i\infty} \Pi_{a,b,c} \frac{\Gamma(\sum_{i=1}^{m} A_{ai} z_i + B_a)}{\Gamma(\sum_{j=1}^{r} C_{bj} z_j + D_b)} dz_c Y^{z_c - 1} ,$$

- And application Cauchy theorem we obtain linear combination of multiple series, or Horn type hypergeometric function.

$$\Phi(n, \vec{x}) \sim \sum_{k_1, \cdots, k_{r+m}=0}^{\infty} \Pi_{a,b} \frac{\Gamma(\sum_{i=1}^{m} \tilde{A}_{ai} k_i + \tilde{B}_a)}{\Gamma(\sum_{j=1}^{r} \tilde{C}_{bj} k_j + \tilde{D}_b)} x_1^{k_1} \cdots x_{r+m}^{k_{r+m}}$$

Horn definition of series representation (hypergeometric function)

- Series $$\Phi(\vec{x}) = \sum C(\vec{m})\vec{x}^{\vec{m}} = \sum_{m_1,m_2,...,m_r} C(m_1,m_2,...,m_r)x_1^{m_1}...x_r^{m_r}$$

is called Horn-type if $$C(\vec{m}) = \Pi_{i=1}^{r}\lambda_i^{m_i}R(\vec{m})\left(\Pi_{j=1}^{N}\Gamma(\mu_j(\vec{m})+\gamma_j+1)\right)^{-1}$$

They obey differential equation

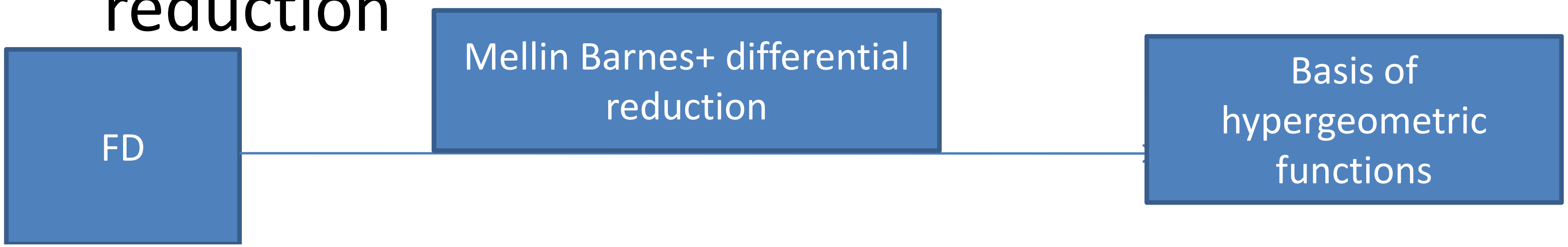
$$Q_j\left(\sum_{k=1}^{r} x_k\frac{\partial}{\partial x_k}\right)\frac{1}{x_j}\Phi(\vec{x}) = P_j\left(\sum_{k=1}^{r} x_k\frac{\partial}{\partial x_k}\right)\Phi(\vec{x})$$

and coefficients have remarkable property

$$\vec{e}_j = (0,\cdots,0,1,0,\cdots,0) \qquad \frac{C(\vec{m}+e_j)}{C(\vec{m})} = \frac{P_j(\vec{m})}{Q_j(\vec{m})}$$

Hypergeometric function approach of multi-loop FD evaluation

- There exist differential step-up and step-down operators, which change parameters of hyp. funct. by unity. $\Phi(\vec{\gamma}+\vec{e}_c;\vec{\sigma};\vec{x}) = U^{+}_{[\gamma_c\to\gamma_c+1]}\Phi(\vec{\gamma},\vec{\sigma},\vec{x})$
- In **N. Takayama, J. App. Math. 6 (1989) 147** the algorithm of inverse operators was developed.
- By this differential operators the parameters of hypergeometric functions can be changed by arbitrary integer numbers---differential reduction



Differential reduction algorithm for $_{p+1}F_p$ hypergeometric funct.

