How to reduce integration-by-part irreducible integrals?

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- **Integration by parts.**
- **Propagators, 3-loop.**
- **Glue-and-Cut relations.**
- **Propagators, 4-loop.**
- **Propagators, 5-loop?**
- **Vertexes?**
- **Finiteness.**
- **Multi-legs–multi-scale diagrams?**
- **Summary.**

Feynman integrals

$$F(\underline{n}, d) = \int d^d p_1 .. d^d p_L / (E_1^{n_1} \cdots E_a^{n_a})$$

$$E_a = A_a^{ik}(p_i p_k) + m_a^2$$

Integration by parts (IBP):

$$0 = \int d^d p_1 .. d^d p_L \ \partial_{p_i}(p_k \cdots)$$

$$\partial_{p_i} \ (p_k \ \cdot) = d \ \delta_k^i + p_k(\partial_{p_i} \ \cdot) = d \ \delta_k^i + (AA)_b^a \ E_a \ (\partial_{E_b} \ \cdot)$$

Recurrence relations:

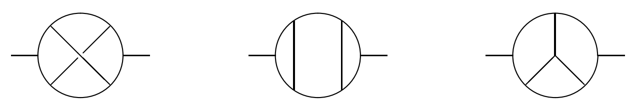
$$0 = d \ \delta_k^i \ F + (AA)_b^a \ E_a \ \partial_{E_b} \ F$$

$$R(I^-, I^+, d) F(n_1, ..., n_k, d) = 0$$

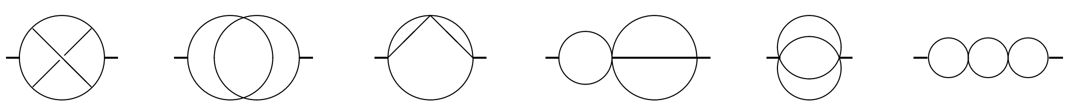
Result of the reduction procedure:

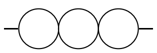
$$F(\underline{n}, d) = C_1(\underline{n}, d)F_1 + ... + C_k(\underline{n}, d)F_k$$

Any 3-loop massless propagator integral

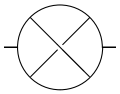


can be reduced to IBP-irreducible integrals:

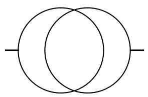




$$= \frac{1}{\epsilon^3}$$



$$= 20\zeta_5 + O(\epsilon)$$



$$= \frac{1}{3\epsilon^3} + \frac{1}{3\epsilon^2} + \frac{1}{\epsilon}(\frac{14}{3}\zeta_3 - \frac{7}{3}) + \frac{14}{3}\zeta_3 + 7\zeta_4 - \frac{67}{3}$$

$$+\epsilon(\frac{86}{3}\zeta_3 + 7\zeta_4 + 126\zeta_5 - \frac{403}{3}) + O(\epsilon^2)$$

$$\ldots$$

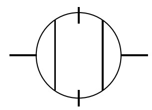
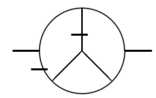
$$M_i = C_{ik}(\epsilon) Z_k \qquad Z_k : \; 1, \; \zeta_3, \; \zeta_4, \; \zeta_5$$

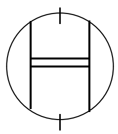
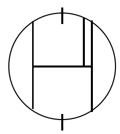
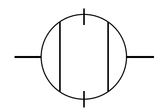
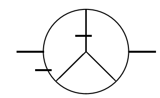
$$\hat{\zeta}_3 = \zeta_3 + \frac{3}{2}\,\epsilon\,\zeta_4 \quad \Rightarrow \; Z_k : \; 1, \; \hat{\zeta}_3, \; \zeta_5$$

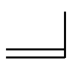
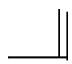
6 IBP-irreducible integrals depends on 3 objects

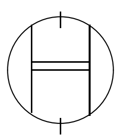
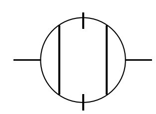
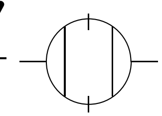
additional relations?

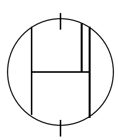
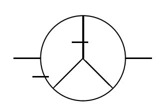
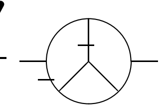
Glue-and-Cut relations. Example.

