New Methods for Feynman Integrals

Feynman Integral Reduction

A.V. Smirnov

Scientific Research Computing Center of Moscow State University

FIRE


FIRE

Feynman Integral REduction

Reduction problem for Feynman integrals

A given Feynman graph $\Gamma \to$ tensor reduction $\to$ various scalar Feynman integrals that have the same structure of the integrand with various distributions of powers of propagators

$$F(a_1, \ldots, a_n) \;=\; \int \cdots \int \frac{\mathrm{d}^d k_1 \ldots \mathrm{d}^d k_h}{E_1^{a_1} \ldots E_n^{a_n}} \,.$$

$d = 4 - 2\epsilon$; the denominators E_r are either quadratic or linear with respect to the loop momenta $p_i = k_i,\; i = 1, \ldots, h$ or to the independent external momenta $p_{h+1} = q_1, \ldots, p_{h+N} = q_N$ of the graph.

An old analytical strategy:
to evaluate, by some methods, every scalar Feynman integral generated by the given graph.

An old analytical strategy:
to evaluate, by some methods, every scalar Feynman integral generated by the given graph.

And what is already a traditional strategy:

to derive, without calculation, and then apply integration by parts (IBP) relations [K.G. Chetyrkin and F.V. Tkachov'81]
between the given family of Feynman integrals as recurrence relations.

An old analytical strategy:
to evaluate, by some methods, every scalar Feynman integral generated by the given graph.

And what is already a traditional strategy:

to derive, without calculation, and then apply integration by parts (IBP) relations [K.G. Chetyrkin and F.V. Tkachov'81]
between the given family of Feynman integrals as recurrence relations.

A general integral from the given family is expressed as a linear combination of some basic (master) integrals.

The whole problem of evaluation falls apart into two parts

The whole problem of evaluation falls apart into two parts

- constructing a reduction procedure

The whole problem of evaluation falls apart into two parts

- constructing a reduction procedure
- evaluating master integrals

The whole problem of evaluation falls apart into two parts

- constructing a reduction procedure
- evaluating master integrals

The talk is devoted to the first part of the problem.

Types of relations

Types of relations

Most commonly used relations: IBP relations.
IBP: [K.G. Chetyrkin and F.V. Tkachov'81]

$$\int \ldots \int \mathrm{d}^d k_1 \ldots \mathrm{d}^d k_n \frac{\partial}{\partial k_i} \left(p_j \frac{1}{E_1^{a_1} \ldots E_n^{a_n}} \right) = 0 \; ,$$

Types of relations

Most commonly used relations: IBP relations.
IBP: [K.G. Chetyrkin and F.V. Tkachov'81]

$$\int \ldots \int \mathrm{d}^d k_1 \ldots \mathrm{d}^d k_n \frac{\partial}{\partial k_i} \left(p_j \frac{1}{E_1^{a_1} \ldots E_n^{a_n}} \right) = 0 \,,$$

$\longrightarrow$

$$\sum \alpha_i F(a_1 + b_{i,1}, \ldots, a_n + b_{i,n}) = 0 \,,$$

Types of relations

Most commonly used relations: IBP relations.
IBP: [K.G. Chetyrkin and F.V. Tkachov'81]

$$\int \ldots \int \mathrm{d}^d k_1 \ldots \mathrm{d}^d k_n \frac{\partial}{\partial k_i} \left(p_j \frac{1}{E_1^{a_1} \ldots E_n^{a_n}} \right) = 0 \, ,$$

$\longrightarrow$

$$\sum \alpha_i F(a_1 + b_{i,1}, \ldots, a_n + b_{i,n}) = 0 \, ,$$

Lorentz-invariance (LI) identities [T. Gehrmann and E. Remiddi'00]

Types of relations

Most commonly used relations: IBP relations.
IBP: [K.G. Chetyrkin and F.V. Tkachov'81]

$$\int \ldots \int \mathrm{d}^d k_1 \ldots \mathrm{d}^d k_n \frac{\partial}{\partial k_i} \left(p_j \frac{1}{E_1^{a_1} \ldots E_n^{a_n}} \right) = 0 \,,$$

$\longrightarrow$

$$\sum \alpha_i F(a_1 + b_{i,1}, \ldots, a_n + b_{i,n}) = 0 \,,$$

Lorentz-invariance (LI) identities [T. Gehrmann and E. Remiddi'00]

$\longrightarrow$ they are a consequence of IBPs [R. Lee'08]

Types of relations

symmetry relations, e.g.,

$$F(a_1,\ldots,a_n) = (-1)^{d_1 a_1+\ldots d_n a_n} F(a_{\sigma(1)},\ldots,a_{\sigma(n)}),$$

Types of relations

symmetry relations, e.g.,

$$F(a_1, \ldots, a_n) = (-1)^{d_1 a_1 + \ldots d_n a_n} F(a_{\sigma(1)}, \ldots, a_{\sigma(n)}),$$

Boundary conditions:

$$F(a_1, a_2, \ldots, a_n) = 0 \text{ when } a_{i_1} \leq 0, \ldots a_{i_k} \leq 0$$

for some subsets of indices i_j;

Types of relations

symmetry relations, e.g.,

$$F(a_1,\ldots,a_n) = (-1)^{d_1 a_1 + \ldots d_n a_n} F(a_{\sigma(1)},\ldots,a_{\sigma(n)}),$$

Boundary conditions:

$$F(a_1,a_2,\ldots,a_n) = 0 \text{ when } a_{i_1} \leq 0,\ldots a_{i_k} \leq 0$$

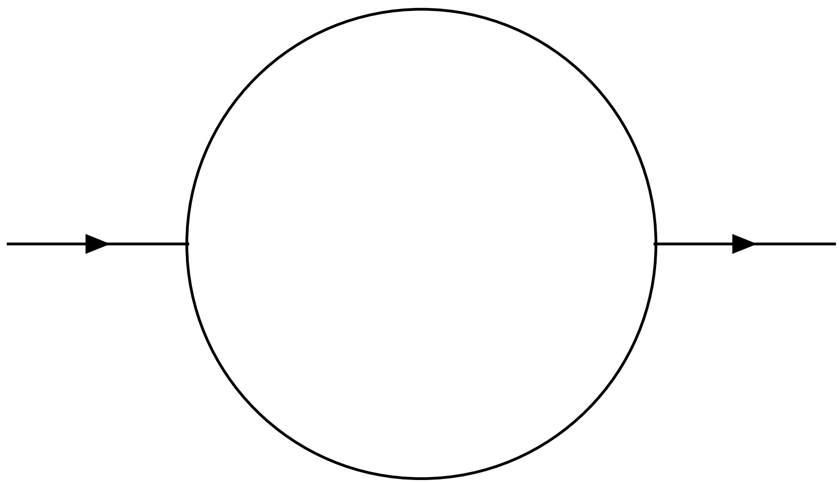
for some subsets of indices i_j;

parity conditions,. . .

