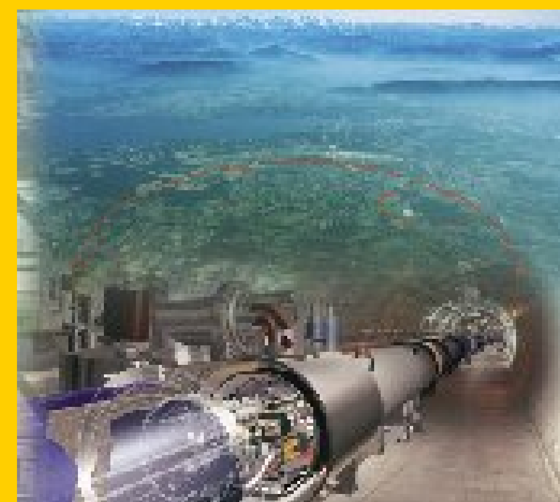
Resummation approach in APT
How many loops do we need to calculate?

A. P. Bakulev

Bogoliubov Lab. Theor. Phys., JINR (Dubna, Russia)



OUTLINE

- Intro: Analytic Perturbation Theory (**APT**) in QCD

OUTLINE

- Intro: Analytic Perturbation Theory (**APT**) in QCD
- Problems of **APT** and their resolution in **FAPT**:

OUTLINE

- Intro: Analytic Perturbation Theory (**APT**) in QCD
- Problems of **APT** and their resolution in **FAPT**:
- Technical development of **FAPT**: thresholds

OUTLINE

- Intro: Analytic Perturbation Theory (**APT**) in QCD
- Problems of **APT** and their resolution in **FAPT**:
- Technical development of **FAPT**: thresholds
- Resummation in **APT**and **FAPT**

OUTLINE

- Intro: Analytic Perturbation Theory (**APT**) in QCD
- Problems of **APT** and their resolution in **FAPT**:
- Technical development of **FAPT**: thresholds
- Resummation in **APT**and **FAPT**
- Applications: Higgs decay $H^0 \to b\bar{b}$

OUTLINE

- Intro: Analytic Perturbation Theory (**APT**) in QCD
- Problems of **APT** and their resolution in **FAPT**:
- Technical development of **FAPT**: thresholds
- Resummation in **APT**and **FAPT**
- Applications: Higgs decay $H^0 \to b\bar{b}$
- Applications: Adler function $D(Q^2)$ in $N_f = 4$ region

OUTLINE

- Intro: Analytic Perturbation Theory (**APT**) in QCD
- Problems of **APT** and their resolution in **FAPT**:
- Technical development of **FAPT**: thresholds
- Resummation in **APT**and **FAPT**
- Applications: Higgs decay $H^0 \to b\bar{b}$
- Applications: Adler function $D(Q^2)$ in $N_f = 4$ region
- Conclusions

Collaborators & Publications

Collaborators:

S. Mikhailov (Dubna), N. Stefanis (Bochum), and
A. Karanikas (Athens)

Collaborators & Publications

Collaborators:

S. Mikhailov (Dubna), N. Stefanis (Bochum), and
A. Karanikas (Athens)

Publications:

- A. B., Mikhailov, Stefanis — **PRD 72 (2005) 074014**
- A. B., Karanikas, Stefanis — **PRD 72 (2005) 074015**
- A. B., Mikhailov, Stefanis — **PRD 75 (2007) 056005**
- A. B.&Mikhailov — “Resummation in (F)APT”, **arXiv:0803.3013 [hep-ph]**
- A. B. — “Global FAPT in QCD with Selected Applications”, **arXiv:0805.0829 [hep-ph]**

Analytic Perturbation Theory

in

QCD

History of APT

Euclidean	**Minkowskian**
$Q^2 = \vec{q}^2 - q_0^2 \geq 0$	$s = q_0^2 - \vec{q}^2 \geq 0$

History of APT

Euclidean

$$Q^2 = \vec{q}^2 - q_0^2 \geq 0$$

Minkowskian

$$s = q_0^2 - \vec{q}^2 \geq 0$$

RG+Analyticity

ghost-free $\overline{\alpha}_{\text{QED}}(Q^2)$

Bogoliubov et al. 1959

History of APT

Euclidean	Minkowskian
$Q^2 = \vec{q}^2 - q_0^2 \geq 0$	$s = q_0^2 - \vec{q}^2 \geq 0$

RG$+$Analyticity

ghost-free $\overline{\alpha}_{\text{QED}}(Q^2)$

Bogoliubov et al. 1959

pQCD$+$RG: resum π^2-terms

Arctg(s), UV Non-Power Series

Radyush., Krasn. & Pivov. 1982

History of APT

Euclidean

$Q^2 = \vec{q}^{\,2} - q_0^2 \geq 0$

Minkowskian

$s = q_0^2 - \vec{q}^{\,2} \geq 0$

RG+Analyticity

ghost-free $\overline{\alpha}_{\text{QED}}(Q^2)$

Bogoliubov et al. 1959

pQCD+RG: resum π^2-terms

Arctg(s), UV Non-Power Series

Radyush., Krasn. & Pivov. 1982

pQCD+renormalons:

Arctg(s) at **LE** region

Ball, Beneke & Braun 1994-95

History of APT

Euclidean

$Q^2 = \vec{q}^2 - q_0^2 \geq 0$

Minkowskian

$s = q_0^2 - \vec{q}^2 \geq 0$

RG+Analyticity

ghost-free $\overline{\alpha}_{\mathsf{QED}}(Q^2)$

Bogoliubov et al. 1959

pQCD+RG: resum π^2-terms

Arctg(s), UV Non-Power Series

Radyush., Krasn. & Pivov. 1982

DispRel+renormalons

IR finite $\alpha_s^{\mathsf{eff}}(Q^2)$

Dokshitzer et al. 1995

pQCD+renormalons

Arctg(s) at **LE** region

Ball, Beneke & Braun 1994-95

History of APT

Euclidean	**Minkowskian**
$Q^2 = \vec{q}^2 - q_0^2 \geq 0$	$s = q_0^2 - \vec{q}^2 \geq 0$

RG+Analyticity

ghost-free $\overline{\alpha}_{\text{QED}}(Q^2)$

Bogoliubov et al. 1959

DispRel+renormalons

IR finite $\alpha_s^{\text{eff}}(Q^2)$

Dokshitzer et al. 1995

pQCD+RG: resum π^2-terms

Arctg(s), UV Non-Power Series

Radyush., Krasn. & Pivov. 1982

pQCD+renormalons

Arctg(s) at **LE** region

Ball, Beneke & Braun 1994-95

Integral Transformation:

$\mathcal{R}\left[\overline{\alpha}_s\right] \longrightarrow$ **Arctg**(s)

Jones & Solovtsov 1995

History of APT

Euclidean	**Minkowskian**
$Q^2 = \vec{q}^2 - q_0^2 \geq 0$	$s = q_0^2 - \vec{q}^2 \geq 0$

RG+Analyticity
ghost-free $\overline{\alpha}_{\text{QED}}(Q^2)$
Bogoliubov et al. 1959

pQCD+RG: resum π^2-terms
Arctg(s), UV Non-Power Series
Radyush., Krasn. & Pivov. 1982

DispRel+renormalons
IR finite $\alpha_s^{\text{eff}}(Q^2)$
Dokshitzer et al. 1995

pQCD+renormalons
Arctg(s) at **LE** region
Ball, Beneke & Braun 1994-95

RG+Analyticity
ghost-free $\alpha_{\text{E}}(Q^2)$
Shirkov & Solovtsov 1996

Integral Transformation:
$\mathcal{R}\left[\overline{\alpha}_s\right] \longrightarrow$ **Arctg**(s)
Jones & Solovtsov 1995

History of APT

Euclidean

$$Q^2 = \vec{q}^2 - q_0^2 \geq 0$$

Minkowskian

$$s = q_0^2 - \vec{q}^2 \geq 0$$

RG+Analyticity

ghost-free $\alpha_{\mathrm{E}}(Q^2)$

Shirkov & Solovtsov 1996

Integral Transformation:

$\mathcal{R}\left[\overline{\alpha}_s\right] \longrightarrow$ **Arctg**(s)

Jones & Solovtsov 1995

History of APT

Euclidean

$Q^2 = \vec{q}^2 - q_0^2 \geq 0$

Minkowskian

$s = q_0^2 - \vec{q}^2 \geq 0$

RG+Analyticity

ghost-free $\alpha_{\mathsf{E}}(Q^2)$

Shirkov & Solovtsov 1996

Integral Transformation:

$\mathcal{R}\left[\overline{\alpha}_s\right] \to$ **Arctg**(s)

Jones & Solovtsov 1995

pQCD+RG+Analiticity

Transforms: $\hat{\mathcal{D}} = \hat{\mathcal{R}}^{-1}$

Couplings: $\alpha_{\mathsf{E}}(Q^2) \Leftrightarrow \alpha_{\mathsf{M}}(s)$

Milton & Solovtsov 1996–97

History of APT

Euclidean

$Q^2 = \vec{q}^2 - q_0^2 \geq 0$

Minkowskian

$s = q_0^2 - \vec{q}^2 \geq 0$

RG+Analyticity

ghost-free $\alpha_{\mathsf{E}}(Q^2)$

Shirkov & Solovtsov 1996

Integral Transformation:

$\mathcal{R}\left[\overline{\alpha}_s\right] \longrightarrow$ **Arctg**(s)

Jones & Solovtsov 1995

pQCD+RG+Analiticity

Transforms: $\hat{\mathcal{D}} = \hat{\mathcal{R}}^{-1}$

Couplings: $\alpha_{\mathsf{E}}(Q^2) \Leftrightarrow \alpha_{\mathsf{M}}(s)$

Milton & Solovtsov 1996–97

Analytic (global) pQCD+Analiticity

Global couplings: $\mathcal{A}_n(Q^2) \Leftrightarrow \mathfrak{A}_n(s)$

Non-Power perturbative expansions

Shirkov 1999–2001

Intro: PT in QCD

- coupling $\alpha_s(\mu^2) = (4\pi/b_0)\, a_s[L]$ with $L = \ln(\mu^2/\Lambda^2)$

Intro: PT in QCD

- coupling $\alpha_s(\mu^2) = (4\pi/b_0)\, a_s[L]$ with $L = \ln(\mu^2/\Lambda^2)$
- RG equation $\dfrac{d\, a_s[L]}{d\, L} = -a_s^2 - c_1\, a_s^3 - \ldots$

Intro: PT in QCD

- coupling $\alpha_s(\mu^2) = (4\pi/b_0)\, a_s[L]$ with $L = \ln(\mu^2/\Lambda^2)$
- RG equation $\dfrac{d\, a_s[L]}{d\, L} = -a_s^2 - c_1\, a_s^3 - \ldots$
- 1-loop solution generates Landau pole singularity: $a_s[L] = 1/L$

Intro: PT in QCD

- coupling $\alpha_s(\mu^2) = (4\pi/b_0)\, a_s[L]$ with $L = \ln(\mu^2/\Lambda^2)$
- RG equation $\dfrac{d\, a_s[L]}{d\, L} = -a_s^2 - c_1\, a_s^3 - \ldots$
- 1-loop solution generates Landau pole singularity: $a_s[L] = 1/L$
- 2-loop solution generates square-root singularity: $a_s[L] \sim 1/\sqrt{L + c_1 \ln c_1}$

Intro: PT in QCD

- coupling $\alpha_s(\mu^2) = (4\pi/b_0)\, a_s[L]$ with $L = \ln(\mu^2/\Lambda^2)$
- RG equation $\dfrac{d\, a_s[L]}{d\, L} = -a_s^2 - c_1\, a_s^3 - \ldots$
- 1-loop solution generates Landau pole singularity: $a_s[L] = 1/L$
- 2-loop solution generates square-root singularity: $a_s[L] \sim 1/\sqrt{L + c_1 \ln c_1}$
- PT series: $D[L] = 1 + d_1 a_s[L] + d_2 a_s^2[L] + \ldots$

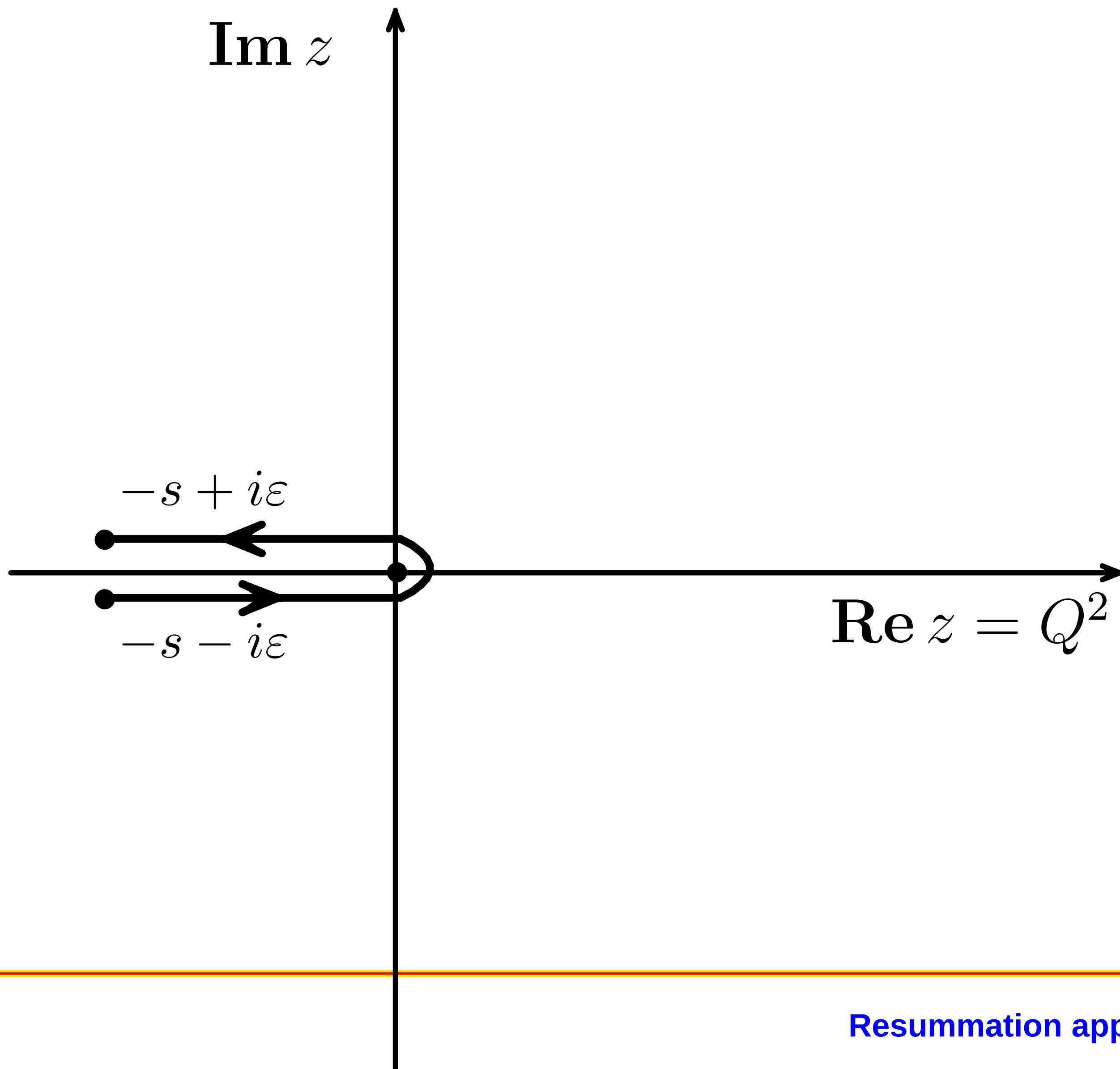
Intro: PT in QCD

- coupling $\alpha_s(\mu^2) = (4\pi/b_0)\, a_s[L]$ with $L = \ln(\mu^2/\Lambda^2)$
- RG equation $\dfrac{d\, a_s[L]}{d\, L} = -a_s^2 - c_1\, a_s^3 - \ldots$
- 1-loop solution generates Landau pole singularity: $a_s[L] = 1/L$
- 2-loop solution generates square-root singularity: $a_s[L] \sim 1/\sqrt{L + c_1 \ln c_1}$
- PT series: $D[L] = 1 + d_1 a_s[L] + d_2 a_s^2[L] + \ldots$
- RG evolution: $B(Q^2) = \left[Z(Q^2)/Z(\mu^2)\right]\, B(\mu^2)$ reduces in 1-loop approximation to

$$Z \sim a^{\nu}[L]\Big|_{\nu\, =\, \nu_0\, \equiv\, \gamma_0/(2b_0)}$$

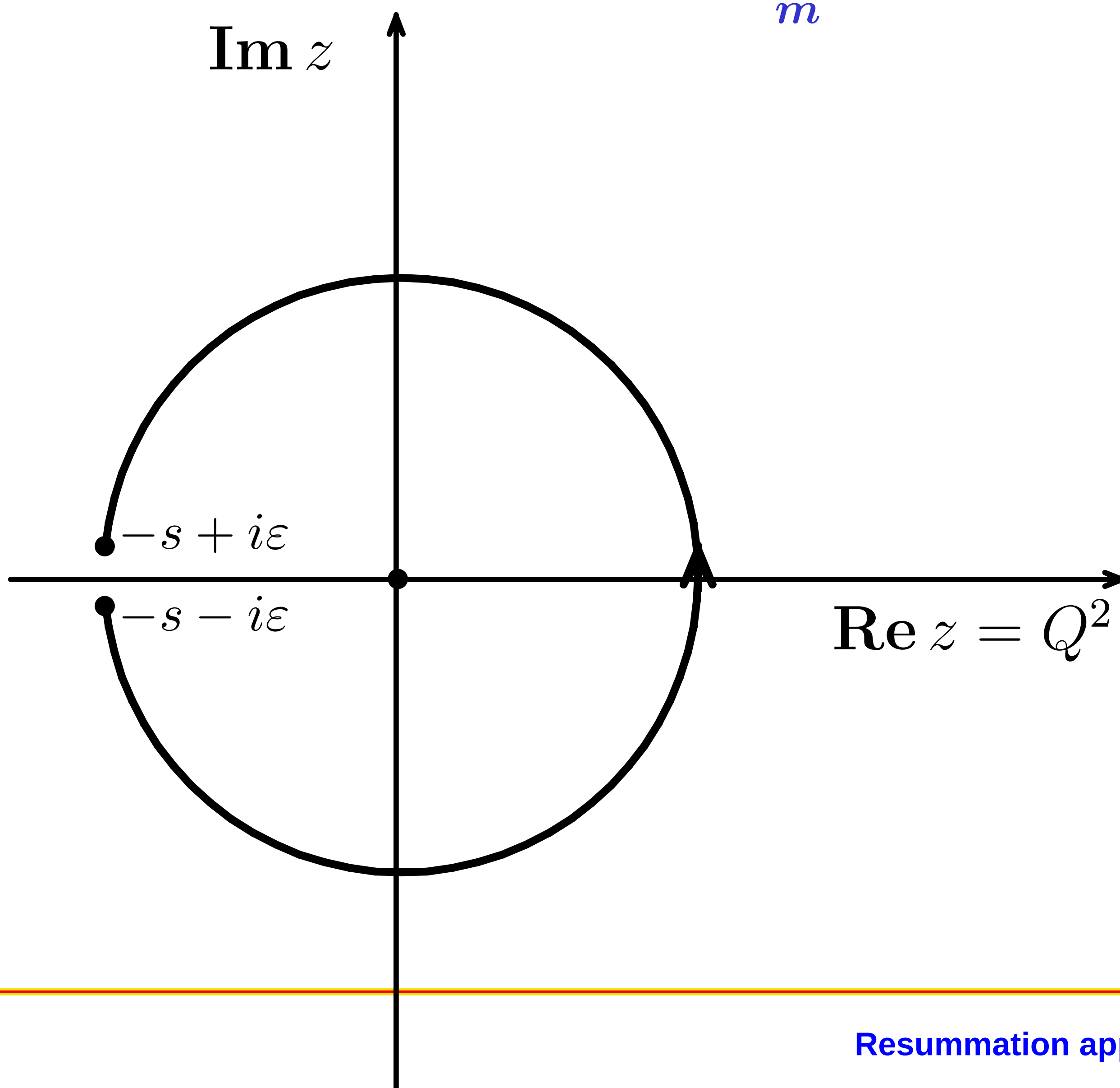
Problem in QCD PT: Minkowski region?