- Definition:

$$_pF_q(\vec{a};\vec{b};z) \equiv {}_pF_q\left(\begin{array}{c}\vec{a}\\ \vec{b}\end{array}\middle| z\right) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \frac{\Pi_{i=1}^{p}(a_i)_k}{\Pi_{j=1}^{q}(b_j)_k}$$

Pochhammer symbol: $(a)_k = \Gamma(a+k)/\Gamma(a)$

- Differential equation:

$$\left[z\Pi_{i=1}^{p}(\theta + a_i) - \theta\Pi_{i=1}^{q}(\theta + b_i - 1)\right] {}_pF_q(\vec{a};\vec{b};z) = 0$$

- Operator: $\theta = z\dfrac{d}{dz}$

Differential reduction algorithm for ${}_{p+1}F_p$ hypergeometric funct.

- Differential identities:

$${}_pF_q(a_1+1,\vec{a};\vec{b};z) = B^+_{a_1}\,{}_pF_q(a_1,\vec{a};\vec{b};z) = \frac{1}{a_1}(\theta+a_1)\,{}_pF_q(a_1,\vec{a};\vec{b};z)$$

$${}_pF_q(\vec{a};b_1-1,\vec{b};z) = H^-_{b_1}\,{}_pF_q(\vec{a};b_1,\vec{b};z) = \frac{1}{b_1-1}(\theta+b_1-1)\,{}_pF_q(\vec{a};b_1,\vec{b};z)$$

$${}_{p+1}F_p(\vec{a};b_i+1,\vec{b};z) = H^+_{b_i}\,{}_{p+1}F_p(\vec{a};b_1,\vec{b};z)\ ,$$

$$H^+_{a_i} = \frac{b_i-1}{d_i}\left[\frac{d}{dz}\Pi_{j\neq i}(\theta+b_j-1) - s_i(\theta)\right]\Bigg|_{b_i\to b_i+1}\ ,$$

$$d_i = \Pi_{j=1}^{p+1}(1+a_j-b_i)\ ,$$

$$s_i(x) = \frac{\Pi_{j=1}^{p+1}(x+a_j)-d_i}{x+b_i-1}\ ,$$

Inverse operators:

$${}_{p+1}F_p(a_i-1,\vec{a};\vec{b};z) = B^-_{a_i}\,{}_{p+1}F_p(a_i,\vec{a};\vec{b};z)\ ,$$

$$B^-_{a_i} = -\frac{a_i}{c_i}\left[t_i(\theta) - z\Pi_{j\neq i}(\theta+a_j)\right]\Big|_{a_i\to a_i-1}\ ,$$

$$c_i = -a_i\Pi_{j=1}^{p}(b_j-1-a_i)\ ,$$

$$t_i(x) = \frac{x\Pi_{j=1}^{p}(x+b_j-1)-c_i}{x+a_i}\ ,$$

Differential reduction algorithm for ${}_{p+1}F_p$ hypergeometric funct.

- By successive implementation of differential operators we can change parameters of ${}_{p+1}F_p$

$$F(\vec{a}+\vec{m};\vec{b}+\vec{n};z) = \left(H^{\pm}_{\{a\}}\right)^{\sum_i m_i} \left(B^{\pm}_{\{b\}}\right)^{\sum_j n_j} F(\vec{a};\vec{b};z)$$

- The max power of θ is $r \equiv \sum_i m_i + \sum_j n_j$.
- ${}_{p+1}F_p$ satisfies the differential equation of order p+1, so

$$ {}_{p+1}F_p(\vec{a}+\vec{m};\vec{b}+\vec{k};z) = $$

$$\left\{ R_1(a_i,b_j,z)\theta^p + R_2(a_i,b_j,z)\theta^{p-1} + \cdots + R_p(a_i,b_j,z)\theta + R_{p+1}(a_i,b_j,z) \right\} {}_{p+1}F_p(\vec{a};\vec{b};z)$$

Differential reduction algorithm for ${}_{p+1}F_p$ hypergeometric funct.