Manual reduction example

Massless one-loop propagator Feynman integrals

$$F(a_1,a_2) = \int \frac{\mathrm{d}^d k}{(k^2)^{a_1}[(q-k)^2]^{a_2}} \; .$$



$a_1 \geq 1$, $a_2 \geq 1$

Reduction problem for Feynman integrals

$$\int \mathrm{d}^d k \frac{\partial}{\partial k} \cdot k \frac{1}{(k^2)^{a_1}[(q-k)^2]^{a_2}} = 0$$

Reduction problem for Feynman integrals

$$\int \mathrm{d}^d k \frac{\partial}{\partial k} \cdot k \frac{1}{(k^2)^{a_1}[(q-k)^2]^{a_2}} = 0$$

$$0 = dF(a_1, a_2) -$$

Reduction problem for Feynman integrals

$$\int \mathrm{d}^d k \frac{\partial}{\partial k} \cdot k \frac{1}{(k^2)^{a_1}[(q-k)^2]^{a_2}} = 0$$

$$0 = dF(a_1, a_2) -$$

$$-2a_1 F(a_1, a_2) -$$

Reduction problem for Feynman integrals

$$\int \mathsf{d}^d k \frac{\partial}{\partial k} \cdot k \frac{1}{(k^2)^{a_1}[(q-k)^2]^{a_2}} = 0$$

$$0 = dF(a_1, a_2) -$$

$$-2a_1 F(a_1, a_2) -$$

$$-a_2(q^2 F(a_1, a_2+1) - F(a_1-1, a_2+1) - F(a_1, a_2))$$

$$\int \mathrm{d}^d k \frac{\partial}{\partial k} \cdot k \frac{1}{(k^2)^{a_1}[(q-k)^2]^{a_2}} = 0$$

$$0 = dF(a_1, a_2) -$$
$$-2a_1 F(a_1, a_2) -$$
$$-a_2(q^2 F(a_1, a_2+1) - F(a_1-1, a_2+1) - F(a_1, a_2))$$

$$F(a_1, a_2) \;=\; -\frac{1}{(a_2-1)q^2}\left[(d - 2a_1 - a_2 + 1)F(a_1, a_2 - 1)\right.$$
$$\left. -(a_2 - 1)F(a_1 - 1, a_2)\right]$$

when $a_2 \neq 1$

To reduce the remaining integrals we use the symmetry condition $F(a_1, a_2) = F(a_2, a_1)$ (just to show an alternative way, we used only one IBP out of two, so IBPs are enough to do the reduction).

To reduce the remaining integrals we use the symmetry condition $F(a_1, a_2) = F(a_2, a_1)$ (just to show an alternative way, we used only one IBP out of two, so IBPs are enough to do the reduction).

Substituting the symmetry condition into the IBP used above we obtain:

$$F(a_1, 1) = -\frac{d - a_1 - 1}{(a_1 - 1)q^2} F(a_1 - 1, 1)$$

for $a_1 \geq 1$

Any integral can be recursively represented as a coefficient times $F(1, 1)$

Relations between Feynman integrals lead to a possibility to express given Feynman integrals in terms of other Feynman integrals.

Relations between Feynman integrals lead to a possibility to express given Feynman integrals in terms of other Feynman integrals.

We have to name certain integrals irreducible (master) and aim to reduce any other integral to those. An attempt to formalize the definition of master integrals was made in

[A.V. Smirnov, JHEP 0604 (2006) 026].

Relations between Feynman integrals lead to a possibility to express given Feynman integrals in terms of other Feynman integrals.

We have to name certain integrals irreducible (master) and aim to reduce any other integral to those. An attempt to formalize the definition of master integrals was made in

[A.V. Smirnov, JHEP 0604 (2006) 026].

The notion of the master (irreducible) integral $\longrightarrow$

Relations between Feynman integrals lead to a possibility to express given Feynman integrals in terms of other Feynman integrals.

We have to name certain integrals irreducible (master) and aim to reduce any other integral to those. An attempt to formalize the definition of master integrals was made in

[A.V. Smirnov, JHEP 0604 (2006) 026].

The notion of the master (irreducible) integral $\rightarrow$

a priority between the points $(a_1, \ldots, a_n) \rightarrow$

Relations between Feynman integrals lead to a possibility to express given Feynman integrals in terms of other Feynman integrals.

We have to name certain integrals irreducible (master) and aim to reduce any other integral to those. An attempt to formalize the definition of master integrals was made in

[A.V. Smirnov, JHEP 0604 (2006) 026].

The notion of the master (irreducible) integral $\rightarrow$

a priority between the points $(a_1, \ldots, a_n) \rightarrow$

an ordering.

How to choose an ordering?

How to choose an ordering?

Feynman integrals are simpler, from the analytic point of view, if they have more non-positive indices.

How to choose an ordering?

Feynman integrals are simpler, from the analytic point of view, if they have more non-positive indices.

Solving IBP relations by hand $\rightarrow$ reducing indices to zero

How to choose an ordering?

Feynman integrals are simpler, from the analytic point of view, if they have more non-positive indices.

Solving IBP relations by hand $\rightarrow$ reducing indices to zero

$\rightarrow$ we come to the notion of sectors

Sectors ('topologies'):
2^n regions labelled by subsets $\nu \subseteq \{1,\ldots,n\}$:
σ_ν = $\{(a_1,\ldots,a_n) : a_i > 0$ if $i \in \nu\,,\;\; a_i \leq 0$ if $i \not\in \nu\}$

Sectors ('topologies'):
2^n regions labelled by subsets $\nu \subseteq \{1, \ldots, n\}$:
$\sigma_\nu = \{(a_1, \ldots, a_n) : a_i > 0$ if $i \in \nu,\ \ a_i \leq 0$ if $i \notin \nu\}$

A sector is σ_ν said to be lower than a sector σ_μ if $\nu \subset \mu$

Sectors ('topologies'):
2^n regions labelled by subsets $\nu \subseteq \{1,\ldots,n\}$:
σ_ν = $\{(a_1,\ldots,a_n) : a_i > 0$ if $i \in \nu\,,\;\; a_i \leq 0$ if $i \notin \nu\}$

A sector is σ_ν said to be lower than a sector σ_μ if $\nu \subset \mu$

$F(a_1,\ldots,a_n) \succ F(a'_1,\ldots,a'_n)$ if the sector of $(a'_1,\ldots,a'_n)$ is lower than the sector of $(a_1,\ldots,a_n)$.