Quantities in Minkowski region $= \oint f(z) D(z) dz$.



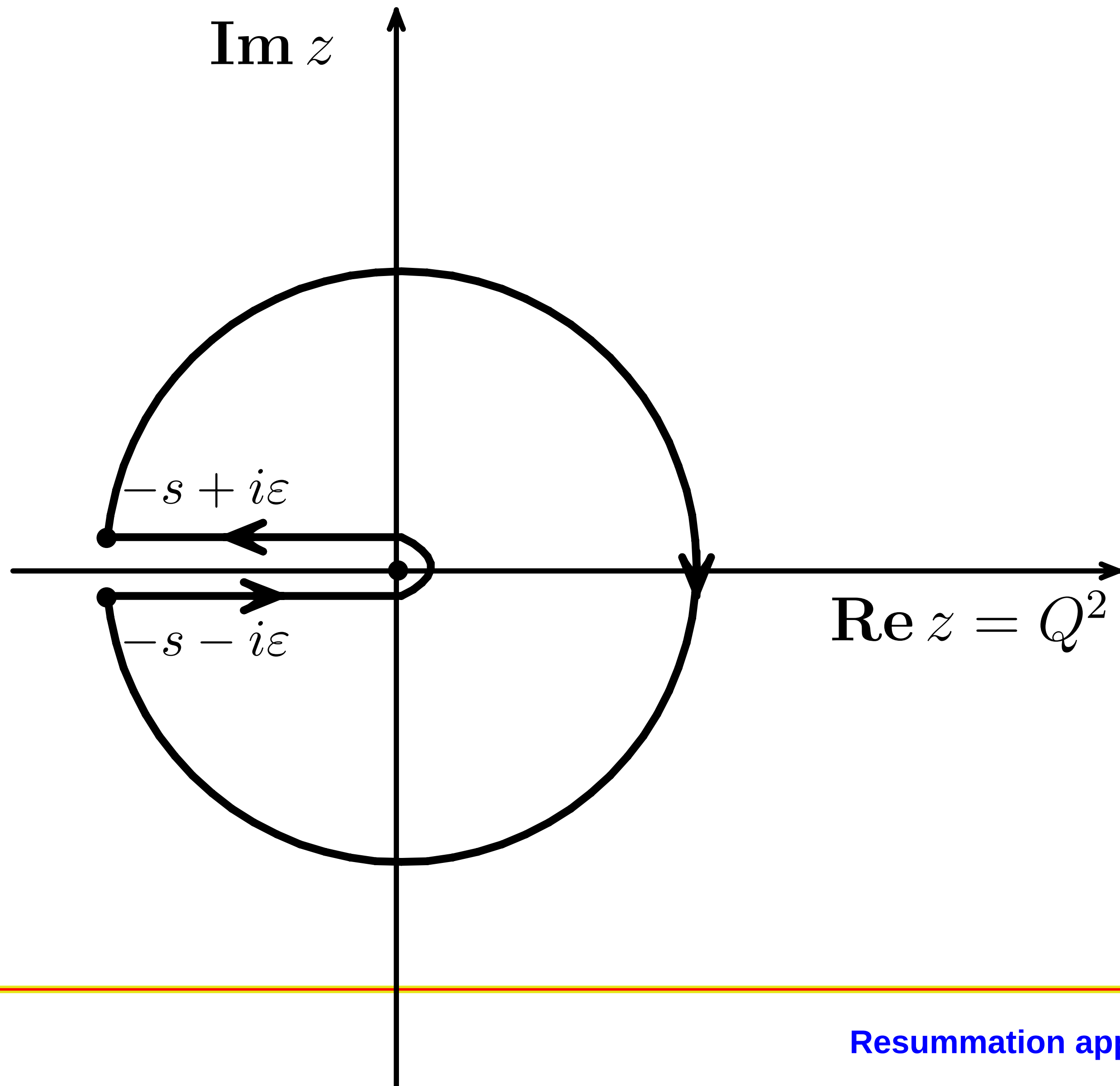
Problem in QCD PT: Minkowski region?

In $\oint f(z)D(z)dz$ one uses $D(z) = \sum_m d_m \alpha_s^m(z)$.



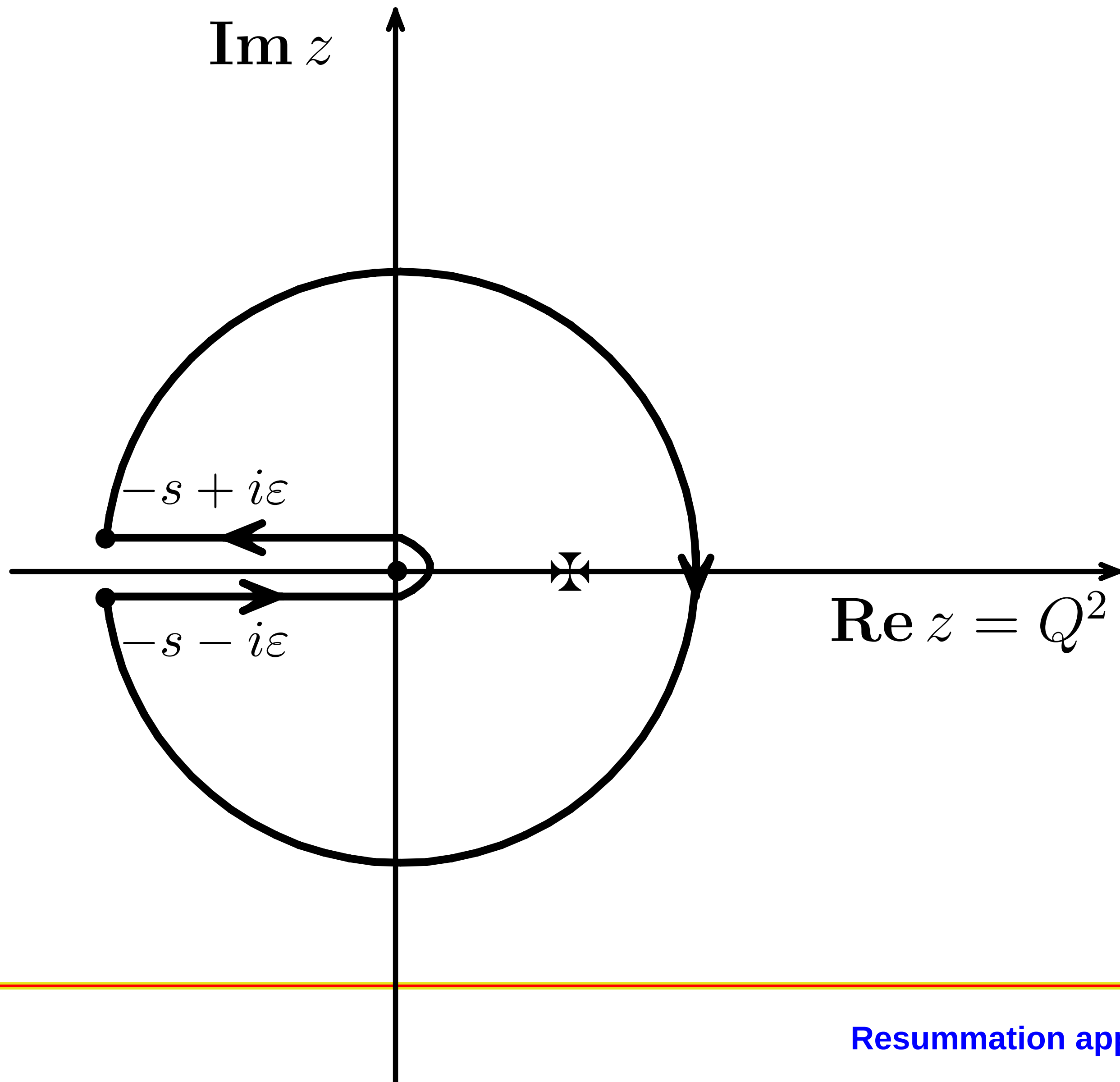
Problem in QCD PT: Minkowski region?

This change of integration contour is legitimate if $D(z)\,f(z)$ is analytic inside



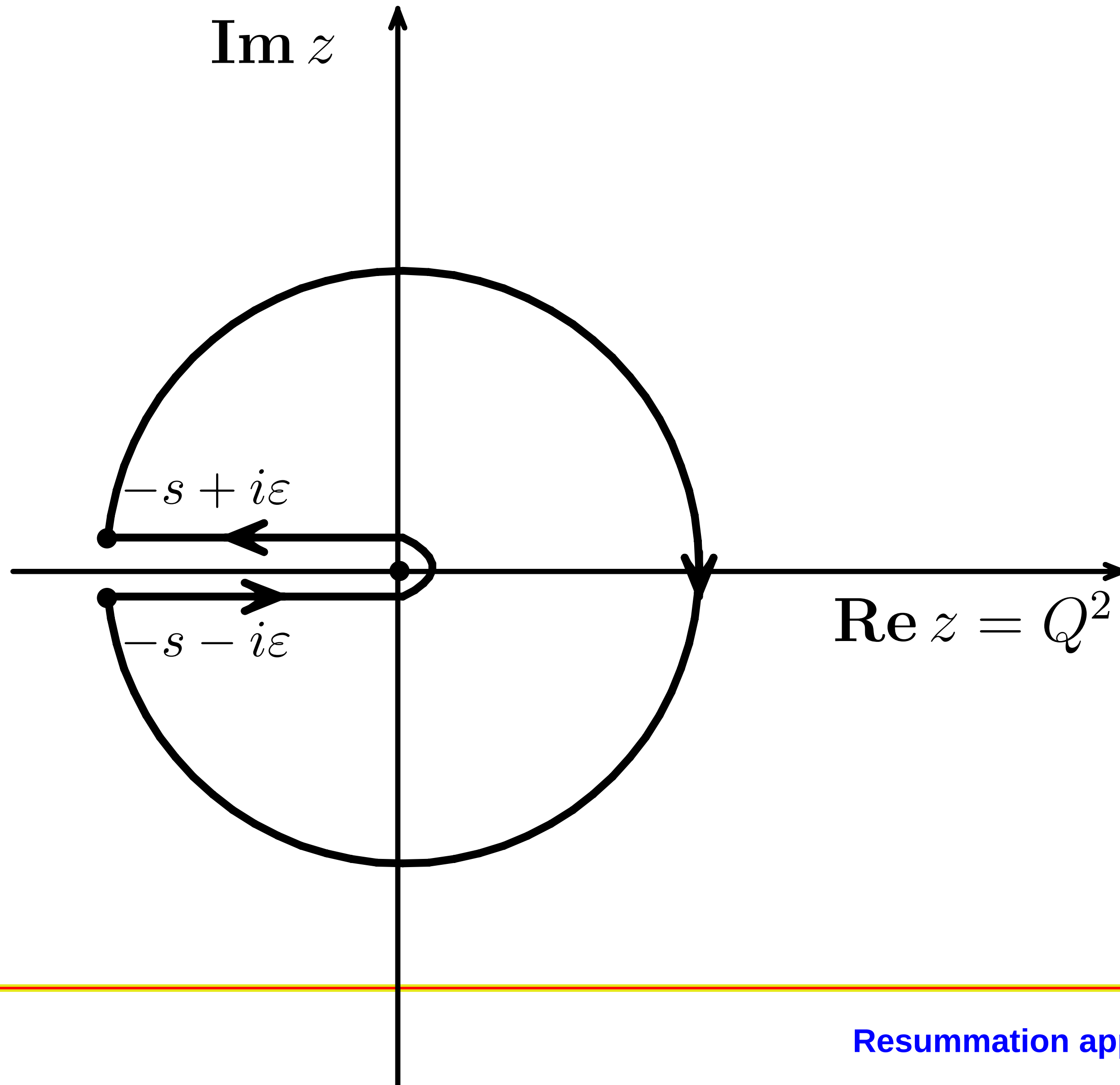
Problem in QCD PT: Minkowski region?

But $\alpha_s(z)$ and hence $D(z)\,f(z)$ have Landau pole singularity just inside!



Problem in QCD PT: Minkowski region?

In **APT** effective couplings $\mathcal{A}_n(z)$ are analytic functions $\Rightarrow$ Problem does not appear! Equivalence to CIPT for $R(s)$.



Basics of ***APT***

- Different effective couplings in **Euclidean** **(S&S)** and **Minkowskian** **(R&K&P)** regions

Basics of ***APT***

- Different effective couplings in **Euclidean** **(S&S)** and **Minkowskian** **(R&K&P)** regions
- Based on **RG** $\oplus$ **Causality**

Basics of APT

- Different effective couplings in **Euclidean** **(S&S)** and **Minkowskian** **(R&K&P)** regions
- Based on **RG** $+$ **Causality**

  $\Downarrow$ UV asymptotics

  $\Downarrow$ Spectrality

Basics of ***APT***

- Different effective couplings in **Euclidean** **(S&S)** and **Minkowskian** **(R&K&P)** regions
- Based on **RG** $\oplus$ **Causality**

  $\Downarrow$ UV asymptotics $\qquad$ $\Downarrow$ Spectrality

- Euclidean: $-q^2 = Q^2, \;\; L = \ln Q^2/\Lambda^2, \;\; \{\mathcal{A}_n(L)\}_{n\in\mathbb{N}}$

Basics of APT

- Different effective couplings in **Euclidean** **(S&S)** and **Minkowskian** **(R&K&P)** regions
- Based on **RG** $\oplus$ **Causality**

  RG $\Downarrow$ UV asymptotics

  Causality $\Downarrow$ Spectrality
- Euclidean: $-q^2 = Q^2,\ \ L = \ln Q^2/\Lambda^2,\ \ \{\mathcal{A}_n(L)\}_{n\in\mathbb{N}}$
- Minkowskian: $q^2 = s,\ \ L_s = \ln s/\Lambda^2,\ \ \{\mathfrak{A}_n(L_s)\}_{n\in\mathbb{N}}$

Basics of ***APT***

- Different effective couplings in **Euclidean (S&S)** and **Minkowskian (R&K&P)** regions

- Based on **RG** $\oplus$ **Causality**

  $\Downarrow$ UV asymptotics $\Downarrow$ Spectrality

- Euclidean: $-q^2 = Q^2, \;\; L = \ln Q^2/\Lambda^2, \;\; \{\mathcal{A}_n(L)\}_{n\in\mathbb{N}}$

- Minkowskian: $q^2 = s, \;\; L_s = \ln s/\Lambda^2, \;\; \{\mathfrak{A}_n(L_s)\}_{n\in\mathbb{N}}$

- PT $\quad \sum_m d_m a_s^m(Q^2) \;\Rightarrow\; \sum_m d_m \mathcal{A}_m(Q^2)$ **APT**

  m is power $\;\Rightarrow\;$ m is **index**

Spectral representation

By **analytization** we mean “Källen–Lehman” representation

$$\left[f(Q^2)\right]_{\text{an}} = \int_0^\infty \frac{\rho_f(\sigma)}{\sigma + Q^2 - i\epsilon}\, d\sigma$$

with spectral density $\rho_f(\sigma) = \textbf{Im}\left[f(-\sigma)\right]/\pi$.

Spectral representation

By **analytization** we mean “Källen–Lehman” representation

$$\left[f(Q^2)\right]_{\text{an}} = \int_0^\infty \frac{\rho_f(\sigma)}{\sigma + Q^2 - i\epsilon}\, d\sigma$$

Then

$$\rho(\sigma) = \frac{1}{L_\sigma^2 + \pi^2}$$

$$\mathcal{A}_1[L] = \int_0^\infty \frac{\rho(\sigma)}{\sigma + Q^2}\, d\sigma = \frac{1}{L} - \frac{1}{e^L - 1}$$

Spectral representation

By **analytization** we mean "Källen–Lehman" representation

$$\left[f(Q^2)\right]_{\text{an}} = \int_0^\infty \frac{\rho_f(\sigma)}{\sigma + Q^2 - i\epsilon}\, d\sigma$$

Then (note here **pole remover**):

$$\rho(\sigma) = \frac{1}{L_\sigma^2 + \pi^2}$$

$$\mathcal{A}_1[L] = \int_0^\infty \frac{\rho(\sigma)}{\sigma + Q^2}\, d\sigma = \frac{1}{L} - \frac{1}{e^L - 1}$$

Spectral representation

By **analytization** we mean "Källen–Lehman" representation

$$\left[f(Q^2)\right]_{\text{an}} = \int_0^\infty \frac{\rho_f(\sigma)}{\sigma + Q^2 - i\epsilon}\, d\sigma$$

Then (note here **pole remover**):

$$\begin{aligned}
\rho(\sigma) &= \frac{1}{L_\sigma^2 + \pi^2} \\
\mathcal{A}_1[L] &= \int_0^\infty \frac{\rho(\sigma)}{\sigma + Q^2}\, d\sigma = \frac{1}{L} - \frac{1}{e^L - 1} \\
\mathfrak{A}_1[L_s] &= \int_s^\infty \frac{\rho(\sigma)}{\sigma}\, d\sigma \quad = \frac{1}{\pi} \arccos \frac{L_s}{\sqrt{\pi^2 + L_s^2}}
\end{aligned}$$

Spectral representation

By analytization we mean “Källen–Lehman” representation

$$\left[f(Q^2)\right]_{\text{an}} = \int_0^\infty \frac{\rho_f(\sigma)}{\sigma + Q^2 - i\epsilon}\, d\sigma$$

with spectral density $\rho_f(\sigma) = \textbf{Im}\left[f(-\sigma)\right]/\pi$. Then:

$$\mathcal{A}_n[L] = \int_0^\infty \frac{\rho_n(\sigma)}{\sigma + Q^2}\, d\sigma = \frac{1}{(n-1)!}\left(-\frac{d}{dL}\right)^{n-1} \mathcal{A}_1[L]$$

Spectral representation

By analytization we mean "Källen–Lehman" representation

$$\left[f(Q^2)\right]_{\text{an}} = \int_0^\infty \frac{\rho_f(\sigma)}{\sigma + Q^2 - i\epsilon}\, d\sigma$$

with spectral density $\rho_f(\sigma) = \textbf{Im}\left[f(-\sigma)\right]/\pi$. Then:

$$\mathcal{A}_n[L] = \int_0^\infty \frac{\rho_n(\sigma)}{\sigma + Q^2}\, d\sigma = \frac{1}{(n-1)!}\left(-\frac{d}{dL}\right)^{n-1} \mathcal{A}_1[L]$$

$$\mathfrak{A}_n[L_s] = \int_s^\infty \frac{\rho_n(\sigma)}{\sigma}\, d\sigma = \frac{1}{(n-1)!}\left(-\frac{d}{dL_s}\right)^{n-1} \mathfrak{A}_1[L_s]$$