- Example of differential reduction:

$$
{}_3F_2\left(\left.\begin{matrix} a_1-1, a_2, a_3 \\ b_1, b_2 \end{matrix}\right| z\right)(b_1-a_1)(b_2-a_1) = \Big\{ (1-z)\theta^2
$$

$$
+ \left[(b_1+b_2-1-a_1) - z(a_2+a_3)\right]\theta + (b_1-a_1)(b_2-a_1) - z a_2 a_3 \Big\} {}_3F_2\left(\left.\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix}\right| z\right)
$$

- In reduction on more units the structure of equality will be the same

Differential reduction algorithm for $_{p+1}F_p$ hypergeometric funct.

- Criterion of **further** reducibility of $_{p+1}F_p$

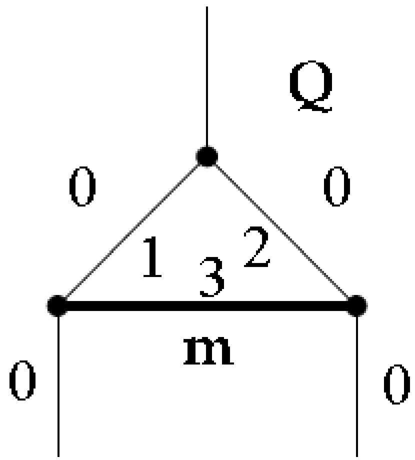
(reducibility means that initial function is expressible in terms of lower order hyp. funct. and (or) derivatives)

- The $_pF_q(\vec{a};\vec{b};z)$, where $a_i = b_i + m_i$, (m_j integer), is expressible in terms of function of lower order

$$
{}_pF_q\left(\begin{matrix} b_1+m_1, a_2, \cdots, a_p \\ b_1, b_2, \cdots b_q \end{matrix}\middle| z\right) = \sum_{j=0}^{m_1} z^j \binom{m_1}{j} \frac{(a_2)_j \cdots (a_p)_j}{(b_1)_j \cdots (b_q)_j} {}_{p-1}F_{q-1}\left(\begin{matrix} a_2+j, \cdots a_p+j \\ b_2+j, \cdots b_q+j \end{matrix}\middle| z\right)
$$

If one of the upper parameters of $_{p+1}F_p$ is an integer, the result of reduction has one less derivative

Examples



C 1

$$\frac{C_1}{i^{1-n}\pi^{\frac{n}{2}}} = (-m^2)^{\frac{n}{2}-j_{123}}$$

$$\left\{ \frac{\Gamma\left(j_{123}-\frac{n}{2}\right)\Gamma\left(\frac{n}{2}-j_{12}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma(j_3)}\, {}_3F_2\left(\left.\begin{matrix} j_{123}-\frac{n}{2}, j_1, j_2 \\ \frac{n}{2}, 1+j_{12}-\frac{n}{2} \end{matrix}\right| -\frac{Q^2}{m^2}\right)\right.$$

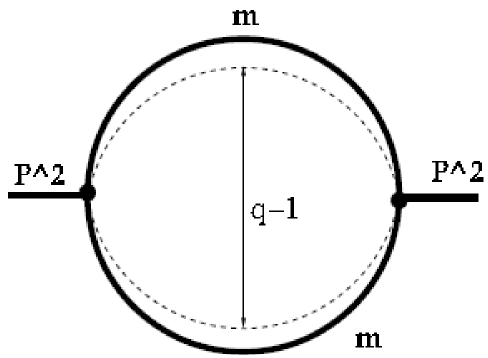
$$+\left(-\frac{Q^2}{m^2}\right)^{\frac{n}{2}-j_{12}} \frac{\Gamma\left(\frac{n}{2}-j_1\right)\Gamma\left(\frac{n}{2}-j_2\right)\Gamma\left(j_{12}-\frac{n}{2}\right)}{\Gamma\left(n-j_{12}\right)\Gamma(j_1)\Gamma(j_2)}$$

$$\left.\times {}_3F_2\left(\left.\begin{matrix} j_3, \frac{n}{2}-j_1, \frac{n}{2}-j_2 \\ n-j_{12}, \frac{n}{2}-j_{12}+1 \end{matrix}\right| -\frac{Q^2}{m^2}\right)\right\}.$$

- By using the differential reduction:

$$ {}_2F_1\left(\left.\begin{matrix} 1, 1 \\ I_1+\frac{n}{2} \end{matrix}\right| z\right) \;,\quad {}_2F_1\left(\left.\begin{matrix} 1, I_1+\frac{n}{2} \\ I_2+n \end{matrix}\right| z\right)$$

- IBP procedure gives us the vertex master-integral, buble (rational function) and massless propagator type diagram.