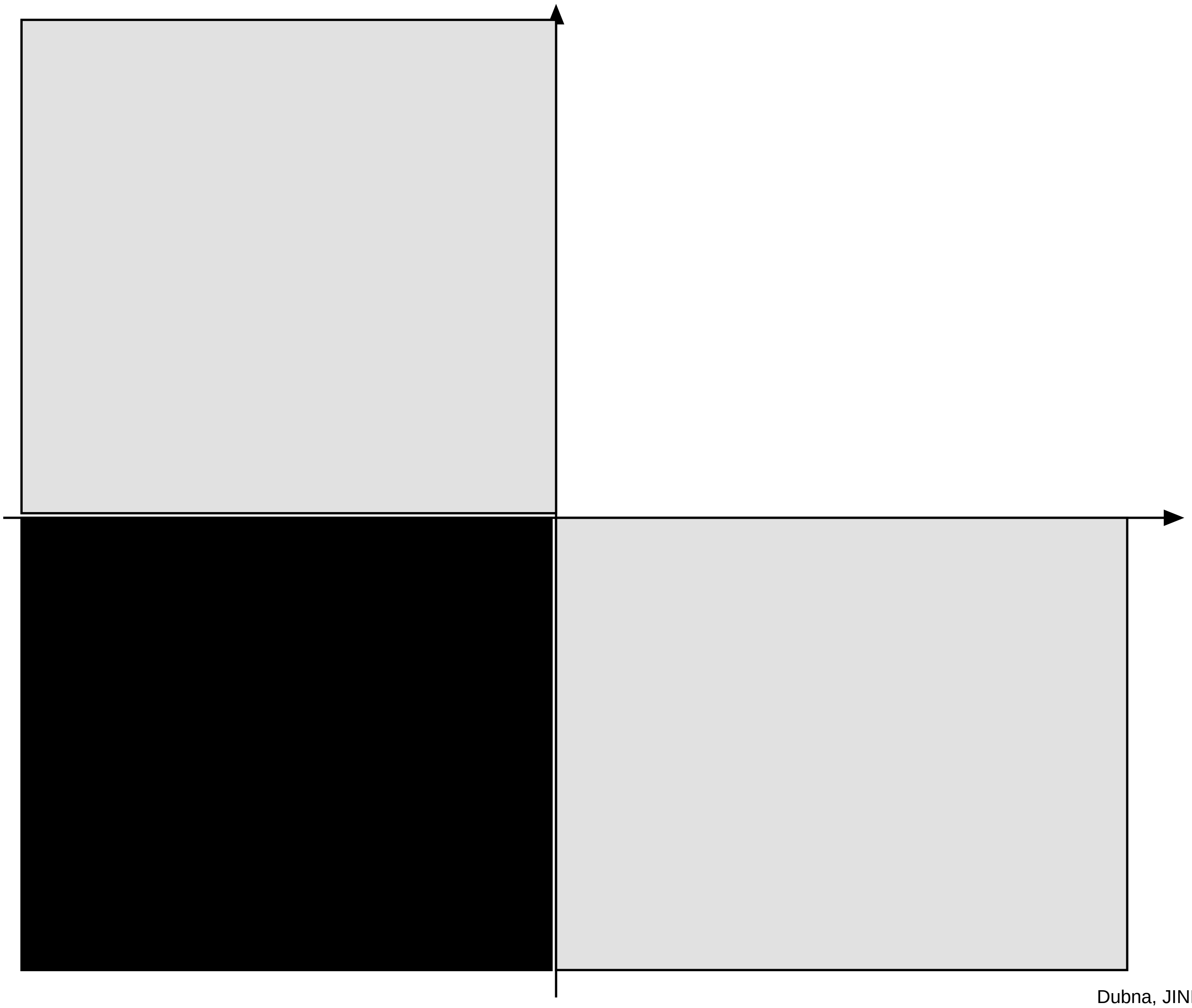
Sectors ('topologies'):
2^n regions labelled by subsets $\nu \subseteq \{1,\ldots,n\}$:
$\sigma_\nu = \{(a_1,\ldots,a_n) : a_i > 0$ if $i \in \nu, \; a_i \leq 0$ if $i \notin \nu\}$

A sector is σ_ν said to be lower than a sector σ_μ if $\nu \subset \mu$

$F(a_1,\ldots,a_n) \succ F(a'_1,\ldots,a'_n)$ if the sector of $(a'_1,\ldots,a'_n)$ is lower than the sector of $(a_1,\ldots,a_n)$.

To define an ordering completely introduce it in some way inside the sectors (to be discussed later).

Reduction problem for Feynman integrals



Laporta algorithm

[S. Laporta and E. Remiddi'96; S. Laporta'00; T. Gehrmann and E. Remiddi'01]

'When increasing the total power of the denominator and numerator, the total number of IBP equations grows faster than the number of independent Feynman integrals. Therefore this system of equations sooner or later becomes overdetermined, and one obtains the possibility to perform a reduction to master integrals'

Various implementations:

- first public version `AIR` [C. Anastasiou and A. Lazopoulos'04]

Various implementations:

- first public version `AIR` [C. Anastasiou and A. Lazopoulos'04]
- several private versions [T. Gehrmann and E. Remiddi, M. Czakon, P. Marquard and D. Seidel, Y. Schröder, C. Sturm, A. Pak, V. Velizhanin . . .]

Various implementations:

- first public version `AIR` [C. Anastasiou and A. Lazopoulos'04]
- several private versions [T. Gehrmann and E. Remiddi, M. Czakon, P. Marquard and D. Seidel, Y. Schröder, C. Sturm, A. Pak, V. Velizhanin . . .]
- new public version `FIRE` [A.Smirnov'08]
  not only a Laporta algorithm!

But initially FIRE originated from the idea to construct bases in all sectors.

But initially FIRE originated from the idea to construct bases in all sectors.

A *basis* in a sector is an iterative instruction how to reduce all integrals in this sector except masters to lower integrals.

But initially FIRE originated from the idea to construct bases in all sectors.

A *basis* in a sector is an iterative instruction how to reduce all integrals in this sector except masters to lower integrals. How does one obtain bases?

And where does this word come from?