Spectral representation

By analytization we mean "Källen–Lehman" representation

$$\left[f(Q^2)\right]_{\text{an}} = \int_0^\infty \frac{\rho_f(\sigma)}{\sigma + Q^2 - i\epsilon}\, d\sigma$$

with spectral density $\rho_f(\sigma) = \textbf{Im}\left[f(-\sigma)\right]/\pi$. Then:

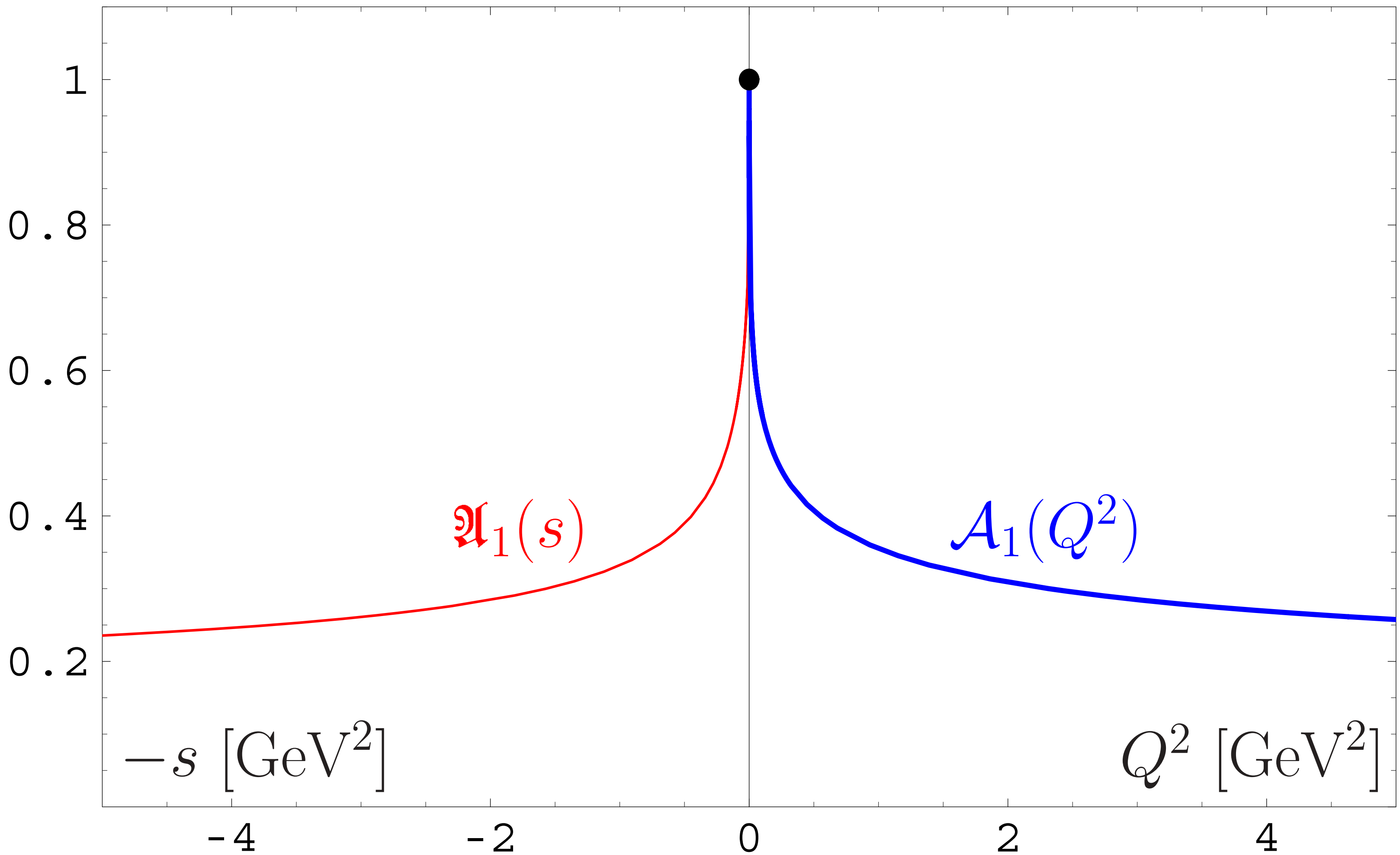
$$\mathcal{A}_n[L] = \int_0^\infty \frac{\rho_n(\sigma)}{\sigma + Q^2}\, d\sigma = \frac{1}{(n-1)!}\left(-\frac{d}{dL}\right)^{n-1} \mathcal{A}_1[L]$$

$$\mathfrak{A}_n[L_s] = \int_s^\infty \frac{\rho_n(\sigma)}{\sigma}\, d\sigma = \frac{1}{(n-1)!}\left(-\frac{d}{dL_s}\right)^{n-1} \mathfrak{A}_1[L_s]$$

$$a_s^n[L] = \frac{1}{(n-1)!}\left(-\frac{d}{dL}\right)^{n-1} a_s[L]$$

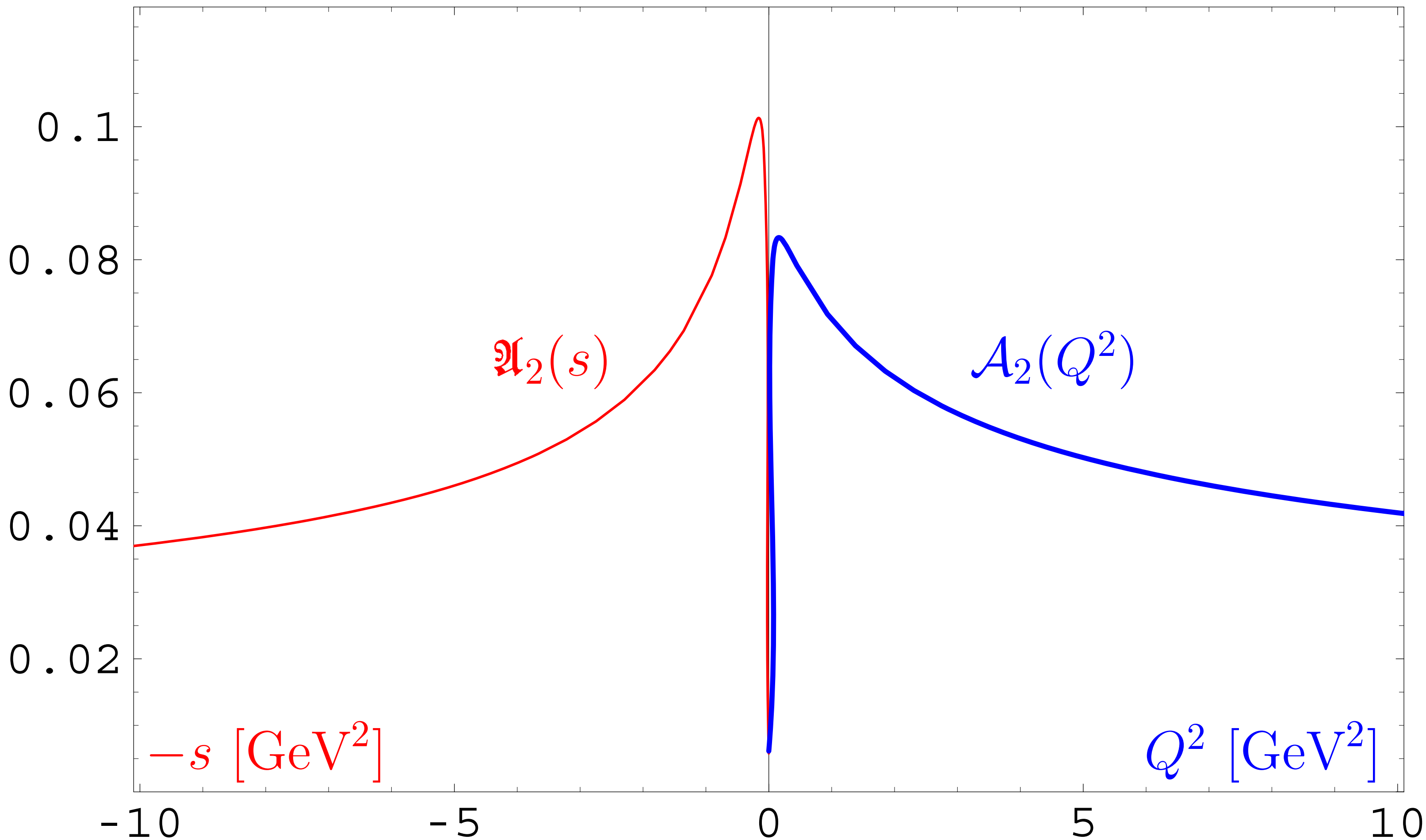
APT graphics: Distorting mirror

First, couplings: $\mathfrak{A}_1(s)$ and $\mathcal{A}_1(Q^2)$



APT graphics: Distorting mirror

Second, square-images: $\mathfrak{A}_2(s)$ and $\mathcal{A}_2(Q^2)$



Problems of APT.
Resolution:
Fractional APT

Problems of APT

Open Questions

- “Analytization” of multi-scale amplitudes beyond LO of pQCD: additional logs depending on scale that serves as **factorization** or **renormalization** scale **[Karanikas&Stefanis – PLB 504 (2001) 225]**

Problems of APT

Open Questions

- “Analytization” of multi-scale amplitudes beyond LO of pQCD: additional logs depending on scale that serves as **factorization** or **renormalization** scale **[Karanikas&Stefanis – PLB 504 (2001) 225]**
- Evolution induces some non-integer, **fractional**, powers of coupling constant

Problems of APT

Open Questions

- “Analytization” of multi-scale amplitudes beyond LO of pQCD: additional logs depending on scale that serves as **factorization** or **renormalization** scale **[Karanikas&Stefanis – PLB 504 (2001) 225]**
- Evolution induces some non-integer, **fractional**, powers of coupling constant
- Resummation of gluonic corrections, giving rise to Sudakov factors, under “Analytization” difficult task **[Stefanis, Schroers, Kim – PLB 449 (1999) 299; EPJC 18 (2000) 137]**

Problems of APT

In standard QCD PT we have not only power series $F[L] = \sum_m f_m \, a_s^m[L]$, but also:

Problems of APT

In standard QCD PT we have not only power series $F[L] = \sum_m f_m \, a_s^m[L]$, but also:

- RG-improvment to account for higher-orders $\longrightarrow$

$$Z[L] = \textbf{exp}\left\{\int^{a_s[L]} \frac{\gamma(a)}{\beta(a)}\, da\right\} \xrightarrow{\textbf{1-loop}} \left[a_s[L]\right]^{\gamma_0/(2\beta_0)}$$

Problems of APT

In standard QCD PT we have not only power series $F[L] = \sum_m f_m\, a_s^m[L]$, but also:

- RG-improvment to account for higher-orders $\longrightarrow$

$$Z[L] = \textbf{exp}\left\{\int^{a_s[L]} \frac{\gamma(a)}{\beta(a)}\, da\right\} \xrightarrow{\textbf{1-loop}} [a_s[L]]^{\gamma_0/(2\beta_0)}$$

- Factorization $\longrightarrow$ $[a_s[L]]^n\, L^m$

Constructing one-loop FAPT

In one-loop **APT** we have a very nice recursive relation

$$\mathcal{A}_n[L] = \frac{1}{(n-1)!}\left(-\frac{d}{dL}\right)^{n-1}\mathcal{A}_1[L]$$

Constructing one-loop FAPT

In one-loop **APT** we have a very nice recursive relation

$$\mathcal{A}_n[L] = \frac{1}{(n-1)!}\left(-\frac{d}{dL}\right)^{n-1}\mathcal{A}_1[L]$$

and the same in Minkowski domain

$$\mathfrak{A}_n[L] = \frac{1}{(n-1)!}\left(-\frac{d}{dL}\right)^{n-1}\mathfrak{A}_1[L]\,.$$

Constructing one-loop FAPT

In one-loop **APT** we have a very nice recursive relation

$$\mathcal{A}_n[L] = \frac{1}{(n-1)!}\left(-\frac{d}{dL}\right)^{n-1}\mathcal{A}_1[L]$$

and the same in Minkowski domain

$$\mathfrak{A}_n[L] = \frac{1}{(n-1)!}\left(-\frac{d}{dL}\right)^{n-1}\mathfrak{A}_1[L]\,.$$

We can use it to construct **FAPT**.

FAPT(E): Properties of $\mathcal{A}_\nu[L]$

First, Euclidean coupling ($L = L(Q^2)$):

$$\mathcal{A}_\nu[L] = \frac{1}{L^\nu} - \frac{F(e^{-L}, 1-\nu)}{\Gamma(\nu)}$$

Here $F(z,\nu)$ is reduced **Lerch** transcendent. function. It is analytic function in ν.

FAPT(E): Properties of $\mathcal{A}_\nu[L]$

First, Euclidean coupling ($L = L(Q^2)$):

$$\mathcal{A}_\nu[L] = \frac{1}{L^\nu} - \frac{F(e^{-L}, 1-\nu)}{\Gamma(\nu)}$$

Here $F(z,\nu)$ is reduced **Lerch** transcendent. function. It is analytic function in ν. Properties:

- $\mathcal{A}_0[L] = 1$;
- $\mathcal{A}_{-m}[L] = L^m$ for $m \in \mathbb{N}$;
- $\mathcal{A}_m[L] = (-1)^m \mathcal{A}_m[-L]$ for $m \geq 2\,,\ m \in \mathbb{N}$;
- $\mathcal{A}_m[\pm\infty] = 0$ for $m \geq 2\,,\ m \in \mathbb{N}$;

FAPT(M): Properties of $\mathfrak{A}_\nu[L]$

Now, Minkowskian coupling ($L = L(s)$):

$$\mathfrak{A}_\nu[L] = \frac{\sin\left[(\nu-1)\arccos\left(L/\sqrt{\pi^2+L^2}\right)\right]}{\pi(\nu-1)\left(\pi^2+L^2\right)^{(\nu-1)/2}}$$

Here we need only elementary functions.

FAPT(M): Properties of $\mathfrak{A}_\nu[L]$

Now, Minkowskian coupling ($L = L(s)$):

$$\mathfrak{A}_\nu[L] = \frac{\sin\left[(\nu-1)\arccos\left(L/\sqrt{\pi^2+L^2}\right)\right]}{\pi(\nu-1)\left(\pi^2+L^2\right)^{(\nu-1)/2}}$$

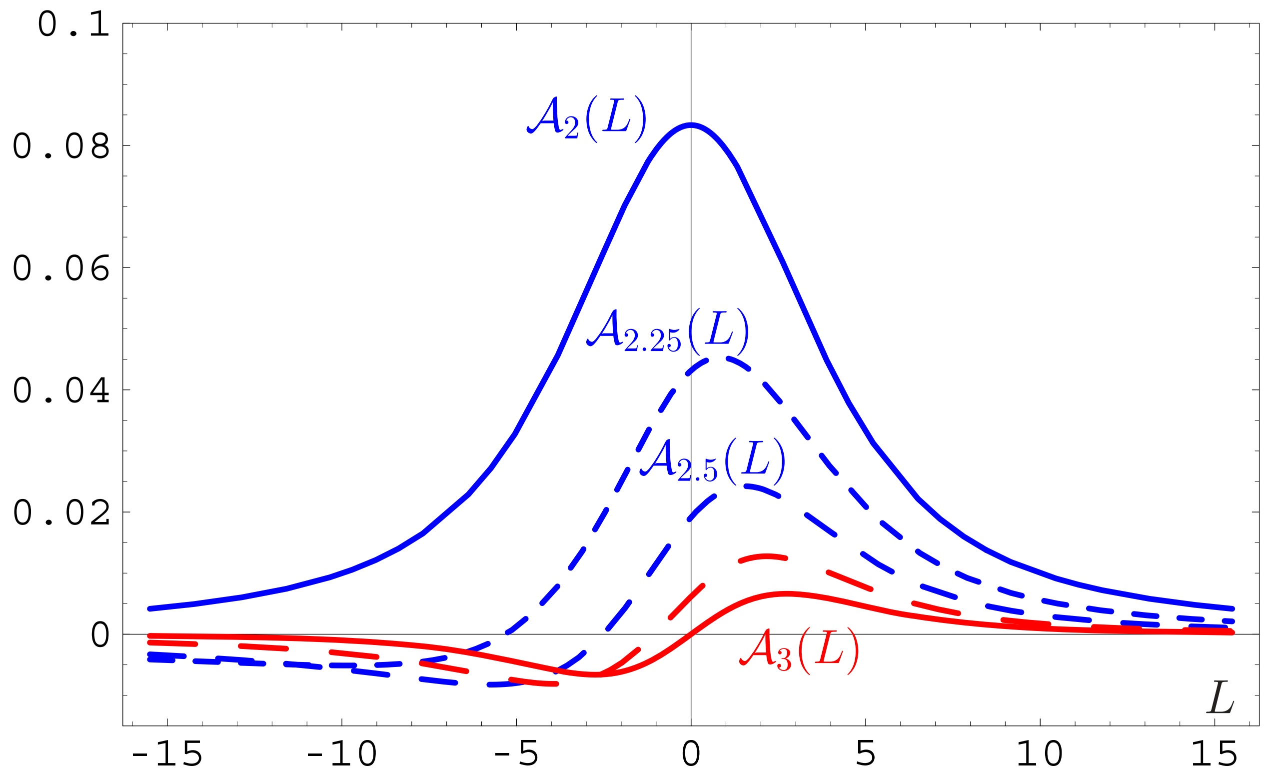
Here we need only elementary functions. Properties:

- $\mathfrak{A}_0[L] = 1$;
- $\mathfrak{A}_{-1}[L] = L$;
- $\mathfrak{A}_{-2}[L] = L^2 - \dfrac{\pi^2}{3}, \quad \mathfrak{A}_{-3}[L] = L\left(L^2-\pi^2\right), \ \ldots$;
- $\mathfrak{A}_m[L] = (-1)^m \mathfrak{A}_m[-L]$ for $m \geq 2, \ m \in \mathbb{N}$;
- $\mathfrak{A}_m[\pm\infty] = 0$ for $m \geq 2, \ m \in \mathbb{N}$