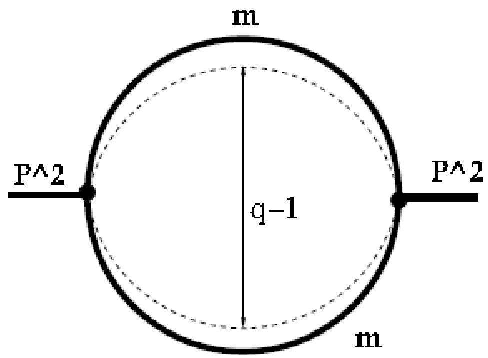
Sunset type diagram J_{22}^q

$$J_{22}^q(m^2,p^2,\alpha_1,\alpha_2,\sigma_1,\cdots,\sigma_{q-1}) = \int \frac{d^n(k_1\cdots k_q)}{[k_1^2]^{\sigma_1}\cdots[k_{q-1}^2]^{\sigma_{q-1}}[k_q^2-m^2]^{\alpha_1}[(k_1+k_2+\cdots+k_q+p)^2-m^2]^{\alpha_2}}$$

- By using Mellin-Barnes representation

$$J_{22}^q(m^2,p^2,\alpha_1,\alpha_2,\sigma_1,\cdots,\sigma_{q-1}) = \frac{(-m^2)^{\frac{n}{2}-\alpha_{1,2}}(p^2)^{\frac{n}{2}(q-1)-\sigma}}{\left[i^{1-n}\pi^{n/2}\right]^{-q}\Gamma(\alpha_1)\Gamma(\alpha_2)}\left\{\Pi_{k=1}^{q-1}\frac{\Gamma(\frac{n}{2}-\sigma_k)}{\Gamma(\sigma_k)}\right\}$$
$$\times\int ds\left(-\frac{p^2}{m^2}\right)^s\frac{\Gamma(\alpha_1+s)\Gamma(\alpha_2+s)\Gamma(\alpha_{1,2}-\frac{n}{2}+s)\Gamma(\frac{n}{2}+s)\Gamma(\sigma-\frac{n}{2}(q-1)-s)}{\Gamma(\alpha_{1,2}+2s)\Gamma(\frac{n}{2}q-\sigma+s)},$$

and Cauchy theorem we obtain



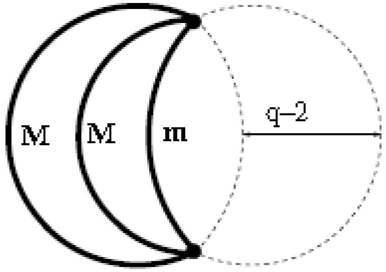
Sunset type diagram J_{22}^{q}

$$J_{22}^{q}(m^2,p^2,\alpha_1,\alpha_2,\sigma_1,\cdots,\sigma_{q-1}) = \left[i^{1-n}\pi^{n/2}\right]^q \frac{(-m^2)^{\frac{n}{2}q-\alpha_{1,2}-\sigma}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \left\{\Pi_{k=1}^{q-1}\frac{\Gamma(\frac{n}{2}-\sigma_k)}{\Gamma(\sigma_k)}\right\}$$

$$\times\frac{\Gamma\left(\alpha_1+\sigma-\frac{n}{2}(q-1)\right)\Gamma\left(\alpha_2+\sigma-\frac{n}{2}(q-1)\right)\Gamma\left(\sigma-\frac{n}{2}(q-2)\right)\Gamma\left(\alpha_{1,2}+\sigma-\frac{n}{2}q\right)}{\Gamma\left(\alpha_{1,2}+2\sigma-n(q-1)\right)\Gamma\left(\frac{n}{2}\right)}$$

$${}_4F_3\left(\begin{matrix}\alpha_1+\sigma-\frac{n}{2}(q-1),\alpha_2+\sigma-\frac{n}{2}(q-1),\sigma-\frac{n}{2}(q-2),\alpha_{1,2}+\sigma-\frac{n}{2}q\\ \frac{n}{2},\frac{1}{2}(\alpha_{1,2}-n(q-1))+\sigma,\frac{1}{2}(1+\alpha_{1,2}-n(q-1))+\sigma,\end{matrix}\middle|\frac{p^2}{4m^2}\right) . ($$