Sector bases

- The initial idea to reduce integrals manually also resulted in a set of reduction rules — now we can call them a manual basis

- The initial idea to reduce integrals manually also resulted in a set of reduction rules — now we can call them a manual basis
- Reduction using Gröbner bases: historically, suggested by O.V. Tarasov [O.V. Tarasov'98], reduce the problem to differential equations by introducing a mass for every line; $a_i \mathbf{i}^+ \rightarrow \frac{\partial}{\partial m_i^2}$

- The initial idea to reduce integrals manually also resulted in a set of reduction rules — now we can call them a manual basis
- Reduction using Gröbner bases: historically, suggested by O.V. Tarasov [O.V. Tarasov'98], reduce the problem to differential equations by introducing a mass for every line; $a_i \mathbf{i}^+ \to \frac{\partial}{\partial m_i^2}$
- Direct application of Groebner bases (without the use of differential equations) [V.P. Gerdt'04, 05]

Sector bases

- The initial idea to reduce integrals manually also resulted in a set of reduction rules — now we can call them a manual basis
- Reduction using Gröbner bases: historically, suggested by O.V. Tarasov [O.V. Tarasov'98], reduce the problem to differential equations by introducing a mass for every line; $a_i\mathbf{i}^+ \rightarrow \frac{\partial}{\partial m_i^2}$
- Direct application of Groebner bases (without the use of differential equations) [V.P. Gerdt'04, 05]
- s-bases — one more approach initially based on Gröbner bases [A.V. Smirnov and V.A. Smirnov'05]

- The initial idea to reduce integrals manually also resulted in a set of reduction rules — now we can call them a manual basis
- Reduction using Gröbner bases: historically, suggested by O.V. Tarasov [O.V. Tarasov'98], reduce the problem to differential equations by introducing a mass for every line; $a_i \mathbf{i}^+ \to \frac{\partial}{\partial m_i^2}$
- Direct application of Groebner bases (without the use of differential equations) [V.P. Gerdt'04, 05]
- s-bases — one more approach initially based on Gröbner bases [A.V. Smirnov and V.A. Smirnov'05]
- Ideas developed by R. Lee [R. Lee'08]

Basic idea of FIRE

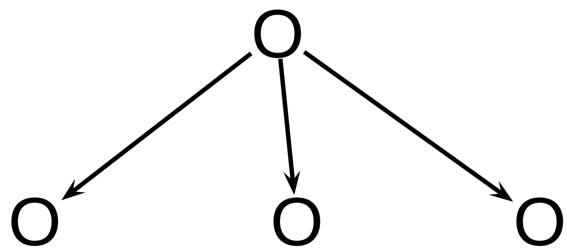
Suppose you have bases constructed everywhere.

Basic idea of FIRE

Suppose you have bases constructed everywhere.

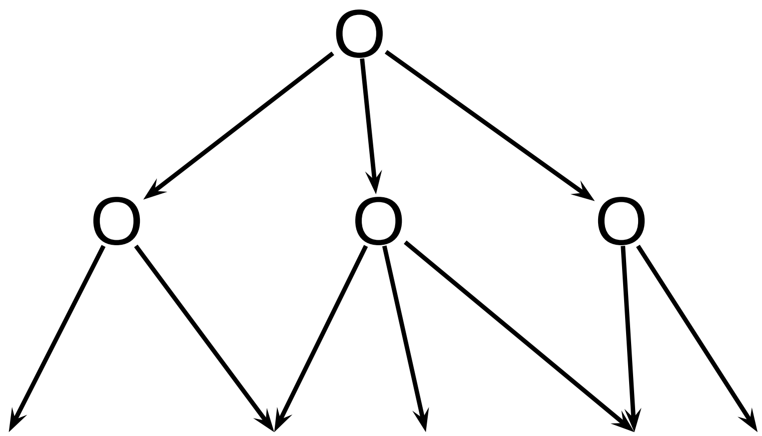
O

Suppose you have bases constructed everywhere.

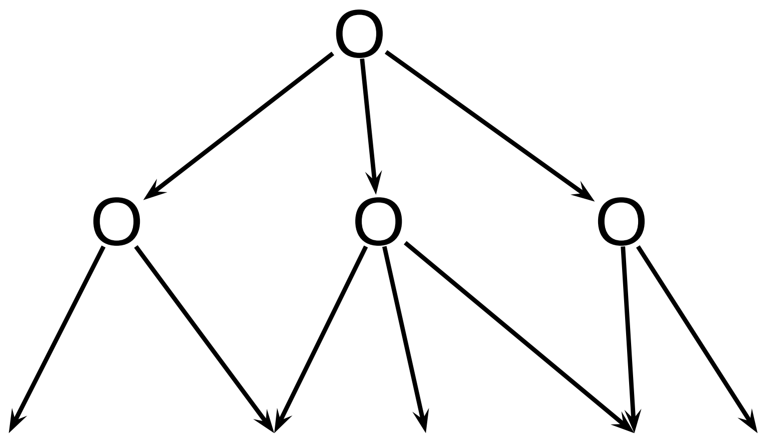


Basic idea of FIRE

Suppose you have bases constructed everywhere.

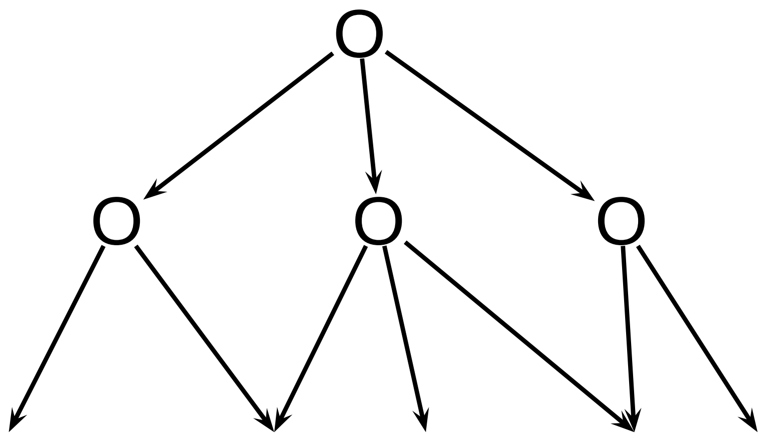


Suppose you have bases constructed everywhere.