FAPT(E): Graphics of $\mathcal{A}_\nu[L]$ vs. L

$$\mathcal{A}_\nu[L] = \frac{1}{L^\nu} - \frac{F(e^{-L}, 1-\nu)}{\Gamma(\nu)}$$

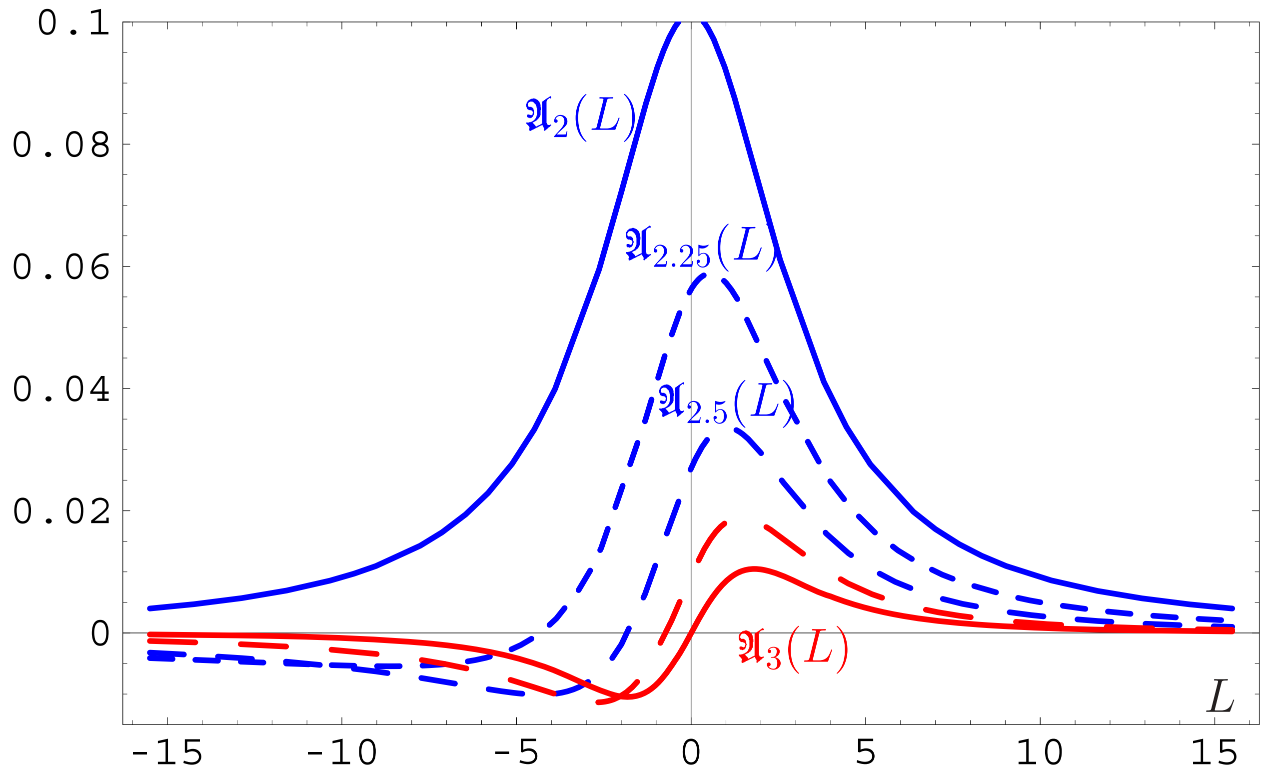
Graphics for fractional $\nu \in [2, 3]$:



FAPT(M): Graphics of $\mathfrak{A}_\nu[L]$ vs. L

$$\mathfrak{A}_\nu[L] = \frac{\sin\left[(\nu-1)\arccos\left(L/\sqrt{\pi^2+L^2}\right)\right]}{\pi(\nu-1)\left(\pi^2+L^2\right)^{(\nu-1)/2}}$$

Compare with graphics in Minkowskian region :



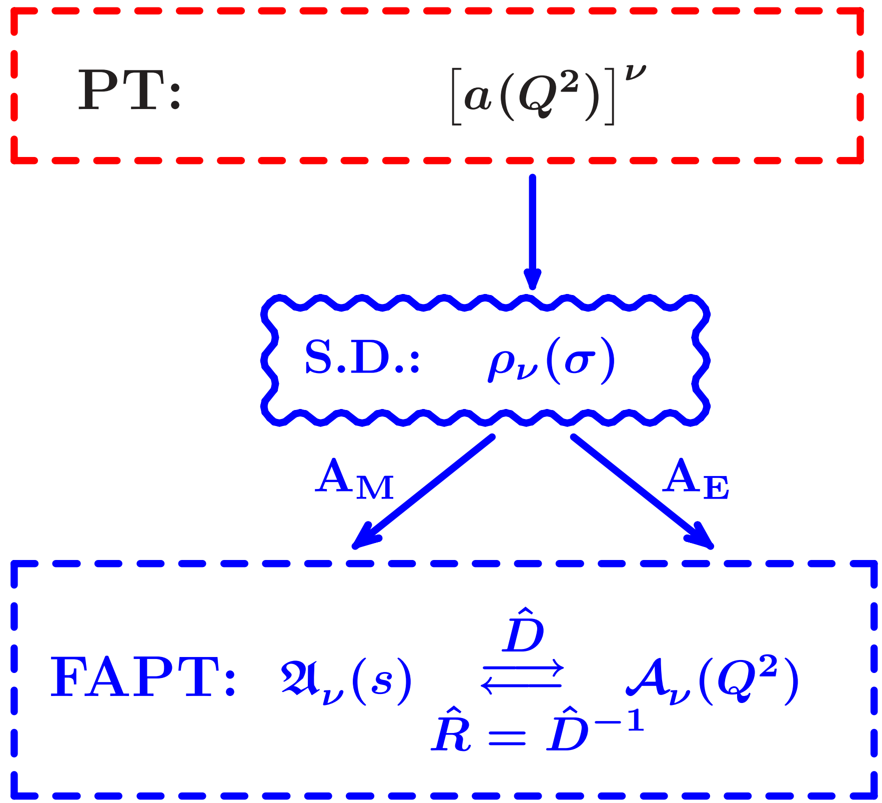
Comparison of PT, APT, and FAPT

Theory	PT	APT	FAPT
Set	$\left\{a^{\nu}\right\}_{\nu\in\mathbb{R}}$	$\left\{\mathcal{A}_m, \mathfrak{A}_m\right\}_{m\in\mathbb{N}}$	$\left\{\mathcal{A}_{\nu}, \mathfrak{A}_{\nu}\right\}_{\nu\in\mathbb{R}}$
Series	$\sum_m f_m\, a^m$	$\sum_m f_m\, \mathcal{A}_m$	$\sum_m f_m\, \mathcal{A}_m$
Inv. powers	$(a[L])^{-m}$	—	$\mathcal{A}_{-m}[L] = L^m$
Products	$a^{\mu} a^{\nu} = a^{\mu+\nu}$	—	—
Index deriv.	$a^{\nu} \ln^k a$	—	$\mathcal{D}^k \mathcal{A}_{\nu}$
Logarithms	$a^{\nu} L^k$	—	$\mathcal{A}_{\nu-k}$

Development of FAPT:

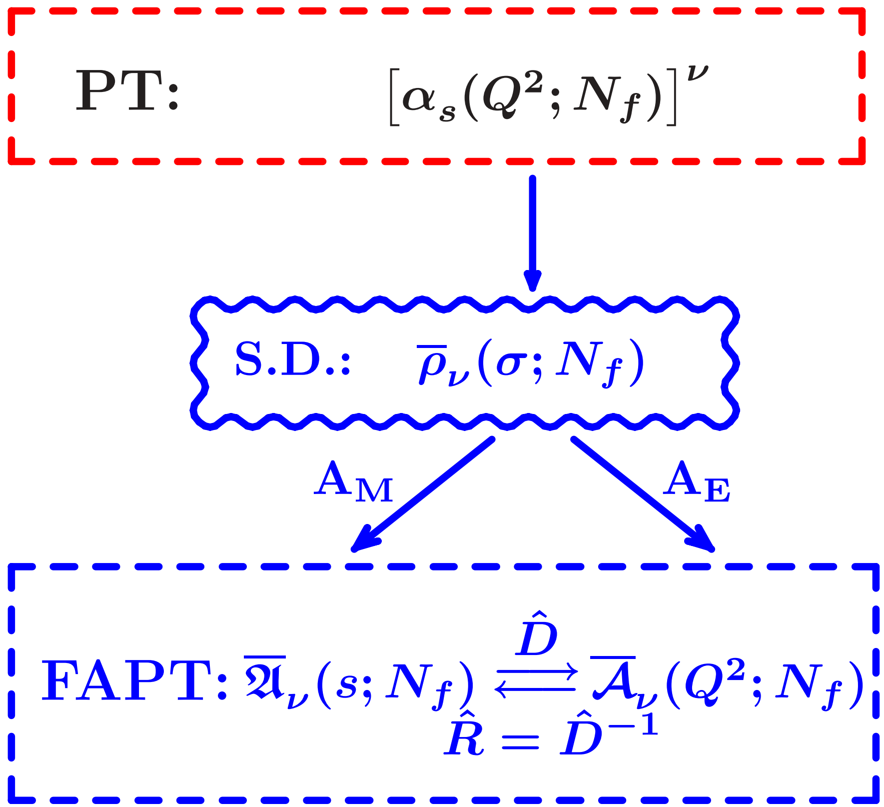
Heavy-Quark Thresholds

Conceptual scheme of ***FAPT***



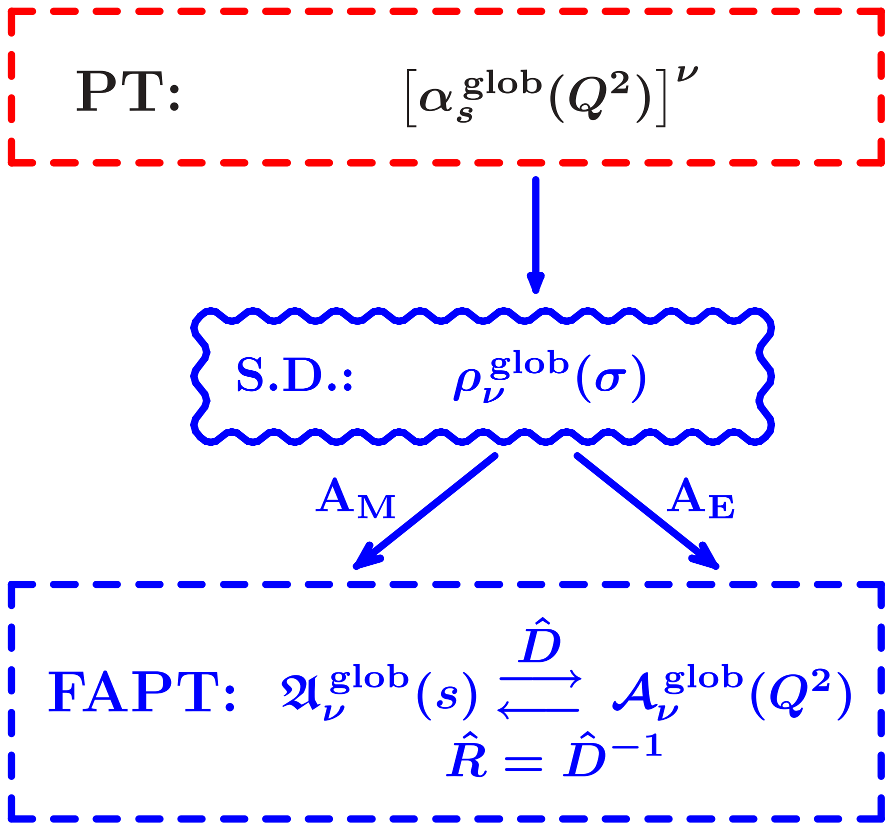
Here N_f is fixed and factorized out.

Conceptual scheme of ***FAPT***



Here N_f is fixed, but not factorized out.

*Conceptual scheme of **FAPT***



Here we see how “analytization” takes into account N_f-dependence.

Global FAPT: Single threshold case

- Consider for simplicity only one threshold at $s = m_c^2$ with transition $N_f = 3 \to N_f = 4$.
- Denote: $L_4 = \ln\left(m_c^2/\Lambda_3^2\right)$ and $\lambda_4 = \ln\left(\Lambda_3^2/\Lambda_4^2\right)$.

Global FAPT: Single threshold case

- Consider for simplicity only one threshold at $s = m_c^2$ with transition $N_f = 3 \to N_f = 4$.
- Denote: $L_4 = \ln(m_c^2/\Lambda_3^2)$ and $\lambda_4 = \ln(\Lambda_3^2/\Lambda_4^2)$.

Then:

$$\mathfrak{A}_\nu^{\text{glob}}[L] = \theta(L < L_4)\left[\overline{\mathfrak{A}}_\nu[L;3] - \overline{\mathfrak{A}}_\nu[L_4;3] + \overline{\mathfrak{A}}_\nu[L_4+\lambda_4;4]\right]$$
$$+\,\theta(L \geq L_4)\,\overline{\mathfrak{A}}_\nu[L+\lambda_4;4]$$

Global FAPT: Single threshold case

- Consider for simplicity only one threshold at $s = m_c^2$ with transition $N_f = 3 \to N_f = 4$.
- Denote: $L_4 = \ln(m_c^2/\Lambda_3^2)$ and $\lambda_4 = \ln(\Lambda_3^2/\Lambda_4^2)$.

Then:

$$\mathfrak{A}_\nu^{\text{glob}}[L] = \theta(L < L_4)\left[\overline{\mathfrak{A}}_\nu[L;3] - \overline{\mathfrak{A}}_\nu[L_4;3] + \overline{\mathfrak{A}}_\nu[L_4+\lambda_4;4]\right] + \theta(L \geq L_4)\,\overline{\mathfrak{A}}_\nu[L+\lambda_4;4]$$

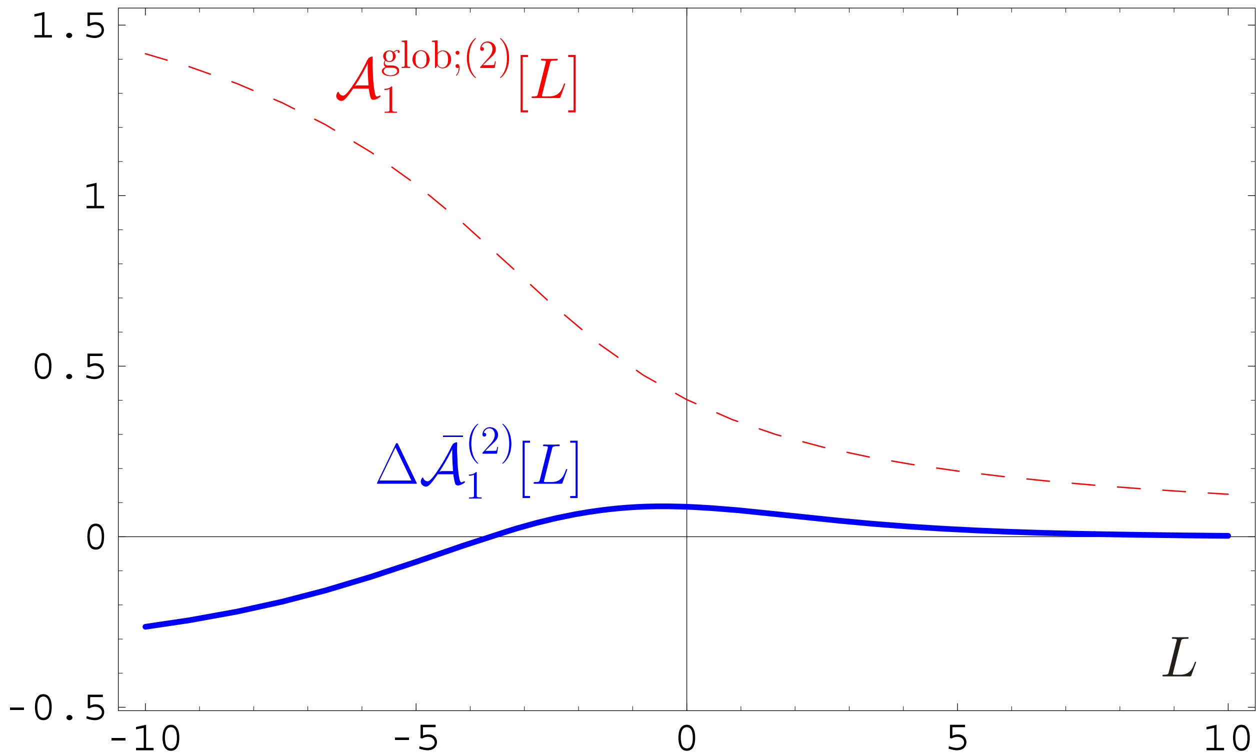
and

$$\mathcal{A}_\nu^{\text{glob}}[L] = \overline{\mathcal{A}}_\nu[L+\lambda_4;4] + \int_{-\infty}^{L_4} \frac{\bar{\rho}_\nu[L_\sigma;3] - \bar{\rho}_\nu[L_\sigma+\lambda_4;4]}{1+e^{L-L_\sigma}}\, dL_\sigma$$

Graphical comparison: Fixed-N_f—***Global***

$$\mathcal{A}_{\nu}^{\textbf{glob}}[L] = \overline{\mathcal{A}}_{\nu}[L + \lambda_4; 4] + \Delta\overline{\mathcal{A}}_{\nu}[L]\,;$$

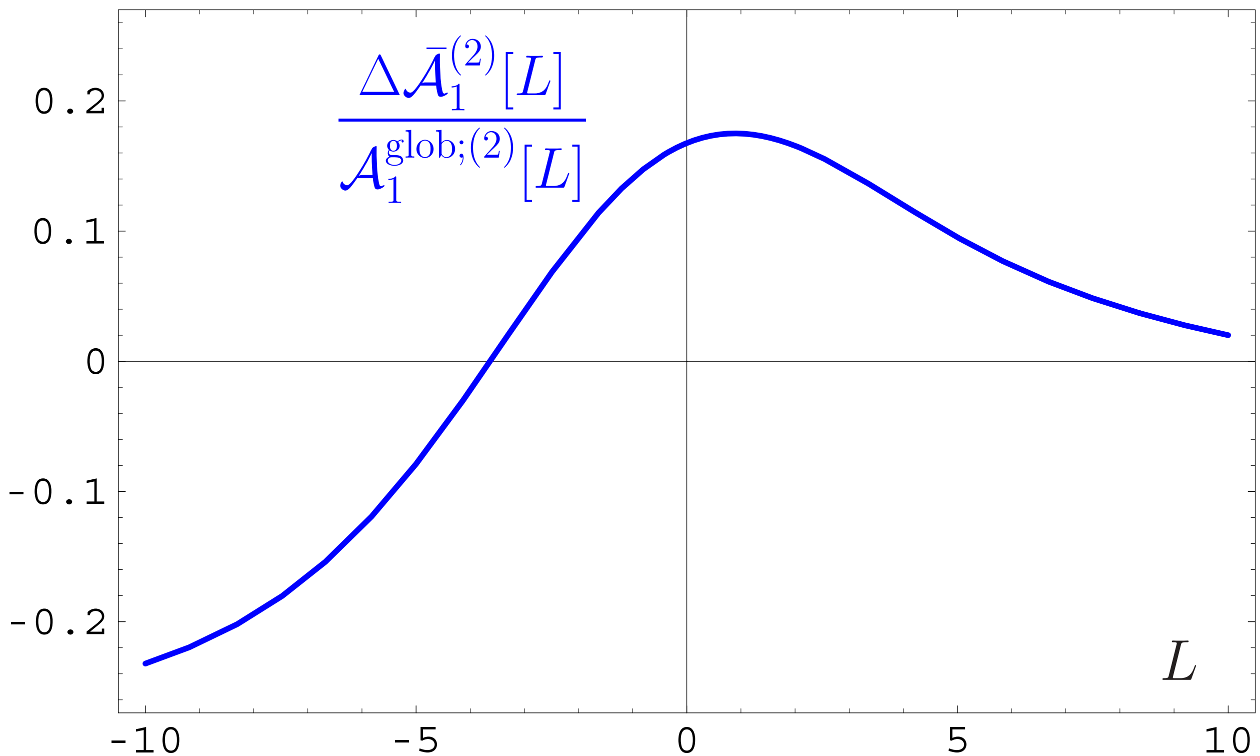
$\Delta\overline{\mathcal{A}}_1[L]$ — **solid**, $\mathcal{A}_1^{\textbf{glob}}[L]$ — **dashed**:



*Graphical comparison: Fixed-N_f—**Global***

$$\mathcal{A}_\nu^{\text{glob}}[L] = \overline{\mathcal{A}}_\nu[L + \lambda_4; 4] + \Delta\overline{\mathcal{A}}_\nu[L]\,;$$

$\Delta\overline{\mathcal{A}}_1[L]/\mathcal{A}_1^{\text{glob}}[L]$ — **solid**:



Resummation
in
one-loop APT and FAPT

Resummation in one-loop APT

Consider series $$\mathcal{D}[L] = d_0 + \sum_{n=1}^{\infty} d_n \, \mathcal{A}_n[L]$$

Resummation in one-loop APT

Consider series $\mathcal{D}[L] = d_0 + \sum_{n=1}^{\infty} d_n \, \mathcal{A}_n[L]$

Let exist the generating function $P(t)$ for coefficients:

$$d_n = d_1 \int_0^{\infty} P(t) \, t^{n-1} dt \quad \text{with} \quad \int_0^{\infty} P(t) \, dt = 1 \, .$$

We define a shorthand notation

$$\langle\langle f(t) \rangle\rangle_{P(t)} \equiv \int_0^{\infty} f(t) \, P(t) \, dt \, .$$

Then coefficients $d_n = d_1 \, \langle\langle t^{n-1} \rangle\rangle_{P(t)}$.