- Criteria of reducibility:
- q=1 $\quad {}_2F_1\left(\begin{matrix}1,I_1-\frac{n}{2},\\ I_2\end{matrix}\middle|z\right)$ IBP gives 1 MI
- q=2 $\quad (1,\theta)\times\ {}_3F_2\left(\begin{matrix}1,I_1-\frac{n}{2},I_2-n\\ I_3+\frac{n}{2},I_4+\frac{1}{2}-\frac{n}{2}\end{matrix}\middle|z\right)$ IBP gives 2 MI
- q=3,4,5…. $\quad (1,\theta,\theta^2)\times\ {}_3F_2\left(\begin{matrix}I_1-\frac{n}{2}(q-1),I_2-\frac{n}{2}(q-2),I_3-\frac{n}{2}q\\ \frac{n}{2},I_4+\frac{1}{2}-\frac{n}{2}(q-1)\end{matrix}\middle|z\right)$ IBP gives ???



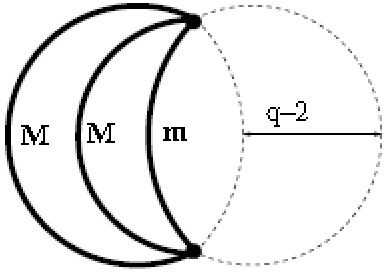
Bubble diagram

- Mellin-Barnes representation
- hypergeometric representation

$$
\begin{aligned}
&B^q_{1220}(m^2, M^2, \alpha_1, \alpha_2, \beta, \sigma_1, \cdots, \sigma_{q-2}) \\
&= \frac{(-M^2)^{\frac{n}{2}q-\alpha_{1,2}-\sigma-\beta}\Gamma\left(\frac{n}{2}-\beta\right)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta)\Gamma\left(\frac{n}{2}\right)\left[i^{1-n}\pi^{\frac{n}{2}}\right]^{-q}} \left\{\Pi_{k=1}^{q-2}\frac{\Gamma(\frac{n}{2}-\sigma_k)}{\Gamma(\sigma_k)}\right\} \\
&\times\left\{ \frac{\Gamma\left(a_{1,\sigma,\beta}-\frac{n}{2}(q-1)\right)\Gamma\left(a_{2,\sigma,\beta}-\frac{n}{2}(q-1)\right)\Gamma\left(a_{1,2,\beta,\sigma}-\frac{n}{2}q\right)\Gamma\left(a_{\sigma,\beta}-\frac{n}{2}(q-2)\right)}{\Gamma\left(\alpha_{1,2}+2\beta+2\sigma-n(q-1)\right)} \right. \\
&\quad {}_4F_3\left(\left.\begin{matrix} a_{1,\beta,\sigma}-\frac{n}{2}(q-1), a_{2,\beta,\sigma}-\frac{n}{2}(q-1), \alpha_{1,2}+\beta+\sigma-\frac{n}{2}q, a_{\sigma,\beta}-\frac{n}{2}(q-2) \\ \frac{1}{2}(\alpha_{1,2}-n(q-1))+\sigma+\beta, \frac{1}{2}(\alpha_{1,2}+1-n(q-1))+\sigma+\beta, 1+\beta-\frac{n}{2} \end{matrix}\right| \frac{m^2}{4M^2}\right) \\
&\quad +\left(\frac{m^2}{M^2}\right)^{\frac{n}{2}-\beta}\frac{\Gamma\left(a_{1,\sigma}-\frac{n}{2}(q-2)\right)\Gamma\left(a_{2,\sigma}-\frac{n}{2}(q-2)\right)\Gamma\left(a_{1,2,\sigma}-\frac{n}{2}(q-1)\right)}{\Gamma\left(\frac{n}{2}-\beta\right)\Gamma\left(\alpha_{1,2}+2\sigma-n(q-2)\right)} \\
&\quad \times\Gamma\left(\sigma-\frac{n}{2}(q-3)\right)\Gamma\left(\beta-\frac{n}{2}\right) \\
&\quad \left. {}_4F_3\left(\left.\begin{matrix} a_{1,\sigma}-\frac{n}{2}(q-2), a_{2,\sigma}-\frac{n}{2}(q-2), a_{1,2,\sigma}-\frac{n}{2}(q-1), \sigma-\frac{n}{2}(q-3) \\ \frac{1}{2}(\alpha_{1,2}-n(q-2))+\sigma, \frac{1}{2}(\alpha_{1,2}+1-n(q-2))+\sigma, 1-\beta+\frac{n}{2} \end{matrix}\right| \frac{m^2}{4M^2}\right)\right\}, \qquad (
\end{aligned}
$$