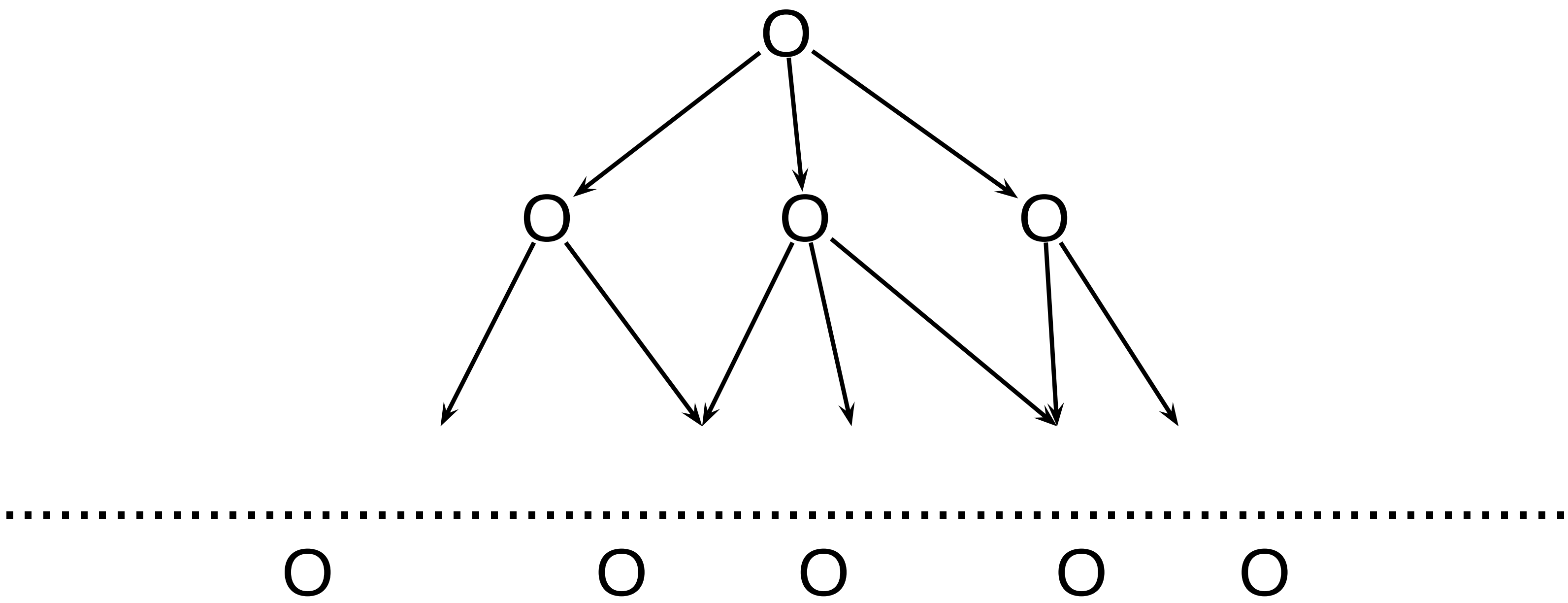


The number of integrals keeps growing, so you cannot substitute, but each expression is short

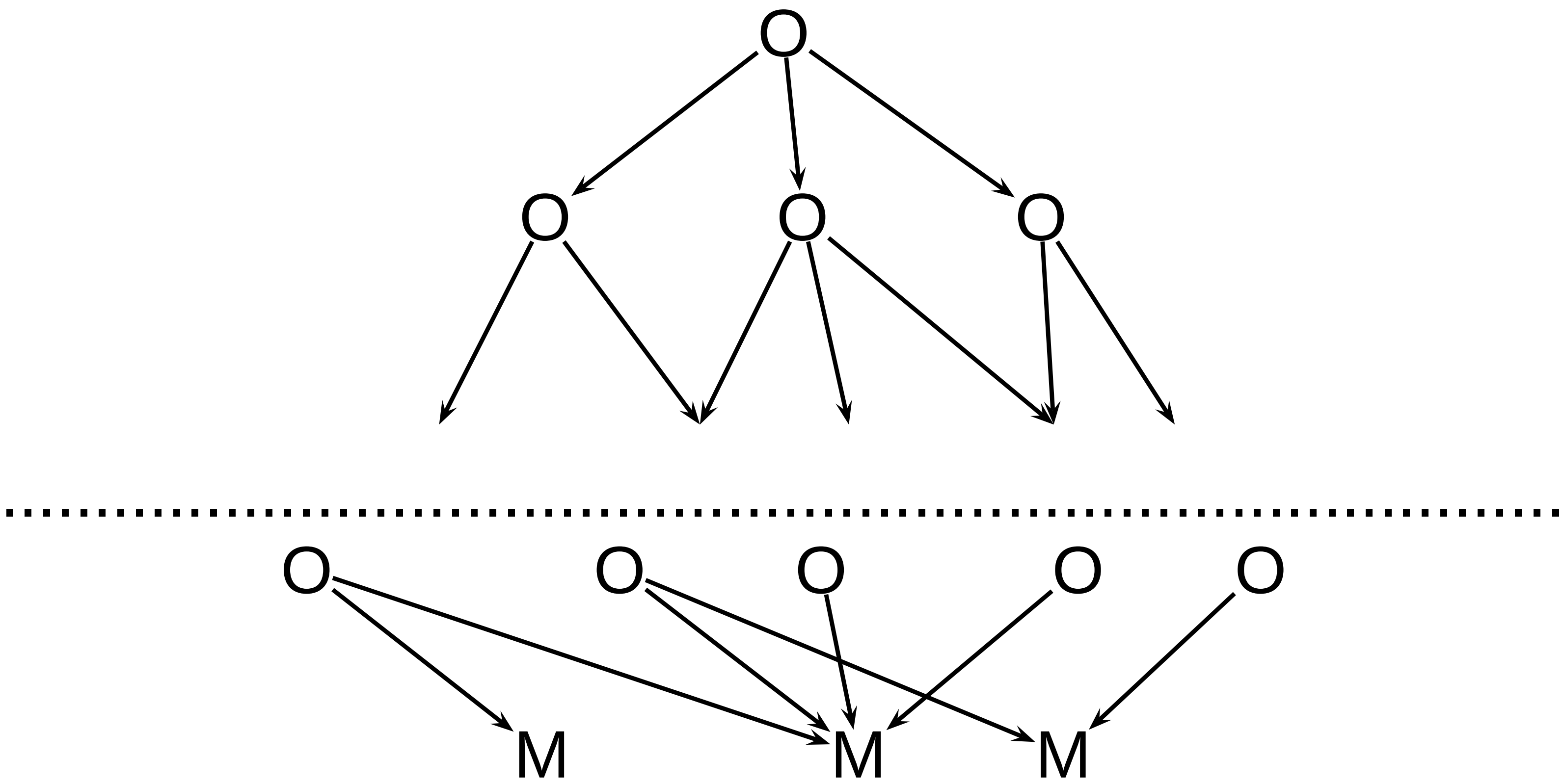
Suppose you have bases constructed everywhere.