Resummation in one-loop APT

Consider series $\mathcal{D}[L] = d_0 + \sum_{n=1}^{\infty} d_n \, \mathcal{A}_n[L]$

with coefficients $d_n = d_1 \, \langle\langle t^{n-1} \rangle\rangle_{P(t)}$.

We have one-loop recurrence relation:

$$\mathcal{A}_{n+1}[L] = \frac{1}{\Gamma(n+1)} \left(-\frac{d}{dL}\right)^n \mathcal{A}_1[L]\,.$$

Resummation in one-loop APT

Consider series $\mathcal{D}[L] = d_0 + \sum_{n=1}^{\infty} d_n \, \mathcal{A}_n[L]$

with coefficients $d_n = d_1 \, \langle\langle t^{n-1} \rangle\rangle_{P(t)}$.
We have one-loop recurrence relation:

$$\mathcal{A}_{n+1}[L] = \frac{1}{\Gamma(n+1)} \left(- \frac{d}{dL} \right)^n \mathcal{A}_1[L] \, .$$

Result:

$$\mathcal{D}[L] = d_0 + d_1 \, \langle\langle \mathcal{A}_1[L-t] \rangle\rangle_{P(t)}$$

Resummation in one-loop APT

Consider series $\mathcal{D}[L] = d_0 + \sum_{n=1}^{\infty} d_n \, \mathcal{A}_n[L]$

with coefficients $d_n = d_1 \, \langle\langle t^{n-1} \rangle\rangle_{P(t)}$.

We have one-loop recurrence relation:

$$\mathcal{A}_{n+1}[L] = \frac{1}{\Gamma(n+1)} \left(-\frac{d}{dL}\right)^n \mathcal{A}_1[L] \, .$$

Result:

$$\mathcal{D}[L] = d_0 + d_1 \, \langle\langle \mathcal{A}_1[L-t] \rangle\rangle_{P(t)}$$

and for Minkowski region:

$$\mathcal{R}[L] = d_0 + d_1 \, \langle\langle \mathfrak{A}_1[L-t] \rangle\rangle_{P(t)}$$

Resummation in Global Minkowskian APT

Consider series $\mathcal{R}[L] = d_0 + \sum_{n=1}^{\infty} d_n\, \mathfrak{A}_n^{\textbf{glob}}[L]$

with coefficients $d_n = d_1 \, \langle\langle t^{n-1} \rangle\rangle_{P(t)}$.

Result:

$$\mathcal{R}[L] = d_0 \; + \; d_1 \langle\langle \theta\,(L\!<\!L_4)\Big[\Delta_4\overline{\mathfrak{A}}_1[t] + \overline{\mathfrak{A}}_1\Big[L - \frac{t}{\beta_3};3\Big]\Big]\rangle\rangle_{P(t)}$$

$$+ \; d_1 \langle\langle \theta\,(L\!\geq\!L_4)\overline{\mathfrak{A}}_1\Big[L + \lambda_4 - \frac{t}{\beta_4};4\Big]\rangle\rangle_{P(t)}\,.$$

where

$$\Delta_4\overline{\mathfrak{A}}_1[t] = \overline{\mathfrak{A}}_1\Big[L_4 + \lambda_4 - \frac{t}{\beta_4};4\Big] - \overline{\mathfrak{A}}_1\Big[L_3 - \frac{t}{\beta_3};3\Big]\,.$$

Resummation in Global Euclidean APT

In Euclidean domain the result is more complicated:

$$\mathcal{D}[L] = d_0 + d_1 \langle\langle \int\limits_{-\infty}^{L_4} \frac{\overline{\rho}_1\,[L_\sigma;3]\;dL_\sigma}{1+e^{L-L_\sigma-t/\beta_3}} \rangle\rangle_{P(t)}$$

$$+ \langle\langle \Delta_4[L,t] \rangle\rangle_{P(t)} + d_1 \langle\langle \int\limits_{L_4}^{\infty} \frac{\overline{\rho}_1\,[L_\sigma+\lambda_4;4]\;dL_\sigma}{1+e^{L-L_\sigma-t/\beta_4}} \rangle\rangle_{P(t)}\,.$$

where

$$\Delta_4[L,t] = \int\limits_0^1 \frac{\overline{\rho}_1\,[L_4+\lambda_4-tx/\beta_4;4]\;t}{\beta_4\,\left[1+e^{L-L_4-t\bar{x}/\beta_4}\right]}\,dx$$

$$-\int\limits_0^1 \frac{\overline{\rho}_1\,[L_3-tx/\beta_3;3]\;t}{\beta_3\,\left[1+e^{L-L_4-t\bar{x}/\beta_3}\right]}\,dx.$$

Resummation in FAPT

Consider seria $\mathcal{R}_\nu[L] = d_0\,\mathfrak{A}_\nu[L] + \sum_{n=1}^{\infty} d_n\,\mathfrak{A}_{n+\nu}[L]$

and $\mathcal{D}_\nu[L] = d_0\,\mathcal{A}_\nu[L] + \sum_{n=1}^{\infty} d_n\,\mathcal{A}_{n+\nu}[L]$

with coefficients $d_n = d_1\,\langle\langle t^{n-1}\rangle\rangle_{P(t)}$.

Result:

$$\begin{aligned}\mathcal{R}_\nu[L] &= d_0\,\mathfrak{A}_\nu[L] + d_1\,\langle\langle\mathfrak{A}_{1+\nu}[L-t]\rangle\rangle_{P_\nu(t)}\,;\\ \mathcal{D}_\nu[L] &= d_0\,\mathcal{A}_\nu[L] + d_1\,\langle\langle\mathcal{A}_{1+\nu}[L-t]\rangle\rangle_{P_\nu(t)}\,.\end{aligned}$$

where $P_\nu(t) = \int_0^1 P\left(\frac{t}{1-z}\right)\nu\,z^{\nu-1}\frac{dz}{1-z}\,.$

Resummation in Global Minkowskian FAPT

Consider series $\mathcal{R}_\nu[L] = d_0\,\mathfrak{A}_\nu^{\textbf{glob}} + \sum_{n=1}^{\infty} d_n\,\mathfrak{A}_{n+\nu}^{\textbf{glob}}[L]$

with coefficients $d_n = d_1\,\langle\langle t^{n-1}\rangle\rangle_{P(t)}$.

Resummation in Global Minkowskian FAPT

Consider series $\mathcal{R}_\nu[L] = d_0\, \mathfrak{A}_\nu^{\text{glob}} + \sum_{n=1}^{\infty} d_n\, \mathfrak{A}_{n+\nu}^{\text{glob}}[L]$

with coefficients $d_n = d_1 \langle\langle t^{n-1} \rangle\rangle_{P(t)}$.

Then result is complete analog of the Global APT(M) result with natural substitutions:

$$\overline{\mathfrak{A}}_1[L] \to \overline{\mathfrak{A}}_{1+\nu}[L] \quad \text{and} \quad P(t) \to P_\nu(t)$$

with $P_\nu(t) = \int_0^1 P\left(\frac{t}{1-z}\right) \nu\, z^{\nu-1} \frac{dz}{1-z}$.

Resummation in Global Euclidean FAPT

Consider series $\mathcal{D}_\nu[L] = d_0\, \mathcal{A}_\nu^{\text{glob}} + \sum_{n=1}^{\infty} d_n\, \mathcal{A}_{n+\nu}^{\text{glob}}[L]$

with coefficients $d_n = d_1 \langle\langle t^{n-1} \rangle\rangle_{P(t)}$.

Then result is complete analog of the Global APT(E) result with natural substitutions:

$$\overline{\rho}_1[L] \to \overline{\rho}_{1+\nu}[L] \quad \text{and} \quad P(t) \to P_\nu(t)$$

with $P_\nu(t) = \int_0^1 P\left(\frac{t}{1-z}\right) \nu\, z^{\nu-1} \frac{dz}{1-z}$.

Resummation
in
2-loop APT

Resummation in 2-loop APT

Consider series $$\mathcal{S}[L] = \sum_{n=1}^{\infty} \langle\langle t^{n-1} \rangle\rangle_{P(t)} \mathcal{F}_n[L].$$

Here $\mathcal{F}_n[L] = \mathcal{A}_n^{(2)}[L]$ or $\mathfrak{A}_n^{(2)}[L]$.

Resummation in 2-loop APT

Consider series $\mathcal{S}[L] = \sum_{n=1}^{\infty} \langle\langle t^{n-1} \rangle\rangle_{P(t)} \, \mathcal{F}_n[L]$.

Here $\mathcal{F}_n[L] = \mathcal{A}_n^{(2)}[L]$ or $\mathfrak{A}_n^{(2)}[L]$.

We have two-loop recurrence relation ($c_1 = b_1/b_0^2$):

$$-\frac{1}{n}\frac{d}{dL}\mathcal{F}_n[L] = \mathcal{F}_{n+1}[L] + c_1\,\mathcal{F}_{n+2}[L]$$

Resummation in 2-loop APT

Consider series $\mathcal{S}[L] = \sum_{n=1}^{\infty} \langle\langle t^{n-1} \rangle\rangle_{P(t)} \mathcal{F}_n[L]$.

Here $\mathcal{F}_n[L] = \mathcal{A}_n^{(2)}[L]$ or $\mathfrak{A}_n^{(2)}[L]$.

We have two-loop recurrence relation ($c_1 = b_1/b_0^2$):

$$-\frac{1}{n}\frac{d}{dL}\mathcal{F}_n[L] = \mathcal{F}_{n+1}[L] + c_1\,\mathcal{F}_{n+2}[L]$$

Result ($\tau(t) = t - c_1 \ln(1 + t/c_1)$):

$$\mathcal{S}[L] = \left\langle\!\!\left\langle \frac{c_1\,\mathcal{F}_1[L] + t\,\mathcal{F}_1[L-\tau(t)]}{c_1 + t} + \frac{c_1\,t}{c_1+t}\,\mathcal{F}_2[L-\tau(t)] \right\rangle\!\!\right\rangle_{P(t)}$$
$$- \left\langle\!\!\left\langle \frac{c_1\,t}{c_1+t}\int_0^t \frac{dt'}{c_1+t'}\,\frac{d\mathcal{F}_1[L+\tau(t')-\tau(t)]}{dL} \right\rangle\!\!\right\rangle_{P(t)}.$$

Resummation in global 2-loop APT

Consider series $\rho_{\Sigma}^{(2)}[L, N_f] =$

$$\beta_f \sum_{n=1}^{\infty} \langle\langle t^{n-1} \rangle\rangle_{P(t)}\, \overline{\rho}_n^{(2)}[L, N_f] = \sum_{n=1}^{\infty} \langle\langle \left[\frac{t}{\beta_f}\right]^{n-1} \rangle\rangle_{P(t)}\, \rho_n^{(2)}[L]$$

Resummation in global 2-loop APT

Thus ($t_f = t/\beta_f$): $\rho_{\Sigma}^{(2)}[L, N_f] = \sum_{n=1}^{\infty} \langle\langle t_f^{n-1} \rangle\rangle_{P(t)}\, \rho_n^{(2)}[L]$

We have two-loop recurrence relation ($c_1 = b_1/b_0^2$):

$$-\frac{1}{n}\,\frac{d}{dL}\,\rho_n^{(2)}[L] = \rho_{n+1}^{(2)}[L] + c_1\,\rho_{n+2}^{(2)}[L]\,.$$

Resummation in global 2-loop APT

Thus ($t_f = t/\beta_f$): $\rho_{\Sigma}^{(2)}[L, N_f] = \sum_{n=1}^{\infty} \langle\langle t_f^{n-1} \rangle\rangle_{P(t)}\, \rho_n^{(2)}[L]$

We have two-loop recurrence relation ($c_1 = b_1/b_0^2$):

$$-\frac{1}{n}\,\frac{d}{dL}\,\rho_n^{(2)}[L] = \rho_{n+1}^{(2)}[L] + c_1\,\rho_{n+2}^{(2)}[L]\,.$$

Result of summation is ($t_f = t/\beta_f$):

$$\rho_{\Sigma}^{(2)}[L, N_f] = \left\langle\!\!\left\langle \frac{c_1\,\rho_1^{(2)}[L] + t_f\,\rho_1^{(2)}[L - \tau(t_f)]}{c_1 + t_f} + \frac{c_1\,t_f}{c_1 + t_f}\,\rho_2^{(2)}[L - \tau(t_f)] \right.\right.$$
$$\left.\left. - \frac{c_1\,t_f}{c_1 + t_f}\int_0^{t_f} \frac{dt'}{c_1 + t'}\,\frac{d\rho_1^{(2)}[L + \tau(t') - \tau(t_f)]}{dL} \right\rangle\!\!\right\rangle_{P(t)}.$$

Resummation in 2-loop FAPT

Consider series $\mathcal{S}_\nu[L] = \sum_{n=1}^{\infty} \langle\langle t^{n-1} \rangle\rangle_{P(t)} \, \mathcal{F}_{n+\nu}[L]$.

Here $\mathcal{F}_n[L] = \mathcal{A}_n^{(2)}[L]$ or $\mathfrak{A}_n^{(2)}[L]$.

Resummation in 2-loop FAPT

Consider series $\mathcal{S}_\nu[L] = \sum_{n=1}^{\infty} \langle\langle t^{n-1} \rangle\rangle_{P(t)} \mathcal{F}_{n+\nu}[L]$.

Here $\mathcal{F}_n[L] = \mathcal{A}_n^{(2)}[L]$ or $\mathfrak{A}_n^{(2)}[L]$.

We have two-loop recurrence relation ($c_1 = b_1/b_0^2$):

$$-\frac{1}{n+\nu}\frac{d}{dL}\mathcal{F}_{n+\nu}[L] = \mathcal{F}_{n+1+\nu}[L] + c_1\,\mathcal{F}_{n+2+\nu}[L]\,.$$

Resummation in 2-loop FAPT

Consider series $\mathcal{S}_\nu[L] = \sum_{n=1}^{\infty} \langle\langle t^{n-1} \rangle\rangle_{P(t)}\, \mathcal{F}_{n+\nu}[L]$.

Here $\mathcal{F}_n[L] = \mathcal{A}_n^{(2)}[L]$ or $\mathfrak{A}_n^{(2)}[L]$.

We have two-loop recurrence relation ($c_1 = b_1/b_0^2$):

$$-\frac{1}{n+\nu}\,\frac{d}{dL}\,\mathcal{F}_{n+\nu}[L] = \mathcal{F}_{n+1+\nu}[L] + c_1\,\mathcal{F}_{n+2+\nu}[L]\,.$$

Result ($\tau(t) = t - c_1 \ln(1 + t/c_1)$):

$$\mathcal{S}[L] = \Big\langle\!\Big\langle \mathcal{F}_{1+\nu}[L] - \tfrac{t^2}{c_1+t} \int_0^1 z^\nu dz\, \dot{\mathcal{F}}_{1+\nu}[L + \tau(t\,z) - \tau(t)] + \tfrac{c_1\,t}{c_1+t} \left\{ \mathcal{F}_{2+\nu}[L] - \int_0^1 dz\, \tfrac{t^2\,z^{\nu+1}}{c_1+t\,z}\, \dot{\mathcal{F}}_{2+\nu}[L + \tau(t\,z) - \tau(t)] \right\} \Big\rangle\!\Big\rangle_{P(t)}$$

Higgs boson decay

$$H^0 \to b\bar{b}$$

Higgs boson decay into $b\bar{b}$-pair

This decay can be expressed in QCD by means of the correlator of quark scalar currents $J_{\rm S}(x) = :\bar{b}(x)b(x):$

$$\Pi(Q^2) = (4\pi)^2 i \int dx e^{iqx} \langle 0|\, T[\, J_{\rm S}(x) J_{\rm S}(0)\,]\, |0\rangle$$

Higgs boson decay into $b\bar{b}$*-pair*

This decay can be expressed in QCD by means of the correlator of quark scalar currents $J_{\mathsf{S}}(x) = :\bar{b}(x)b(x):$

$$\Pi(Q^2) = (4\pi)^2 i \int dx e^{iqx} \langle 0|\, T[\, J_{\mathsf{S}}(x) J_{\mathsf{S}}(0)\,]\,|0\rangle$$

in terms of discontinuity of its imaginary part

$$R_{\mathsf{S}}(s) = \mathbf{Im}\,\Pi(-s - i\epsilon)/(2\pi\, s)\,,$$

so that

$$\Gamma_{\mathsf{H}\to b\bar{b}}(M_{\mathsf{H}}) = \frac{G_F}{4\sqrt{2}\pi} M_{\mathsf{H}}\, m_b^2(M_{\mathsf{H}})\, R_{\mathsf{S}}(s = M_{\mathsf{H}}^2)\,.$$

FAPT(M) analysis of R_S

Running mass $m(Q^2)$ is described by the RG equation

$$m^2(Q^2) = \hat{m}^2 \left[\frac{\alpha_s(Q^2)}{\pi}\right]^{\nu_0} \left[1 + \frac{c_1 b_0 \alpha_s(Q^2)}{4\pi^2}\right]^{\nu_1} .$$

with RG-invariant mass $\hat{m}^2$ (for b-quark $\hat{m}_b \approx 14.6$ **GeV**) and $\nu_0 = 1.04$, $\nu_1 = 1.86$.