- For q=3 $\quad (1,\theta)\times\ {}_2F_1\left(\left.\begin{matrix} I_1-n, I_2-\frac{3n}{2} \\ \frac{1}{2}+I_3-n \end{matrix}\right| z\right)\ ,\quad (1,\theta)\times\ {}_3F_2\left(\left.\begin{matrix} 1, I_1-n, I_2-\frac{n}{2} \\ \frac{1}{2}+I_3-\frac{n}{2}, I_4+\frac{n}{2} \end{matrix}\right| z\right)$

Bubble diagram



- Mellin-Barnes representation
- hypergeometric representation

$$B^3_{1220}(m^2, M^2, 1,1,1,1) = (M^2)^{2-3\varepsilon} \frac{\Gamma^3(1+\varepsilon)(i\pi^{2-\varepsilon})^3}{\varepsilon^3(1-\varepsilon)^2(1-2\varepsilon)}$$

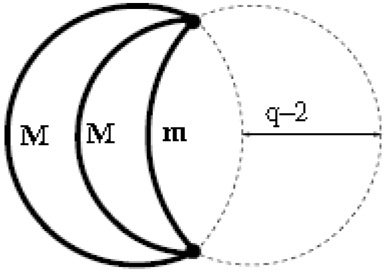
$$\times \left\{ \frac{2(1-\varepsilon)(1-4\varepsilon)\Gamma^2(1+2\varepsilon)\,\Gamma(1+3\varepsilon)\,\Gamma(1-\varepsilon)}{3(1-3\varepsilon)(2-3\varepsilon)\Gamma(1+4\varepsilon)\Gamma^2(1+\varepsilon)} \; {}_2F_1\left(\begin{matrix} -1+2\varepsilon, -2+3\varepsilon \\ -\frac{1}{2}+2\varepsilon \end{matrix} \middle| \frac{m^2}{4M^2} \right) \right.$$

$$\left. + \left(\frac{m^2}{M^2}\right)^{1-\varepsilon} \; {}_3F_2\left(\begin{matrix} 1, \varepsilon, -1+2\varepsilon \\ 2-\varepsilon, \frac{1}{2}+\varepsilon \end{matrix} \middle| \frac{m^2}{4M^2} \right) \right\},$$

- By using the differential reduction we can change the value of the parameters

$$\frac{2z(1-2\varepsilon)(1-3\varepsilon)}{1-\varepsilon} \; {}_3F_2\left(\begin{matrix} 1, \varepsilon, 2\varepsilon \\ 2-\varepsilon, \frac{1}{2}+\varepsilon \end{matrix} \middle| z \right) = 1-2\varepsilon$$

$$+ [2\varepsilon(1-4z)-(1-2z)] \; {}_3F_2\left(\begin{matrix} 1, \varepsilon, 2\varepsilon \\ 1-\varepsilon, \frac{1}{2}+\varepsilon \end{matrix} \middle| z \right) + \frac{8z(1-z)\varepsilon^2}{(1-\varepsilon)(1+2\varepsilon)} \; {}_3F_2\left(\begin{matrix} 2, 1+\varepsilon, 1+2\varepsilon \\ 2-\varepsilon, \frac{3}{2}+\varepsilon \end{matrix} \middle| z \right)$$