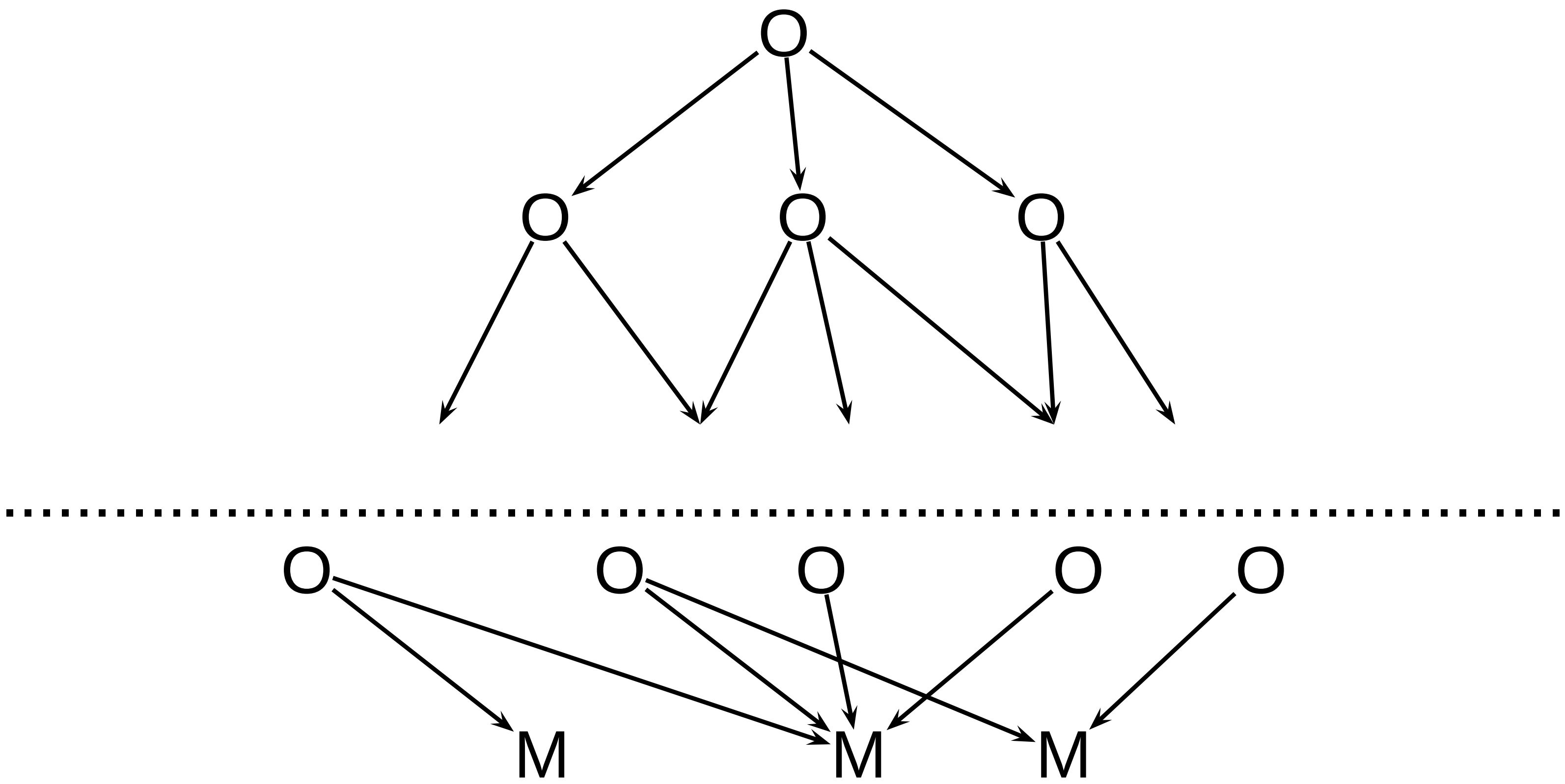
Suppose you have bases constructed everywhere.



Suppose you have bases constructed everywhere.



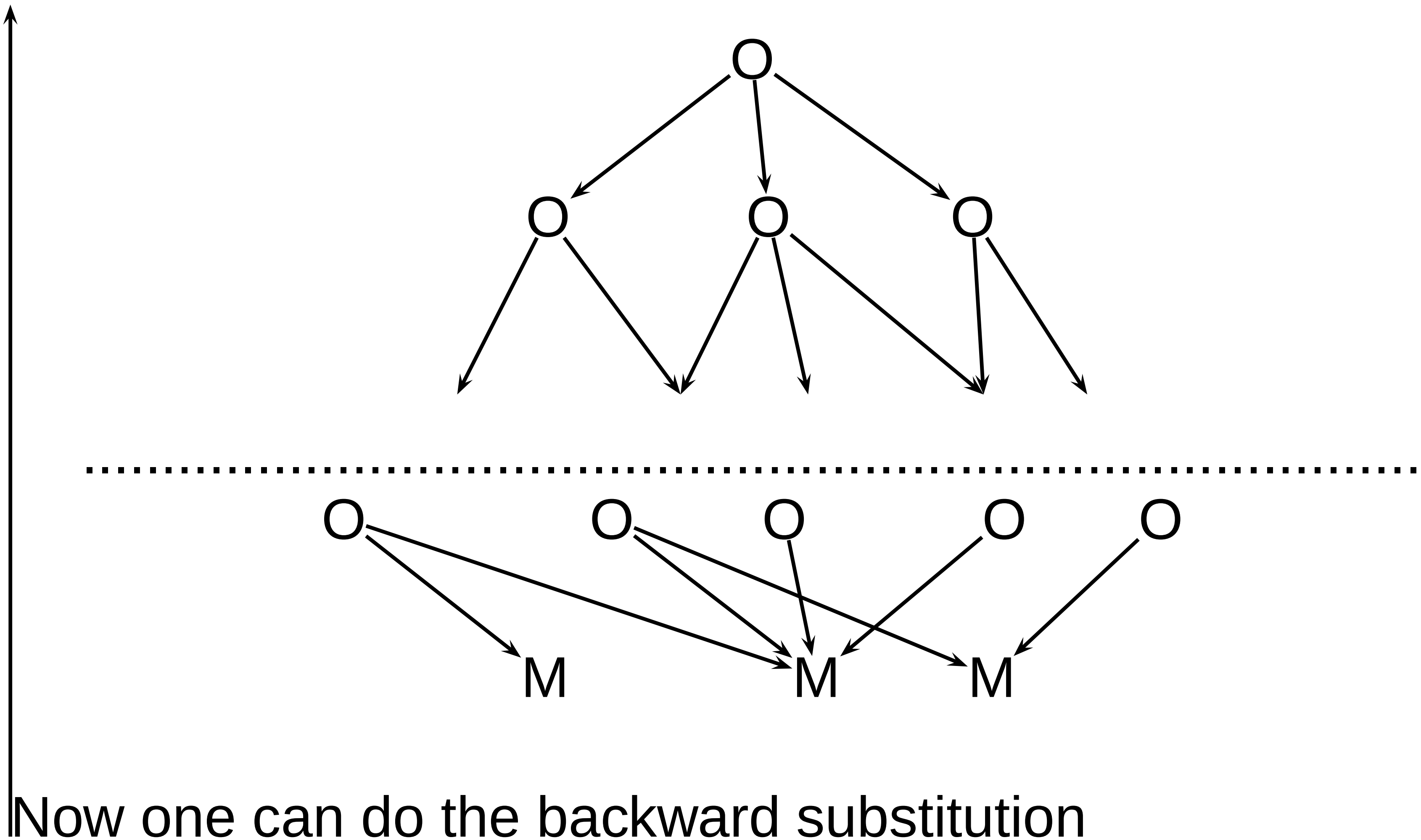
Suppose you have bases constructed everywhere.