FAPT(M) analysis of R_S

Running mass $m(Q^2)$ is described by the RG equation

$$m^2(Q^2) = \hat{m}^2 \left[\frac{\alpha_s(Q^2)}{\pi}\right]^{\nu_0} \left[1 + \frac{c_1 b_0 \alpha_s(Q^2)}{4\pi^2}\right]^{\nu_1} .$$

with RG-invariant mass $\hat{m}^2$ (for b-quark $\hat{m}_b \approx 14.6$ **GeV**) and $\nu_0 = 1.04$, $\nu_1 = 1.86$. This gives us

$$\left[3\,\hat{m}_b^2\right]^{-1} \widetilde{D}_{\mathbf{S}}(Q^2) = \left(\frac{\alpha_s(Q^2)}{\pi}\right)^{\nu_0} + \sum_{m>0} d_m \left(\frac{\alpha_s(Q^2)}{\pi}\right)^{m+\nu_0}$$

FAPT(M) analysis of R_S

Running mass $m(Q^2)$ is described by the RG equation

$$m^2(Q^2) = \hat{m}^2 \left[\frac{\alpha_s(Q^2)}{\pi}\right]^{\nu_0} \left[1 + \frac{c_1 b_0 \alpha_s(Q^2)}{4\pi^2}\right]^{\nu_1} .$$

with RG-invariant mass $\hat{m}^2$ (for b-quark $\hat{m}_b \approx 14.6$ **GeV**) and $\nu_0 = 1.04$, $\nu_1 = 1.86$. This gives us

$$\left[3\,\hat{m}_b^2\right]^{-1} \widetilde{D}_{\mathbf{S}}(Q^2) = \left(\frac{\alpha_s(Q^2)}{\pi}\right)^{\nu_0} + \sum_{m>0} d_m \left(\frac{\alpha_s(Q^2)}{\pi}\right)^{m+\nu_0}$$

In FAPT(M) we obtain

$$\widetilde{\mathcal{R}}_{\mathbf{S}}^{(l);N}[L] = \frac{3\hat{m}^2}{\pi^{\nu_0}} \left[\mathfrak{A}_{\nu_0}^{(l);\text{glob}}[L] + \sum_{m>0}^{N} \frac{d_m^{(l)}}{\pi^m} \mathfrak{A}_{m+\nu_0}^{(l);\text{glob}}[L]\right]$$

Model for perturbative coefficients

Let us have a look to coefficients of our series, $\tilde{d}_m = d_m/d_1$, with $d_1 = 17/3$.

Model	$\tilde{d}_1$	$\tilde{d}_2$	$\tilde{d}_3$	$\tilde{d}_4$	$\tilde{d}_5$
pQCD	1	7.42	62.3		—

Model for perturbative coefficients

Let us have a look to coefficients of our series, $\tilde{d}_m = d_m/d_1$, with $d_1 = 17/3$.

Model	$\tilde{d}_1$	$\tilde{d}_2$	$\tilde{d}_3$	$\tilde{d}_4$	$\tilde{d}_5$
pQCD	1	7.42	62.3		—
$c = 2.5,\ \beta = -0.48$	1	7.42	62.3		

We use model $\tilde{d}_n^{\text{mod}} = \dfrac{c^{n-1}(\beta\,\Gamma(n) + \Gamma(n+1))}{\beta + 1}$

with parameters β and c estimated by known $\tilde{d}_n$ and with use of **Lipatov** asymptotics.

Model for perturbative coefficients

Let us have a look to coefficients of our series, $\tilde{d}_m = d_m/d_1$, with $d_1 = 17/3$.

Model	$\tilde{d}_1$	$\tilde{d}_2$	$\tilde{d}_3$	$\tilde{d}_4$	$\tilde{d}_5$
pQCD	1	7.42	62.3	620	—
$c = 2.5,\ \beta = -0.48$	1	7.42	62.3	662	—

We use model $\tilde{d}_n^{\text{mod}} = \dfrac{c^{n-1}(\beta\,\Gamma(n) + \Gamma(n+1))}{\beta + 1}$

with parameters β and c estimated by known $\tilde{d}_n$ and with use of **Lipatov** asymptotics.

Model for perturbative coefficients

Let us have a look to coefficients of our series, $\tilde{d}_m = d_m/d_1$, with $d_1 = 17/3$.

Model	$\tilde{d}_1$	$\tilde{d}_2$	$\tilde{d}_3$	$\tilde{d}_4$	$\tilde{d}_5$
pQCD	1	7.42	62.3	620	—
$c = 2.5,\ \beta = -0.48$	1	7.42	62.3	662	—
$c = 2.4,\ \beta = -0.52$	1	7.50	61.1	625	

We use model $\tilde{d}_n^{\text{mod}} = \dfrac{c^{n-1}(\beta\,\Gamma(n) + \Gamma(n+1))}{\beta + 1}$

with parameters β and c estimated by known $\tilde{d}_n$ and with use of **Lipatov** asymptotics.

Model for perturbative coefficients

Let us have a look to coefficients of our series, $\tilde{d}_m = d_m/d_1$, with $d_1 = 17/3$.

Model	$\tilde{d}_1$	$\tilde{d}_2$	$\tilde{d}_3$	$\tilde{d}_4$	$\tilde{d}_5$
pQCD	1	7.42	62.3	620	—
$c = 2.5,\ \beta = -0.48$	1	7.42	62.3	$\mathbf{662}$	—
$c = 2.4,\ \beta = -0.52$	1	7.50	61.1	$\mathbf{625}$	$\mathbf{7826}$

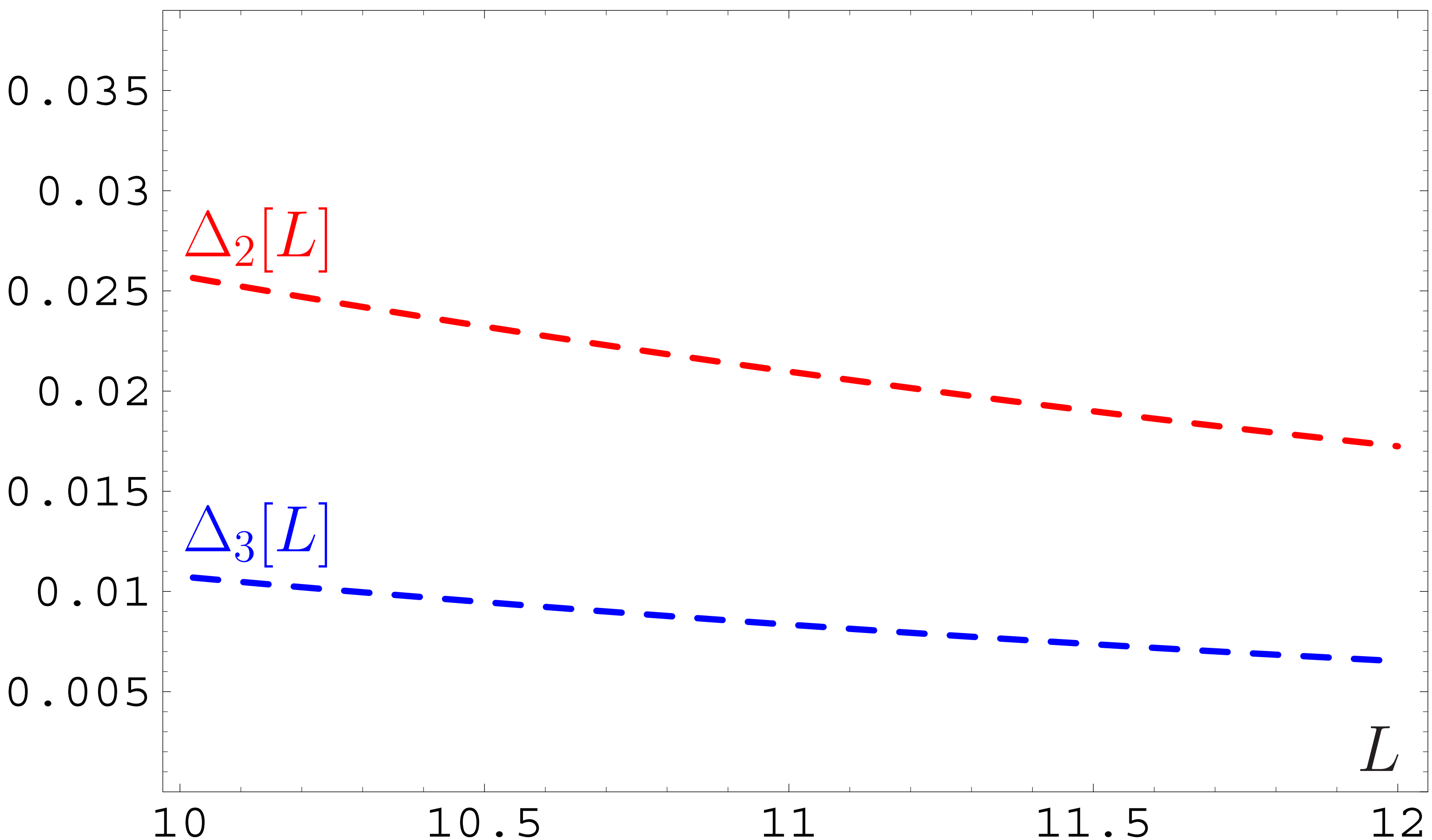
We use model $\tilde{d}_n^{\text{mod}} = \dfrac{c^{n-1}(\beta\,\Gamma(n) + \Gamma(n+1))}{\beta + 1}$

with parameters β and c estimated by known $\tilde{d}_n$ and with use of **Lipatov** asymptotics.

FAPT(M) for $\widetilde{R}_S$: Truncation errors

We define relative errors of series truncation at Nth term:

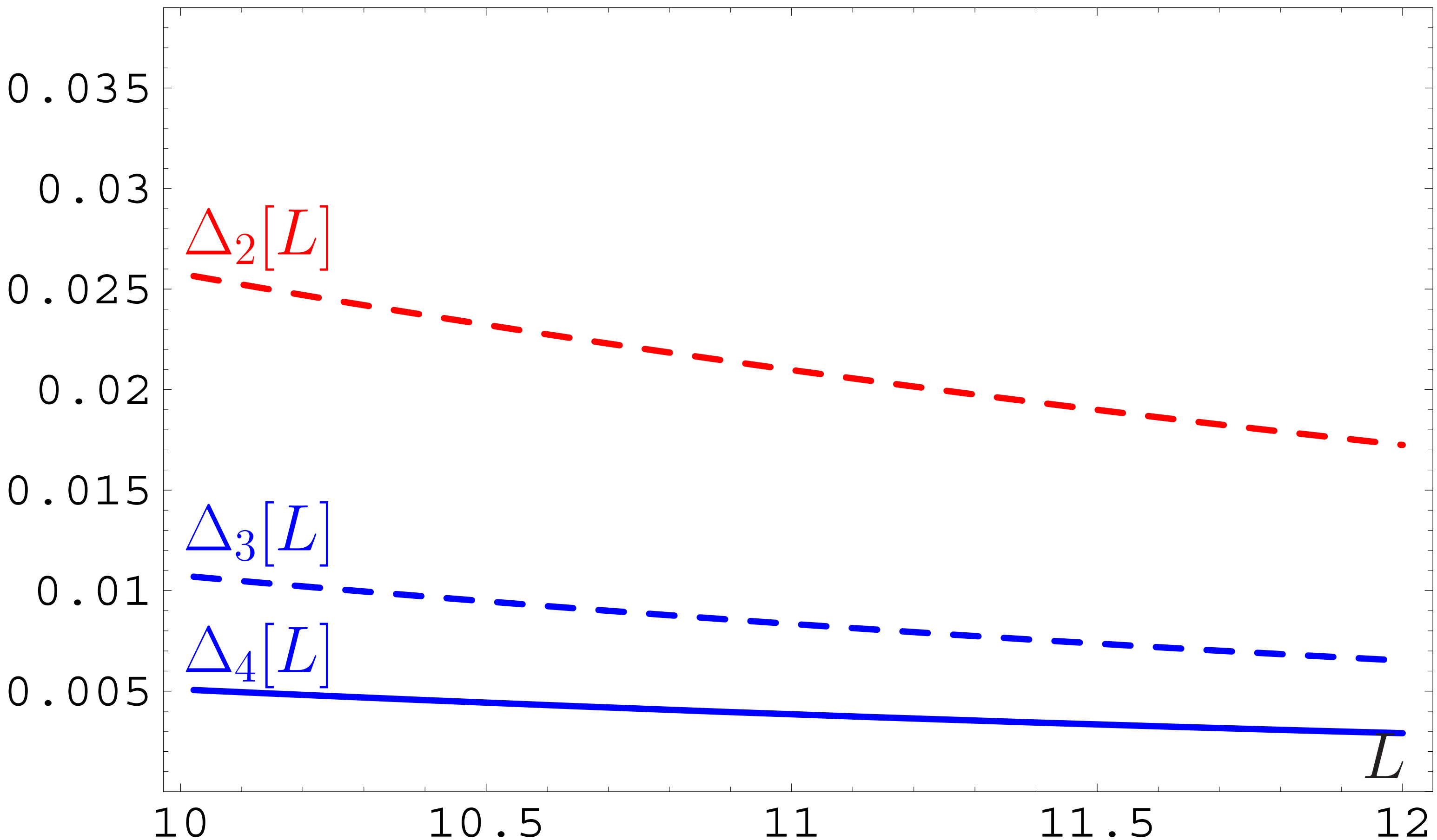
$$\Delta_N[L] = 1 - \widetilde{\mathcal{R}}_{\mathbf{S}}^{(1;N)}[L] / \widetilde{\mathcal{R}}_{\mathbf{S}}^{(1;\infty)}[L]$$



FAPT(M) for $\widetilde{R}_S$: Truncation errors

We define relative errors of series truncation at Nth term:

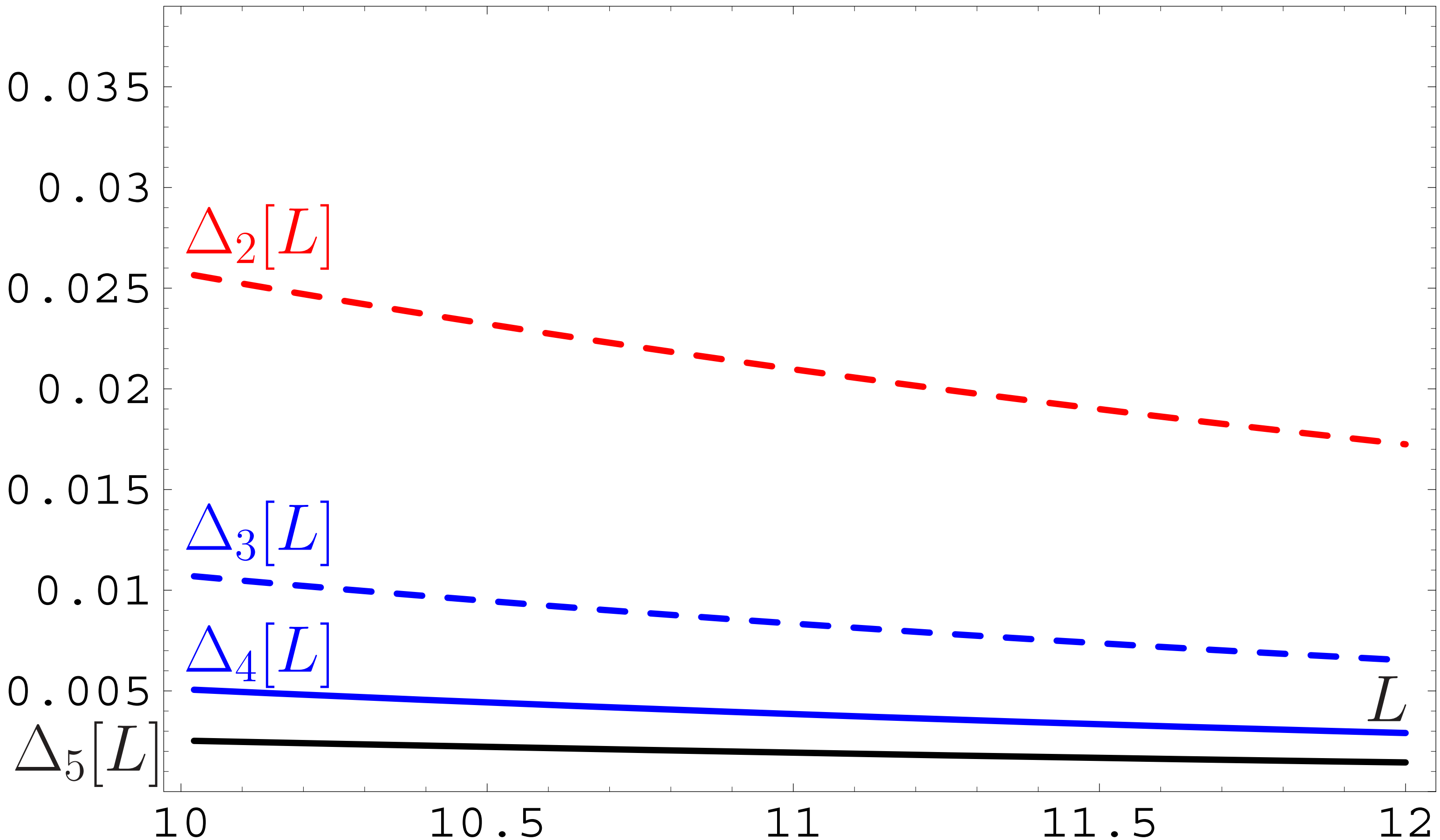
$$\Delta_N[L] = 1 - \widetilde{\mathcal{R}}_{\mathbf{S}}^{(1;N)}[L]/\widetilde{\mathcal{R}}_{\mathbf{S}}^{(1;\infty)}[L]$$



FAPT(M) for $\widetilde{R}_S$: Truncation errors

We define relative errors of series truncation at Nth term:

$$\Delta_N[L] = 1 - \widetilde{\mathcal{R}}_{\mathbf{S}}^{(1;N)}[L] / \widetilde{\mathcal{R}}_{\mathbf{S}}^{(1;\infty)}[L]$$



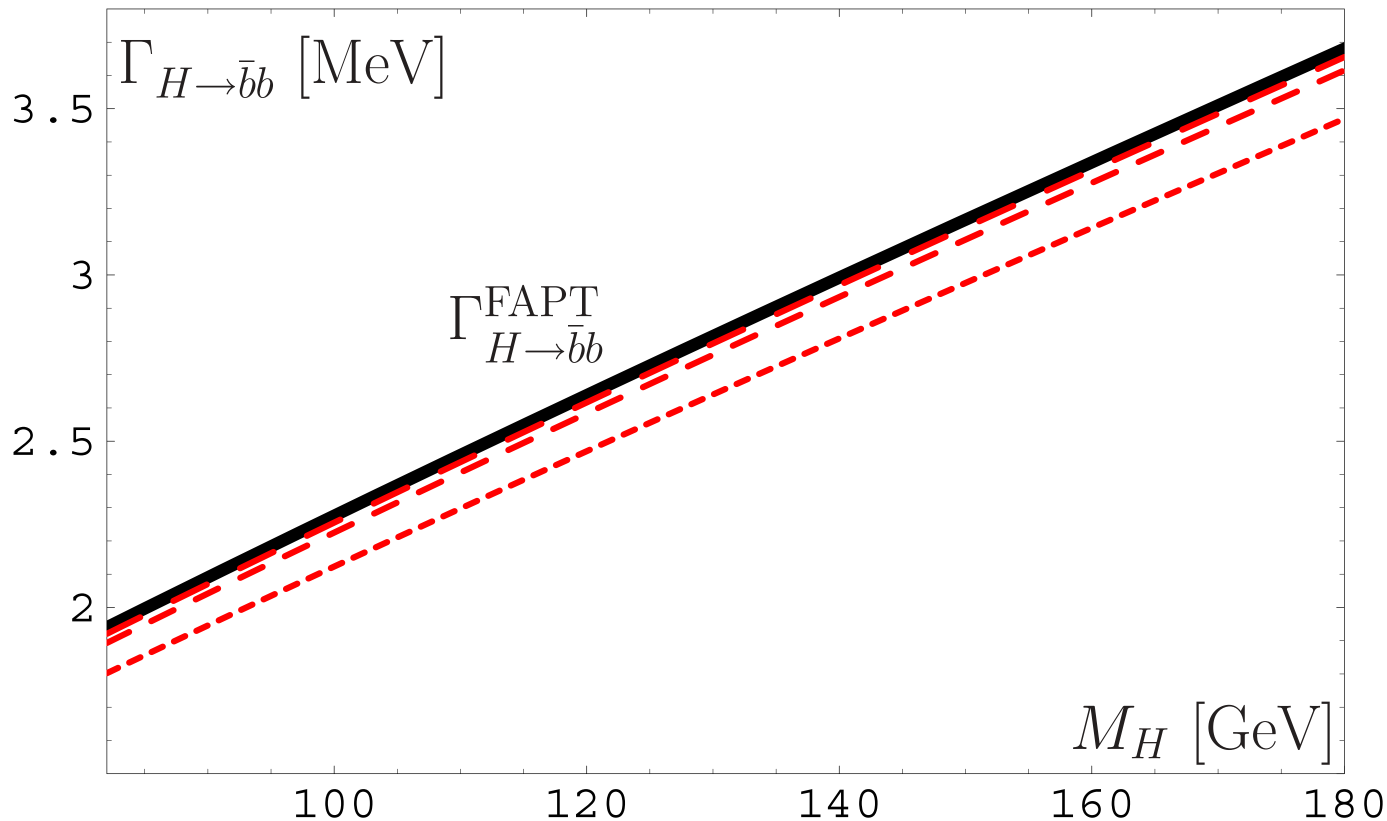
FAPT(M) for $\widetilde{R}_S$: Truncation errors

Conclusion: If we need accuracy better than 0.5% — only then we need to calculate the 5-th correction.