 $=$  $+\, O(\epsilon)$

$\mathbf{S.p.}($  $-$  $) = 0 = \mathbf{F.p.}($  $-$  $)$

$($  $\mathbf{k}$ $\mathbf{p}$ $-$  $) = \frac{1}{(p^2+1)k^2} - \frac{1}{p^2(k^2+1)} = \frac{k^2-p^2}{(p^2+1)p^2(k^2+1)k^2}$

 $= a \int \frac{d^D q}{q^2+1}$  $= \frac{a'}{\epsilon}$ 

 $= a \int \frac{d^D q}{q^2+1}$  $= \frac{a'}{\epsilon}$ 

Glue-and-Cut relations for massless propagators.

Massless propagators which can be obtained by cutting of some massless scalar vacuum diagram

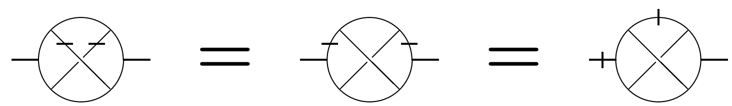
1) without any sub-divergences and

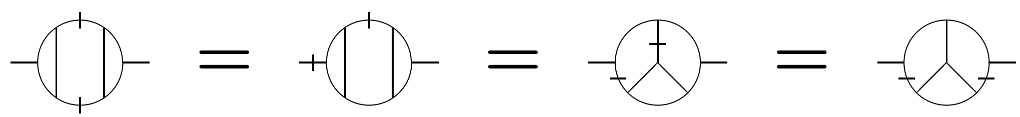
2) with the superficial divergence index 0

are finite and equal to each other.

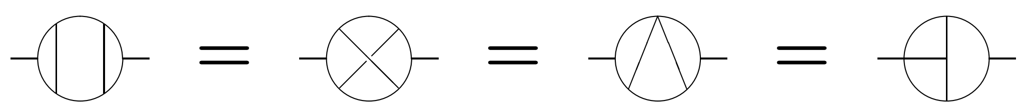
Glue-and-Cut relations. 3-loop.

Relations with dim-2 numerator:



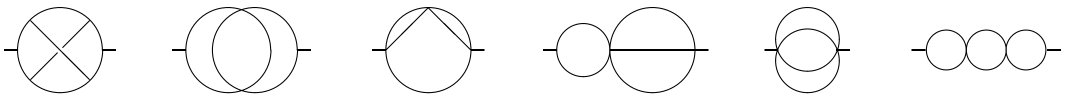


Relations with shrunken line:

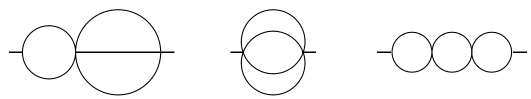


Glue-and-Cut relations. 3-loop. Results.