Now one can do the backward substitution

Basic idea of FIRE

Suppose you have bases constructed everywhere.



Now one can do the backward substitution

- Unfortunately, one can’t construct the bases everywhere.

- Unfortunately, one can’t construct the bases everywhere.
- Still, an automatic construction in many sectors may be done by FIRE

- Unfortunately, one can’t construct the bases everywhere.
- Still, an automatic construction in many sectors may be done by FIRE
- Other improvements — region bases, manual rules

- Unfortunately, one can’t construct the bases everywhere.
- Still, an automatic construction in many sectors may be done by FIRE
- Other improvements — region bases, manual rules
- If nothing helps, FIRE starts the Laporta algorithm inside a sector

- Unfortunately, one can’t construct the bases everywhere.
- Still, an automatic construction in many sectors may be done by FIRE
- Other improvements — region bases, manual rules
- If nothing helps, FIRE starts the Laporta algorithm inside a sector
- Tail-masking has to be performed

Another improvements:

Another improvements:

- Ideas of R. Lee to use less IBPs

Another improvements:

- Ideas of R. Lee to use less IBPs
- Usage of QLink to store large tables on disk

Another improvements:

- Ideas of R. Lee to use less IBPs
- Usage of QLink to store large tables on disk
- Usage of FLink to improve evaluation speed

But still... what are s-bases and how to construct them?

Let's try to give an idea...

Although I would need several talks like that to explain comprehensively what Gröbner bases are

Suppose first that we are interested in expressing any integral in the positive sector $\sigma_{\{1,\ldots,n\}}$ as a linear combination of a finite number of integrals in it.

$$\int \ldots \int \mathrm{d}^d k_1 \ldots \mathrm{d}^d k_N \frac{\partial}{\partial k_i} \left(p_j \frac{1}{E_1^{a_1} \ldots E_N^{a_N}} \right) = 0$$

$\longrightarrow$

$$\sum c_i F(a_1 + b_{i,1}, \ldots, a_n + b_{i,n}) = 0$$

The left-hand sides of IBP relations can be expressed in terms of operators of multiplication by the indices a_i and shift operators $Y_i = \mathbf{i}^+, Y_i^- = \mathbf{i}^-$, where

$$(Y_i^{\pm} \cdot F)(a_1, \ldots, a_n) = F(a_1, \ldots, a_i \pm 1, \ldots, a_n)$$

The left-hand sides of IBP relations can be expressed in terms of operators of multiplication by the indices a_i and shift operators $Y_i = \mathbf{i}^+, Y_i^- = \mathbf{i}^-$, where

$$(Y_i^{\pm} \cdot F)(a_1, \ldots, a_n) = F(a_1, \ldots, a_i \pm 1, \ldots, a_n)$$

Thus, one can choose certain elements f_i corresponding to IBP relations and write

$$(f_i \cdot F)(a_1, \ldots, a_n) = 0$$

The choice is not unique, we will get rid of Y_i^-

For example, the relation we had before

$$0 = dF(a_1, a_2) - 2a_1 F(a_1, a_2) -$$
$$-a_2(q^2 F(a_1, a_2 + 1) - F(a_1 - 1, a_2 + 1) - F(a_1, a_2))$$

For example, the relation we had before

$$0 = dF(a_1, a_2) - 2a_1 F(a_1, a_2) -$$
$$-a_2(q^2 F(a_1, a_2 + 1) - F(a_1 - 1, a_2 + 1) - F(a_1, a_2))$$

can be rewritten as

$$((d - 2A_1 - A_2(q^2 Y_2^+ - Y_1^- Y_2^+)) \cdot F)(a_1, a_2) = 0$$

For example, the relation we had before

$$0 = dF(a_1, a_2) - 2a_1 F(a_1, a_2) -$$

$$-a_2(q^2 F(a_1, a_2+1) - F(a_1-1, a_2+1) - F(a_1, a_2))$$

can be rewritten as

$$((d - 2A_1 - A_2(q^2 Y_2^+ - Y_1^- Y_2^+)) \cdot F)(a_1, a_2) = 0$$

or after multiplying by Y_1^+ as

$$((dY_1^+ - 2(A_1 - 1)Y_1^+ - A_2(q^2 Y_2^+ Y_1^+ - Y_2^+)) \cdot F)(a_1, a_2) = 0$$

The algebra $\mathcal{A}^0$ generated by shift operators Y_i^+ and multiplication operators A_i. It acts on the field of functions $\mathcal{F}$ of n integer variables.
The ideal $\mathcal{I}$ of IBP relations generated by the elements f_i.
For any element $X \in \mathcal{I}$ we have

$$(XF)(1, 1, \ldots, 1) = 0 \, .$$

The algebra $\mathcal{A}^0$ generated by shift operators Y_i^+ and multiplication operators A_i. It acts on the field of functions $\mathcal{F}$ of n integer variables.
The ideal $\mathcal{I}$ of IBP relations generated by the elements f_i.
For any element $X \in \mathcal{I}$ we have

$$(XF)(1,1,\ldots,1) = 0\ .$$

Also we have

$$F(a_1,a_2,\ldots,a_n) = (Y_1^{a_1-1}\ldots Y_n^{a_n-1}F)(1,1,\ldots,1)$$

The algebra $\mathcal{A}^0$ generated by shift operators Y_i^+ and multiplication operators A_i. It acts on the field of functions $\mathcal{F}$ of n integer variables.
The ideal $\mathcal{I}$ of IBP relations generated by the elements f_i.
For any element $X \in \mathcal{I}$ we have

$$(XF)(1,1,\ldots,1) = 0\;.$$

Also we have

$$F(a_1, a_2, \ldots, a_n) = (Y_1^{a_1-1} \ldots Y_n^{a_n-1} F)(1,1,\ldots,1)$$

The idea of the algorithm is to reduce the element in the right-hand side of the equation using the elements of the ideal $\mathcal{I}$.