FAPT(M) for $\widetilde{R}_S$: Truncation errors

Conclusion: If we need accuracy better than 0.5% — only then we need to calculate the 5-th correction.

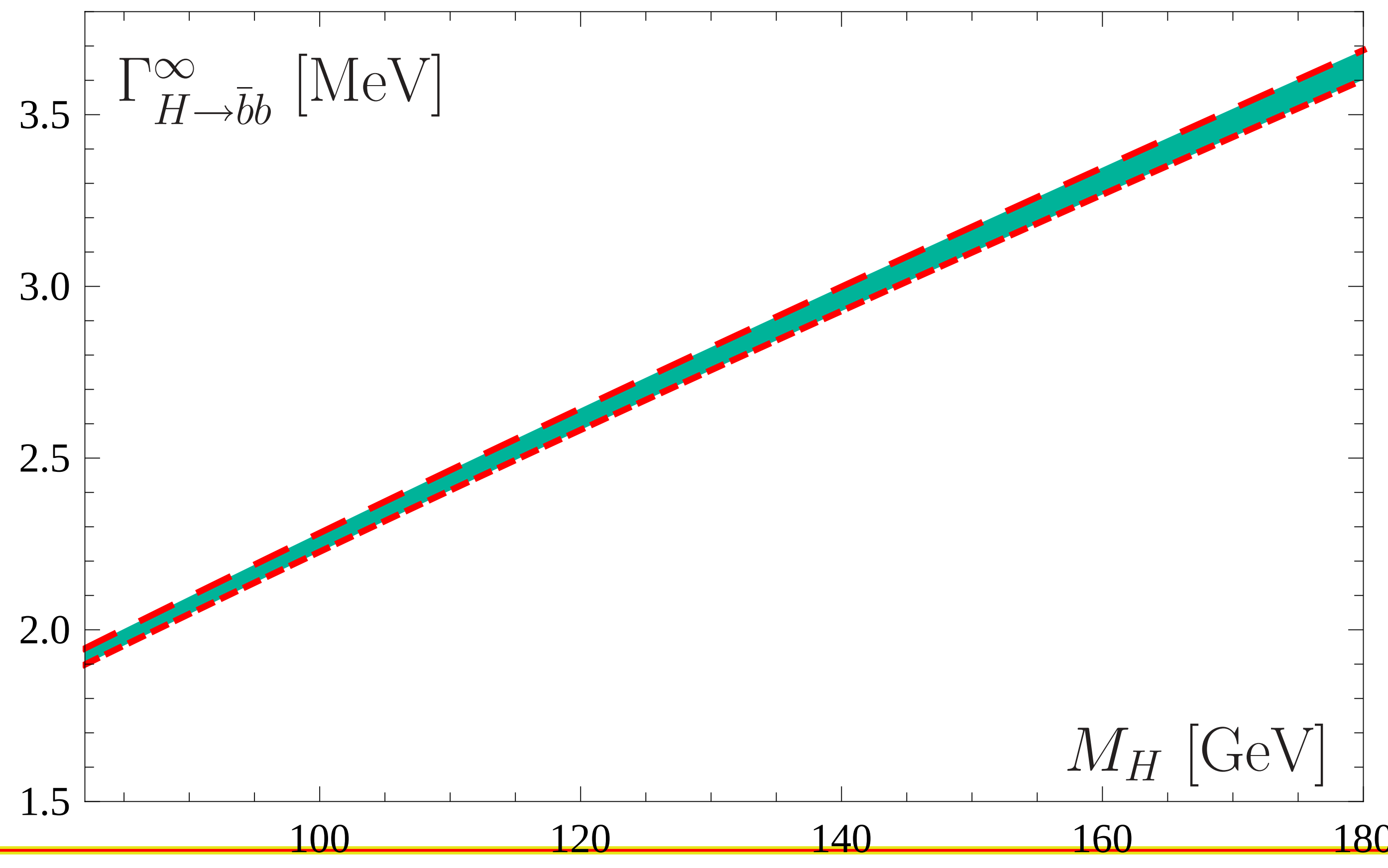
But profit will be tiny — instead of 0.5% one'll obtain 0.3%!



FAPT(M) for $\widetilde{R}_S$: Truncation errors

Conclusion: If we need accuracy better than 0.5% — only then we need to calculate the 5-th correction.

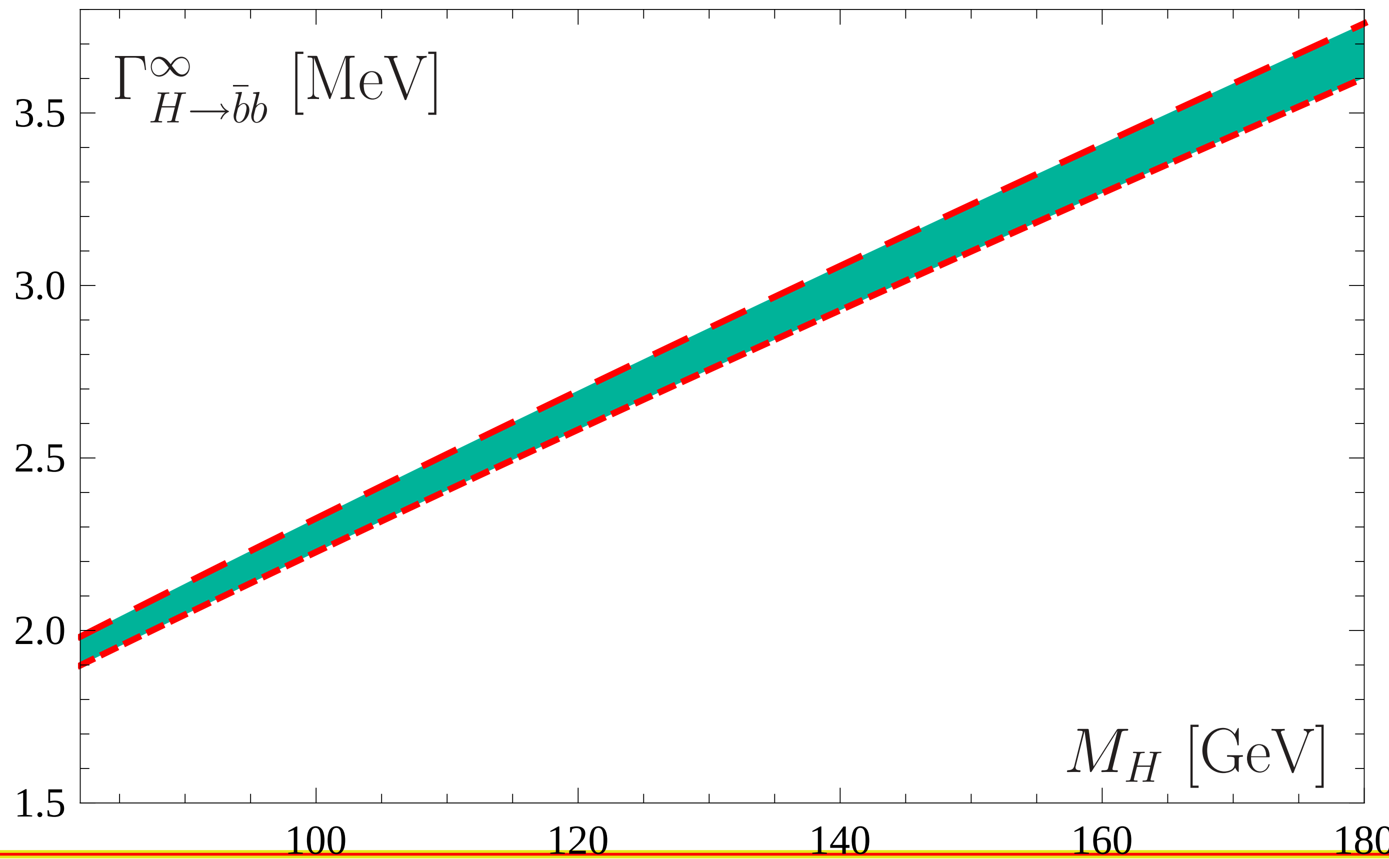
Note: uncertainty due to $P(t)$-modelling is small $\lesssim 0.4\%$.



FAPT(M) for $\widetilde{R}_S$: Truncation errors

Conclusion: If we need accuracy better than 0.5% — only then we need to calculate the 5-th correction.

Note: main is on-shell mass uncertainty $\sim 4\%$.



Adler function $D(Q^2)$ and ratio $R(s)$

Adler function $D(Q^2)$ in vector channel

Adler function $D(Q^2)$ can be expressed in QCD by means of the correlator of quark vector currents

$$\Pi_{\mathsf{V}}(Q^2) = \frac{(4\pi)^2}{3q^2}\, i \int dx\, e^{iqx} \langle 0|\, T[\, J_\mu(x) J^\mu(0)\,]\, |0\rangle$$

in terms of discontinuity of its imaginary part

$$R_{\mathsf{V}}(s) = \frac{1}{\pi}\, \mathrm{Im}\, \Pi_{\mathsf{V}}(-s - i\epsilon)\,,$$

so that

$$D(Q^2) = Q^2 \int_0^\infty \frac{R_{\mathsf{V}}(\sigma)}{(\sigma + Q^2)^2}\, d\sigma\,.$$

APT analysis of $D(Q^2)$ *and* $R_V(s)$

QCD PT gives us

$$D(Q^2) = 1 + \sum_{m>0} \frac{d_m}{\pi^m} \left(\frac{\alpha_s(Q^2)}{\pi} \right)^m .$$

APT analysis of $D(Q^2)$ *and* $R_V(s)$

QCD PT gives us

$$D(Q^2) = 1 + \sum_{m>0} \frac{d_m}{\pi^m} \left(\frac{\alpha_s(Q^2)}{\pi} \right)^m .$$

In **APT**(E) we obtain

$$\mathcal{D}_N(Q^2) = 1 + \sum_{m>0}^{N} \frac{d_m}{\pi^m} \mathcal{A}_m^{\text{glob}}(Q^2)$$

APT analysis of $D(Q^2)$ *and* $R_V(s)$

QCD PT gives us

$$D(Q^2) = 1 + \sum_{m>0} \frac{d_m}{\pi^m} \left(\frac{\alpha_s(Q^2)}{\pi} \right)^m .$$

In **APT**(E) we obtain

$$\mathcal{D}_N(Q^2) = 1 + \sum_{m>0}^{N} \frac{d_m}{\pi^m} \mathcal{A}_m^{\text{glob}}(Q^2)$$

and in **APT**(M)

$$\mathcal{R}_{\mathbf{V};N}(s) = 1 + \sum_{m>0}^{N} \frac{d_m}{\pi^m} \mathfrak{A}_m^{\text{glob}}(s)$$

Model for perturbative coefficients

Let us have a look to coefficients d_m of the PT series.

Model	d_1	d_2	d_3	d_4	d_5
pQCD results with $N_f = 4$	1	1.52	2.59		—

Model for perturbative coefficients

Let us have a look to coefficients d_m of the PT series.

Model	d_1	d_2	d_3	d_4	d_5
pQCD results with $N_f = 4$	1	1.52	2.59		—
$c = 3.467,\ \beta = 1.325$	1	1.50	2.62		

We use model $d_n^{\text{mod}} = \dfrac{c^{n-1}(\beta^{n+1} - n)}{\beta^2 - 1}\,\Gamma(n)$

with parameters β and c estimated by known $\tilde{d}_n$ and with use of **Lipatov** asymptotics.

Model for perturbative coefficients

Let us have a look to coefficients d_m of the PT series.

Model	d_1	d_2	d_3	d_4	d_5
pQCD results with $N_f = 4$	1	1.52	2.59	27.4	—
$c = 3.467,\ \beta = 1.325$	1	1.50	2.62	27.8	

We use model $d_n^{\text{mod}} = \dfrac{c^{n-1}(\beta^{n+1} - n)}{\beta^2 - 1}\,\Gamma(n)$

with parameters β and c estimated by known $\tilde{d}_n$ and with use of **Lipatov** asymptotics.

Model for perturbative coefficients

Let us have a look to coefficients d_m of the PT series.

Model	d_1	d_2	d_3	d_4	d_5
pQCD results with $N_f = 4$	1	1.52	2.59	27.4	—
$c = 3.467,\ \beta = 1.325$	1	1.50	2.62	27.8	
$c = 3.456,\ \beta = 1.325$	1	1.49	2.60	27.5	

We use model $d_n^{\text{mod}} = \dfrac{c^{n-1}(\beta^{n+1} - n)}{\beta^2 - 1}\,\Gamma(n)$

with parameters β and c estimated by known $\tilde{d}_n$ and with use of **Lipatov** asymptotics.

Model for perturbative coefficients

Let us have a look to coefficients d_m of the PT series.

Model	d_1	d_2	d_3	d_4	d_5
pQCD results with $N_f = 4$	1	1.52	2.59	27.4	—
$c = 3.467,\ \beta = 1.325$	1	1.50	2.62	$\mathbf{27.8}$	$\mathbf{1888}$
$c = 3.456,\ \beta = 1.325$	1	1.49	2.60	$\mathbf{27.5}$	$\mathbf{1865}$

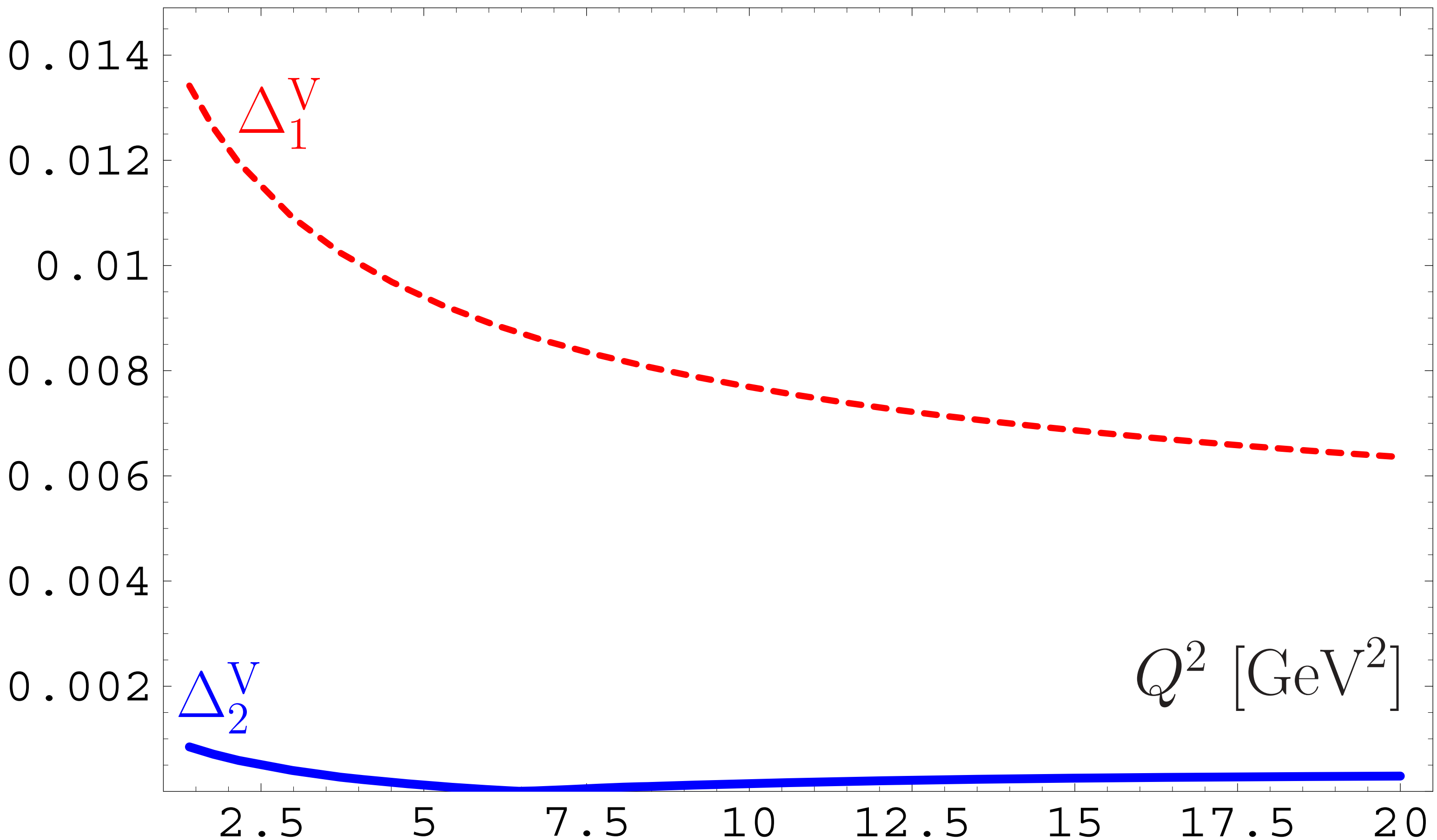
We use model $d_n^{\text{mod}} = \dfrac{c^{n-1}(\beta^{n+1} - n)}{\beta^2 - 1}\,\Gamma(n)$

with parameters β and c estimated by known $\tilde{d}_n$ and with use of **Lipatov** asymptotics.