Bubble diagram

- And using the ε-expansion for hyp. funct. we obtain the explicit answer for integral:

$$\frac{B^3_{1220}(m^2,M^2,1,1,1,1)}{\Gamma^3(1+\varepsilon)(i\pi^{2-\varepsilon})^3} = (M^2)^{2-3\varepsilon}\Bigg(\frac{1+2z}{3\varepsilon^3}+\frac{1}{\varepsilon^2}\left\{\frac{7}{6}+\frac{8}{3}z-\frac{1}{12}z^2-z\ln z\right\}$$

$$+\frac{1}{\varepsilon}\left\{\frac{25}{12}+\frac{20}{3}z-\frac{5}{8}z^2+\frac{1}{4}z\ln z\left[z+2\ln z-16\right]\right\}$$

$$+\frac{8}{3}\zeta_3(1-z)-\frac{5}{24}+\frac{35}{3}z-\frac{145}{48}z^2-\frac{1}{6}z\ln^3 z+\frac{(16-z)}{8}z\ln^2 z-\frac{(80-15z)}{8}z\ln z$$

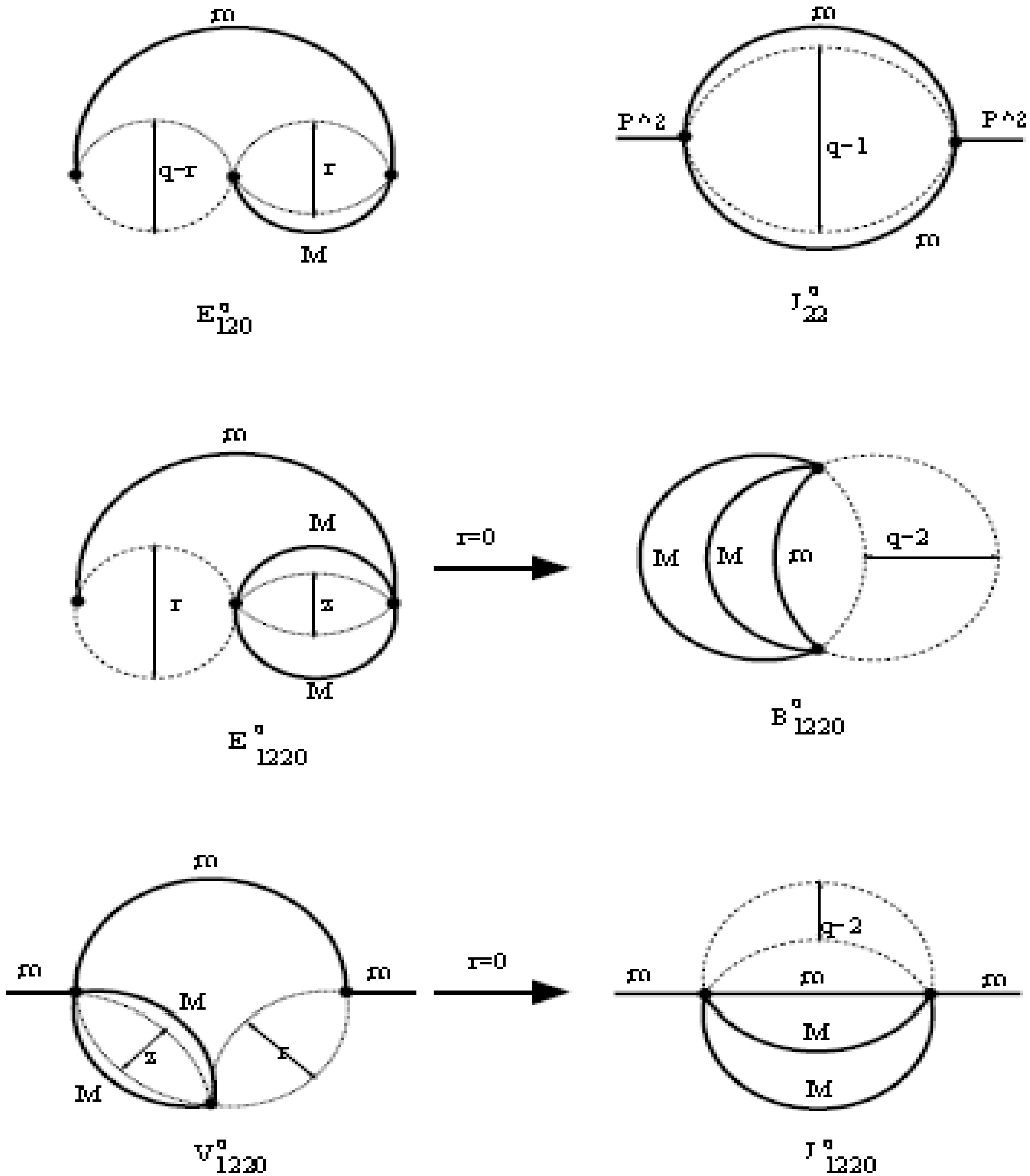
$$-4(1-z)\left[\mathrm{S}_{1,2}(1-y)+\ln y\,\mathrm{Li}_2(1-y)\right]-(1-z)\ln^2 y\left[\ln z+\frac{2}{3}\ln y\right]$$