Glue-and-Cut relations <u>with</u> IBP relations reduce



to

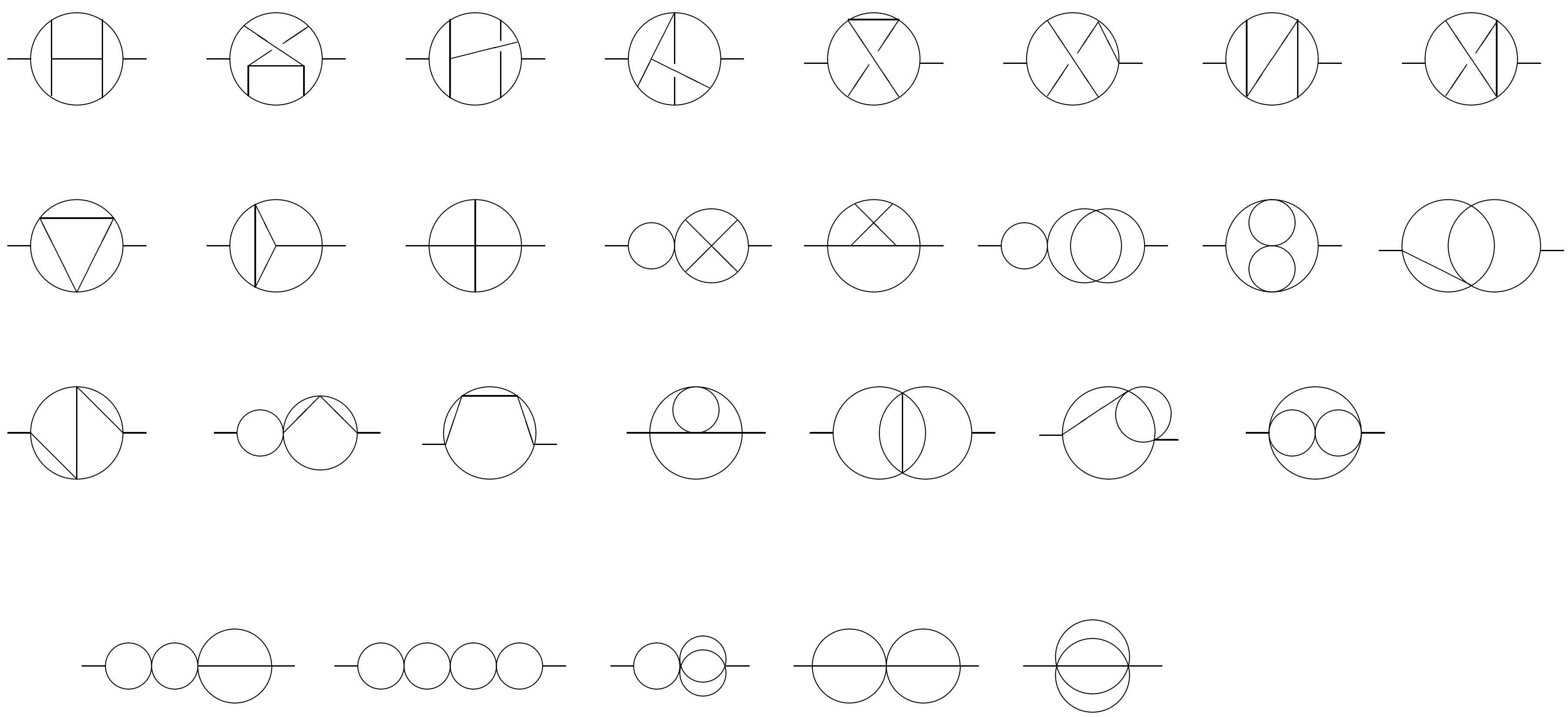


We have 3 "GaC+IBP irreducible" integrals made from 3 irrational "numbers":

$$1 \qquad \hat{\zeta}_3 = \zeta_3 + \frac{3}{2}\epsilon\,\zeta_4 \qquad \zeta_5$$

That is reduction is "complete".

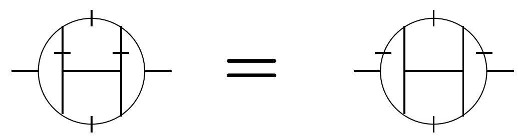
4-loop propagators. 28 IBP irreducible integrals:



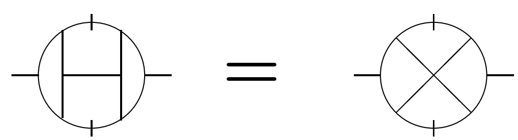
4-loop propagators.

Examples of Glue-and-Cut relations.

Relation with dim-4 numerators:



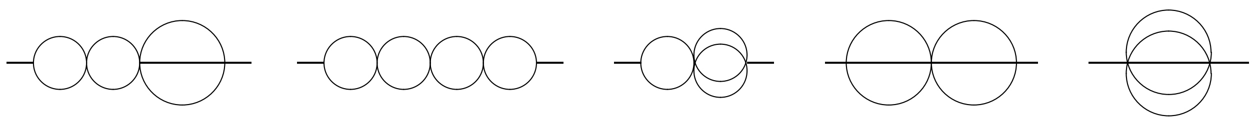
Relation with shrunken line:



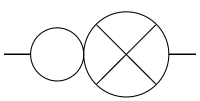
Glue-and-Cut relations. 4-loop. Results.