APT(E) for $\mathcal{D}(Q^2)$: Truncation errors

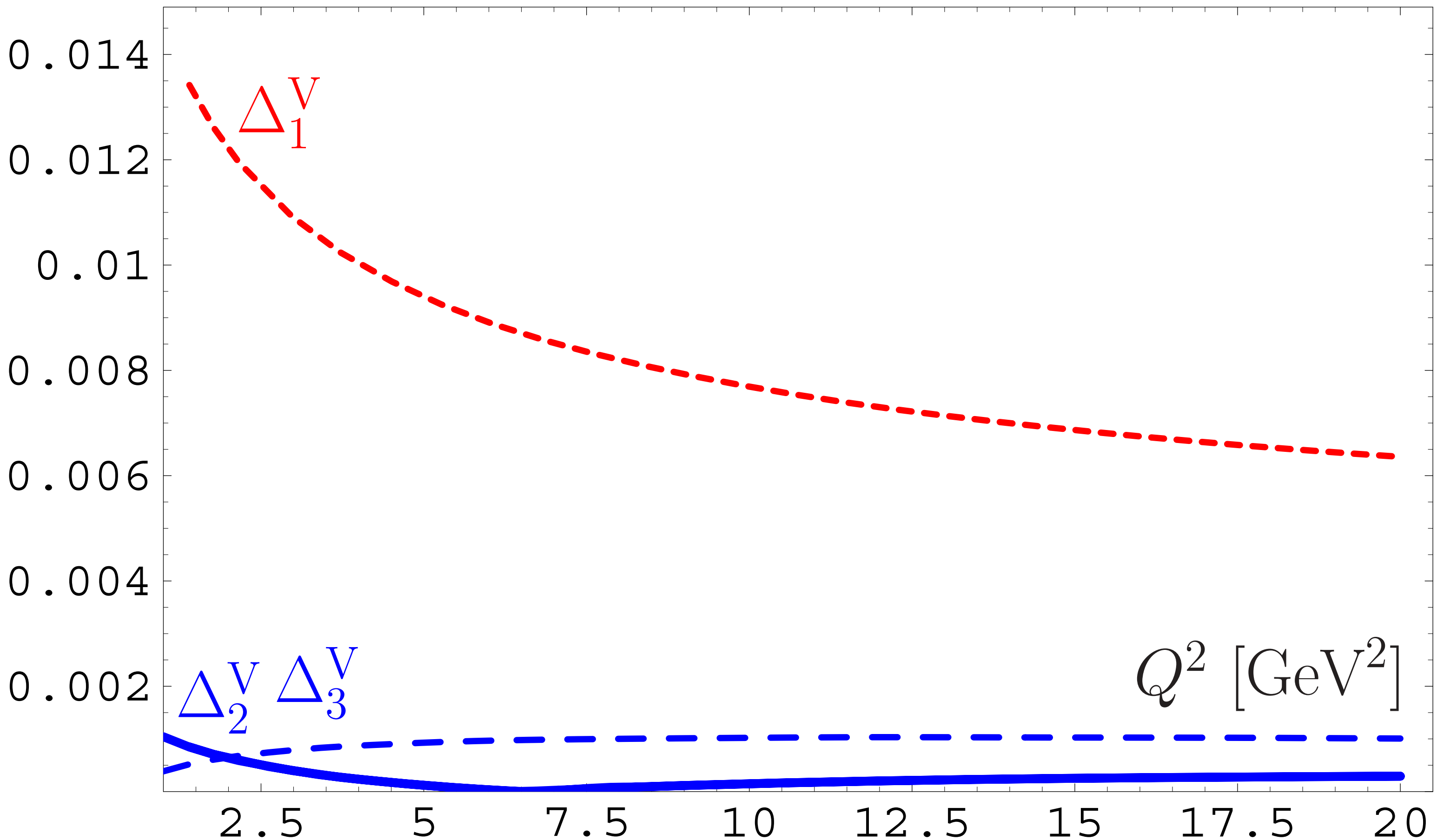
We define relative errors of series truncation at Nth term:

$$\Delta_N^{\mathsf{V}}[L] = 1 - \mathcal{D}_N[L]/\mathcal{D}_\infty[L]$$



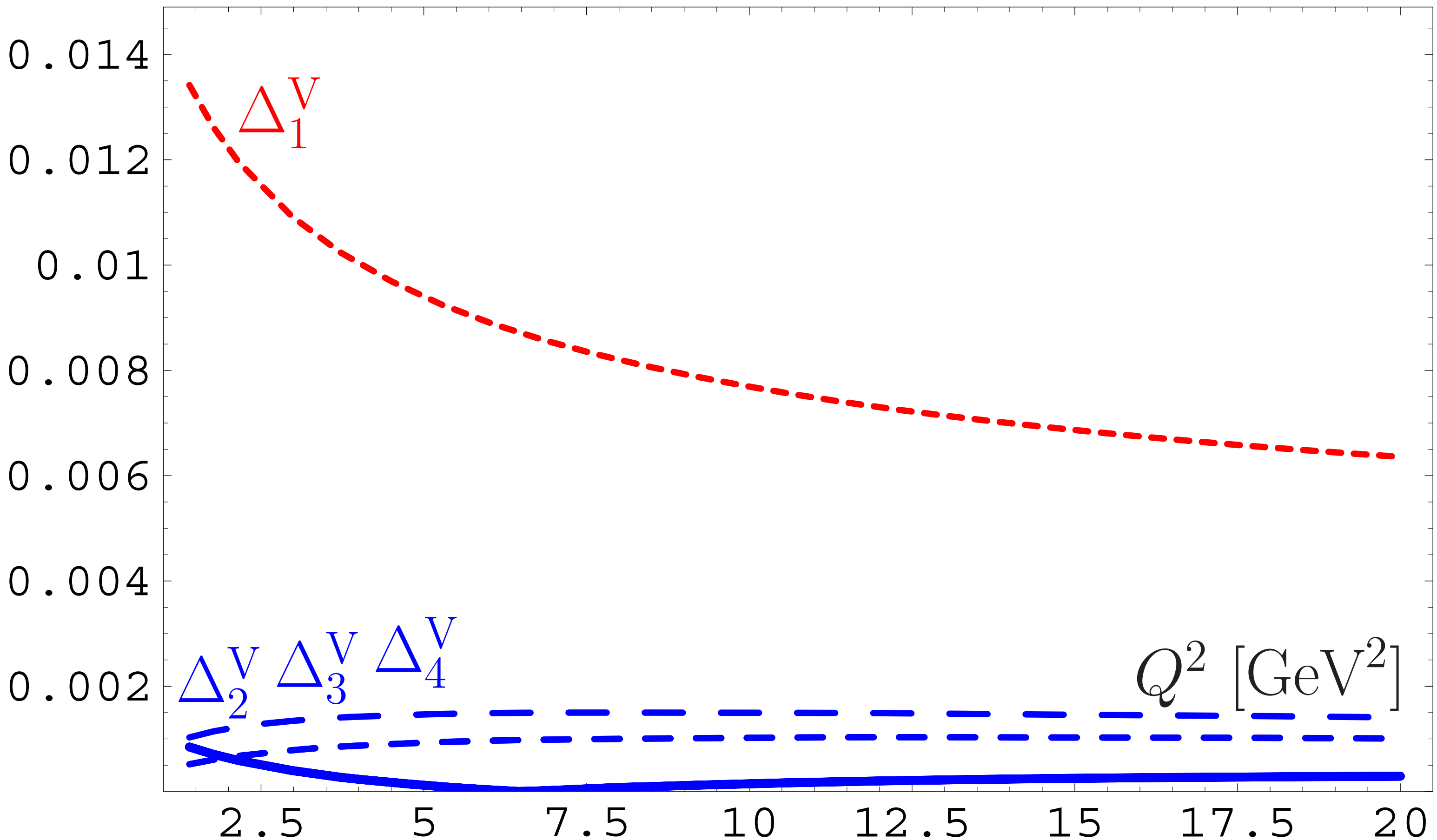
APT(E) for $\mathcal{D}(Q^2)$: Truncation errors

Conclusion: The best accuracy (better than 0.1%) is achieved for **N^2LO** approximation.



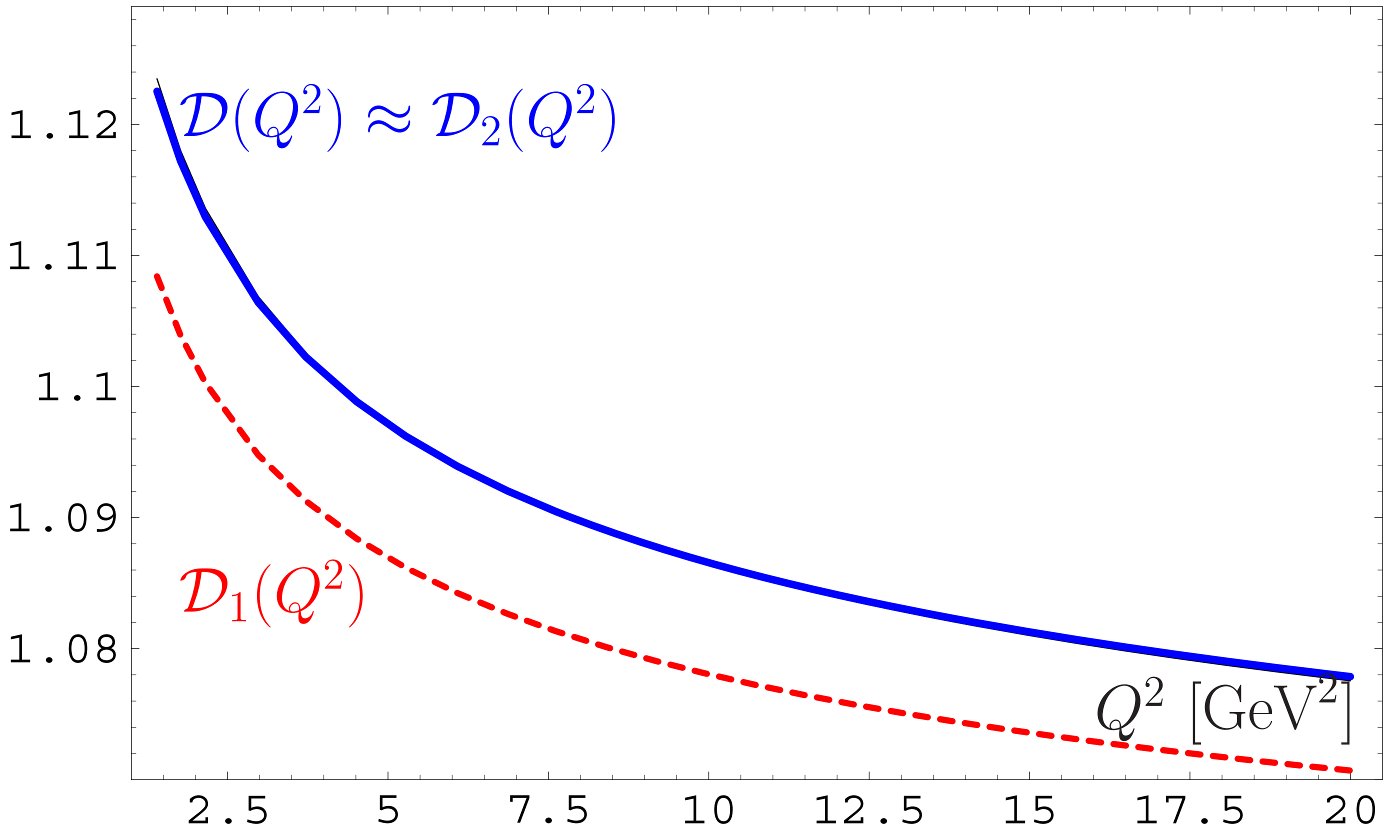
APT(E) for $\mathcal{D}(Q^2)$: Truncation errors

Conclusion: If we add more terms **N^{3}LO** — truncation error increases.



APT(E) for $\mathcal{D}(Q^2)$: Truncation errors

Conclusion: The best accuracy (better than 0.1%) is achieved for **N^{2}LO** approximation.



APT(E) for $\mathcal{D}(Q^2)$: Errors of modelling $P(t)$

We use model $d_n^{\text{mod}} = \dfrac{c^{n-1}(\beta^{n+1} - n)}{\beta^2 - 1}\,\Gamma(n)$ with parameters $\beta = 1.325$ and $c = 3.456$ estimated by known $\tilde{d}_n$ and with use of **Lipatov** asymptotics.

We apply it to resum **APT** series and obtain $\mathcal{D}(Q^2)$.

APT(E) for $\mathcal{D}(Q^2)$: Errors of modelling $P(t)$

We use model $d_n^{\text{mod}} = \dfrac{c^{n-1}(\beta^{n+1} - n)}{\beta^2 - 1}\,\Gamma(n)$

with parameters $\beta = 1.325$ and $c = 3.456$ estimated by known $\tilde{d}_n$ and with use of **Lipatov** asymptotics.

We apply it to resum **APT** series and obtain $\mathcal{D}(Q^2)$.

We deform our model for d_n by using coefficients $\beta_{\text{NNA}} = 1.322$ and $c_{\text{NNA}} = 3.885$

that deforms $d_4 = 27.5 \to d_4^{\text{NNA}} = 20.4$

APT(E) for $\mathcal{D}(Q^2)$: Errors of modelling $P(t)$

We use model $d_n^{\text{mod}} = \dfrac{c^{n-1}(\beta^{n+1} - n)}{\beta^2 - 1}\,\Gamma(n)$

with parameters $\beta = 1.325$ and $c = 3.456$ estimated by known $\tilde{d}_n$ and with use of **Lipatov** asymptotics.

We apply it to resum **APT** series and obtain $\mathcal{D}(Q^2)$.

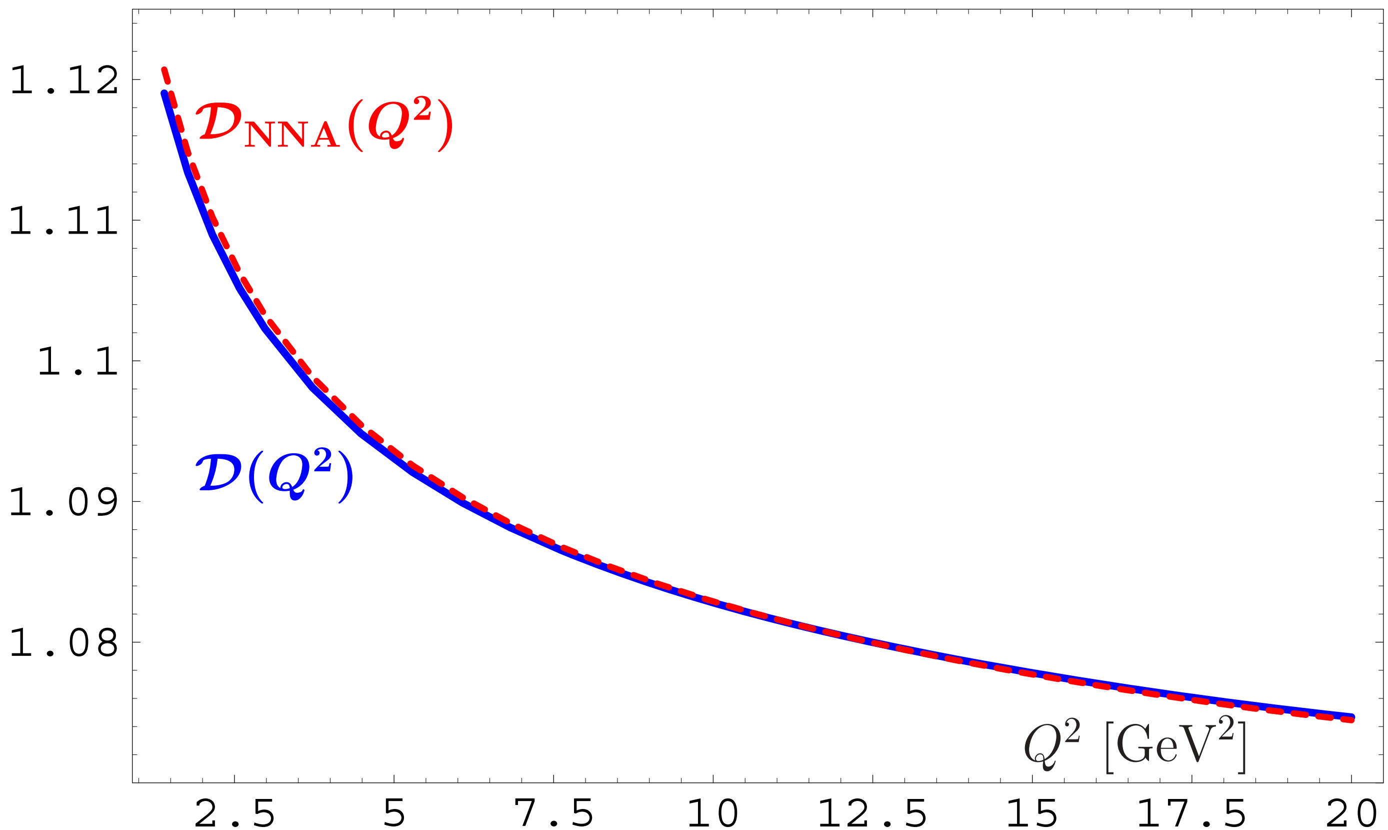
We deform our model for d_n by using coefficients $\beta_{\text{NNA}} = 1.322$ and $c_{\text{NNA}} = 3.885$

that deforms $d_4 = 27.5 \to d_4^{\text{NNA}} = 20.4$

We apply it to resum **APT** series and obtain $\mathcal{D}_{\text{NNA}}(Q^2)$.

APT(E) for $\mathcal{D}(Q^2)$: Errors of modelling $P(t)$

Conclusion: The result of resummation is stable to the variations of higher-order coefficients: deviation is of the order of 0.1%.



CONCLUSIONS

- **APT** provides natural way to Minkowski region for coupling and related quantities.

CONCLUSIONS

- **APT** provides natural way to Minkowski region for coupling and related quantities.
- **FAPT** provides effective tool to apply **APT** approach for renormgroup improved perturbative amplitudes.

CONCLUSIONS

- **APT** provides natural way to Minkowski region for coupling and related quantities.
- **FAPT** provides effective tool to apply **APT** approach for renormgroup improved perturbative amplitudes.
- Both **APT** and **FAPT** produce finite resummed answers for perturbative quantities if we know generating function $P(t)$ for PT coefficients.

CONCLUSIONS

- **APT** provides natural way to Minkowski region for coupling and related quantities.
- **FAPT** provides effective tool to apply **APT** approach for renormgroup improved perturbative amplitudes.
- Both **APT** and **FAPT** produce finite resummed answers for perturbative quantities if we know generating function $P(t)$ for PT coefficients.
- Using quite simple model generating function $P(t)$ for Higgs boson decay $H \to \bar{b}b$ we see that at **N^{3}LO** we have accuracy of the order 1%...

CONCLUSIONS

- **APT** provides natural way to Minkowski region for coupling and related quantities.
- **FAPT** provides effective tool to apply **APT** approach for renormgroup improved perturbative amplitudes.
- Both **APT** and **FAPT** produce finite resummed answers for perturbative quantities if we know generating function $P(t)$ for PT coefficients.
- Using quite simple model generating function $P(t)$ for Higgs boson decay $H \to \bar{b}b$ we see that at **N^{3}LO** we have accuracy of the order 1%...
- ...and for Adler function $\mathcal{D}(Q^2)$ — we have accuracy of the order 0.1% already at **N^{2}LO**.