$$-(8+2z-z^2)\frac{(1-y)}{(1+y)}\left[\mathrm{Li}_2(1-y)+\frac{1}{2}\ln z\ln y+\frac{1}{4}\ln^2 y\right]+\mathcal{O}(\varepsilon)\Bigg),$$

$$z = m^2/M^2 \qquad y=\frac{1-\sqrt{\frac{z}{z-4}}}{1+\sqrt{\frac{z}{z-4}}}$$

Examples of topologies

- Mellin-Barnes Representation
- Hypergeometric Representation
- Differential reduction to the minimal basis
- ε –expansion



Implementation of algorithm

- The package called HYPERDIRE (HYPERgeometric DIFFerential REduction), based on language of program ''Mathematica''
- Key feature is that the product of non-commutative step-up and step-down operators of differential reduction turn into product of special 2-dimensional matrices which greatly simplify and reduce the time of calculation
- Implemented all step-up and step-down operators and reducibility formulas, which permits one to reduce any hyp. func. to predefined basis of functions

Example of HYPERDIRE

$${}_3F_2\left(\begin{matrix} 2+\frac{3}{2}eps, 3+eps, 2 \\ 4-eps, 2+\frac{5}{2}eps \end{matrix}\middle| x\right)$$

- **vector={{2+3/2 eps,3+ eps,2},{4- eps,2+5/2 eps },x}**
- **ToGroebnerBasis[vector]**

$$C(A+B\theta){}_3F_2\left(\begin{matrix} 1+\frac{3}{2}eps, 1+eps, 1 \\ 2-eps, 2+\frac{5}{2}eps \end{matrix}\middle| x\right)+D$$

$$B=\frac{(1-3eps+2eps^2)(4(-1+x)+2eps(8-8x+x^2)+eps^2(-7-x+3x^2)}{2(-1+x)x^2}$$

- **explicitFormGroebnerBasis[answer]**

Summary

- Differential reduction gives us the possibility to reduce the initial Feynman integral to some pre-define basis without using of IBP procedure
- The reduction to MI is done without ε-expansion
- It is possible in advance to say how much MI contains initial FD
- Now the ε-expansion is known for hyp. funct. of one variable, so the explicit answer in terms of multiple polylogaritms can be obtained
- For z=1 (argument of hyp. func.) the differential reduction algorithm can not give explicit result, but it is possible to consider limit z->1 for any order of ε expansion

$$Re \sum_{j=1}^{p} b_j - Re \sum_{j=1}^{p+1} a_j > 0$$

$$_pF_q \left(\begin{array}{c|} \vec{a} \\ \vec{b} \end{array} \; z \right)$$