Glue-and-Cut relations <u>with</u> IBP reduce

28 "IBP-irreducible" integrals to only 5:



For example



$$= \frac{20\zeta_5}{\epsilon} + 2(34\zeta_3^2 - 40\zeta_5 + 25\zeta_6)$$

$$+2\epsilon(-136\zeta_3^2 + 102\zeta_3\zeta_4 + 40\zeta_5 - 100\zeta_6 + 225\zeta_7) + O(\epsilon^2)$$

Integrals consist of 8 irrational numbers:

$$1,\ \zeta_3,\ \zeta_4,\ \zeta_5,\ \zeta_6,\ \zeta_3^2,\ \zeta_7,\ \zeta_3\zeta_4.$$

ϵ-coefficients are dependent. "Basis" can be chosen as

$$1,\quad \hat{\zeta}_3,\quad \hat{\zeta}_5,\quad \hat{\zeta}_3{}^2,\quad \zeta_7,$$

where

$$\hat{\zeta}_3 = \zeta_3 + \frac{3}{2}\,\epsilon\,\zeta_4 - \frac{5}{2}\,\epsilon^3\,\zeta_6$$

$$\hat{\zeta}_5 = \zeta_5 + \frac{5}{2}\,\epsilon\,\zeta_6$$

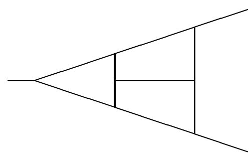
We have 5 "GaC+IBP irreducible" integrals constructed from 5 irrational "numbers", reduction is "complete".

5-loop propagators?

There is a hope that GaC+IBP relations will reduce all 5-loop propagators to already known 2-loop integrals with insertions (non-integer powers of propagators).

GaC relations can be obtained, but IBP reduction not yet available.

Vertex-type integrals?



$p_1^2 = 0$

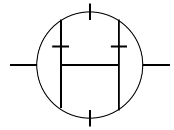
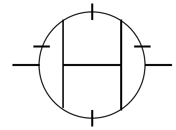
$p_2^2 = 0$

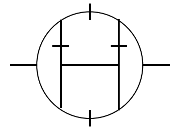
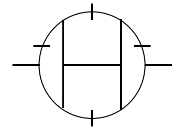
22 IBP-irreducible integrals, made from 9 irrationals:

$$1, \quad \zeta_2, \quad \zeta_3, \quad \zeta_4, \quad \zeta_5, \quad \zeta_2\zeta_3, \quad \zeta_6, \quad \zeta_2\zeta_4, \quad \zeta_3^2.$$

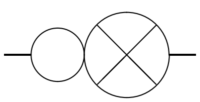
Some additional relations can exist.

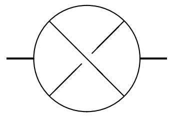
Glue-and-Cut ? It is unclear, what one should cut ...

Glue-and-Cut relation  $=$  $+\, O(\epsilon)$

"Finiteness" relation  ;  $=\ O(1)$

F-relations include GaC-relations with ϵ-precision less by 1 as the subset. Hence the results also have ϵ-precision less by 1, for example:

 $= \frac{20\zeta_5}{\epsilon} + 2(34\zeta_3^2 - 40\zeta_5 + 25\zeta_6) + O(\epsilon)$

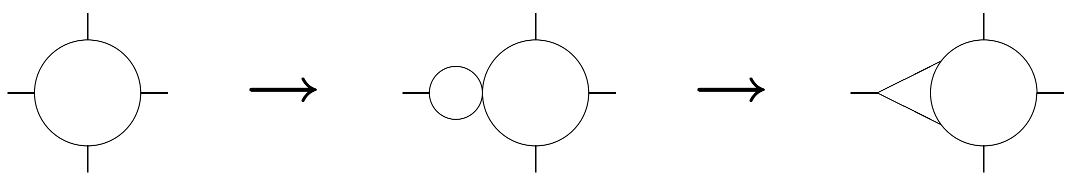
 $=$

3-loop GaC+IBP $\quad 20\hat{\zeta}_5$

4-loop F+IBP $\quad + 2\epsilon(34\hat{\zeta}_3^2 - 40\hat{\zeta}_5)$

4-loop GaC+IBP $\quad + 2\epsilon^2(-136\hat{\zeta}_3^2 + 40\hat{\zeta}_5 + 225\zeta_7)$

"Finiteness" is less powerful then "Glue-and-Cut", but can be used for even multi-leg-multi-scale diagram:



Price: to calculate L loop one need IBP for $L+1$ loop.

Very efficient IBP reduction is neccesary!

Summary

- **"Glue-and-Cut" relations reduce all 4-loop propagators to "watermelons".**
- **"Finiteness" relations hopefully will reduce number of "irreducible" integrals for more complicated cases.**
- **Efficient IBP-reduction algorithms can help even with "non-algebraic" problems (like calculation of IBP-irreducible integrals).**