Suppose we are interested in an integral

$$F(a_1, a_2, \ldots, a_n) = (Y_1^{a_1-1} \ldots Y_n^{a_n-1} F)(1, 1, \ldots, 1)$$

Suppose we are interested in an integral

$$F(a_1, a_2, \ldots, a_n) = (Y_1^{a_1-1} \ldots Y_n^{a_n-1} F)(1, 1, \ldots, 1)$$

The reduction problem $\rightarrow$
reduce the monomial $Y_1^{a_1-1} \ldots Y_n^{a_n-1}$ modulo the ideal of the IBP relations

$$Y_1^{a_1-1} \ldots Y_n^{a_n-1} = \sum r_i f_i + \sum c_{i_1,\ldots,i_n} Y_1^{i_1-1} \ldots Y_n^{i_n-1}$$

Suppose we are interested in an integral

$$F(a_1, a_2, \ldots, a_n) = (Y_1^{a_1-1} \ldots Y_n^{a_n-1} F)(1, 1, \ldots, 1)$$

The reduction problem $\rightarrow$
reduce the monomial $Y_1^{a_1-1} \ldots Y_n^{a_n-1}$ modulo the ideal of the IBP relations

$$Y_1^{a_1-1} \ldots Y_n^{a_n-1} = \sum r_i f_i + \sum c_{i_1,\ldots,i_n} Y_1^{i_1-1} \ldots Y_n^{i_n-1}$$

Apply to F at $a_1 = 1, \ldots, a_n = 1$ to obtain

$$F(a_1, a_2, \ldots, a_n) = \sum c_{i_1,\ldots,i_n} F(i_1, i_2, \ldots, i_n)\,,$$

where integrals on the right-hand side are “master integrals”.

To do the reduction we need an ordering of monomials of operators Y_i or, similarly, an ordering of points $(a_1, \ldots, a_n)$ in the sector:

For two monomials $M_1 = Y_1^{i_1-1} \ldots Y_n^{i_n-1}$ and $M_2 = Y_1^{j_1-1} \ldots Y_n^{j_n-1}$

$(M_1 \cdot F)(1, \ldots, 1) \succ (M_2 \cdot F)(1, \ldots, 1)$ if and only if $M_1 \succ M_2$

Then the reduction procedure becomes similar to the division of polynomials. But one needs to introduce a proper ordering.

Orderings on the algebra of operators

Monomials $\rightarrow$ degrees (sets of n non-negative integers).
We require the following properties:

i) for any $a \in \mathbb{N}^n$ not equal to $(0, \ldots 0)$ one has $(0, \ldots 0) \prec a$
ii) for any $a, b, c \in \mathbb{N}^n$ one has $a \prec b$ if and only if $a + c \prec b + c$.

Orderings on the algebra of operators

Monomials $\to$ degrees (sets of n non-negative integers).
We require the following properties:

i) for any $a \in \mathbb{N}^n$ not equal to $(0, \ldots 0)$ one has $(0, \ldots 0) \prec a$
ii) for any $a, b, c \in \mathbb{N}^n$ one has $a \prec b$ if and only if $a + c \prec b + c$.

E.g., lexicographical ordering:
A set $(i_1, \ldots, i_n)$ is higher than a set $(j_1, \ldots, j_n)$,
$(i_1, \ldots, i_n) \succ (j_1, \ldots, j_n)$
if there is $l \leq n$ such that $i_1 = j_1$, $i_2 = j_2$, . . . , $i_{l-1} = j_{l-1}$ and $i_l > j_l$.

Orderings on the algebra of operators

Monomials $\to$ degrees (sets of n non-negative integers).
We require the following properties:

i) for any $a \in \mathbb{N}^n$ not equal to $(0, \ldots 0)$ one has $(0, \ldots 0) \prec a$
ii) for any $a, b, c \in \mathbb{N}^n$ one has $a \prec b$ if and only if $a + c \prec b + c$.

E.g., lexicographical ordering:
A set $(i_1, \ldots, i_n)$ is higher than a set $(j_1, \ldots, j_n)$,
$(i_1, \ldots, i_n) \succ (j_1, \ldots, j_n)$
if there is $l \leq n$ such that $i_1 = j_1$, $i_2 = j_2$, ..., $i_{l-1} = j_{l-1}$ and $i_l > j_l$.

Degree-lexicographical ordering: $(i_1, \ldots, i_n) \succ (j_1, \ldots, j_n)$ if $\sum i_k > \sum j_k$, or $\sum i_k = \sum j_k$ and $(i_1, \ldots, i_n) \succ (j_1, \ldots, j_n)$ in the sense of the lexicographical ordering.

An ordering can be defined by a matrix.

Lexicographical, degree-lexicographical and reverse degree-lexicographical ordering for $n = 5$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- But the reduction does not always lead to a reasonable number of irreducible integrals $\rightarrow$ one has to build a special basis first.

- But the reduction does not always lead to a reasonable number of irreducible integrals $\rightarrow$ one has to build a special basis first.
- Having elements with lowest possible degrees $\leftrightarrow$ obtaining master integrals with minimal possible degrees.

- But the reduction does not always lead to a reasonable number of irreducible integrals $\rightarrow$ one has to build a special basis first.
- Having elements with lowest possible degrees $\leftrightarrow$ obtaining master integrals with minimal possible degrees.
- One needs to build special bases $\rightarrow$ an algorithm based on the Buchberger algorithm - S-polynomials, reductions.

- But the reduction does not always lead to a reasonable number of irreducible integrals $\longrightarrow$ one has to build a special basis first.
- Having elements with lowest possible degrees $\longleftrightarrow$ obtaining master integrals with minimal possible degrees.
- One needs to build special bases $\longrightarrow$ an algorithm based on the Buchberger algorithm - S-polynomials, reductions.
- Moreover, we must keep in mind that we are interested in integrals not only in the positive sector.

Our algorithm [A.S.& V.S'05] : to build a set of special bases of the ideal of IBP relations (s-bases).

- sectors
  σ_ν = $\{(a_1, \ldots, a_n) : a_i > 0$ if $i \in \nu\,, \quad a_i \leq 0$ if $i \notin \nu\}$
- In the sector $\sigma_{\{1,\ldots,n\}}$, consider Y_i as basic operators.
  In the sector σ_ν, consider Y_i for $i \in \nu$ and Y_i^- for other i as basic operators.
- Construct sector bases (s-bases), rather than Gröbner bases for all the sectors.
  An s-basis for a sector σ_ν is a set of elements of a basis which provides the possibility of a reduction to master integrals *and* integrals whose indices lie in *lower* sectors, i.e. $\sigma_{\nu'}$ for $\nu' \subset \nu$. (It is most complicated to construct s-bases for minimal sectors.)

- The construction — close to the Buchberger algorithm but it can be terminated when the 'current' basis already provides us the needed reduction.
- The basic operations are the same, i.e. calculating S-polynomials and reducing them modulo current basis, with a chosen ordering.

After constructing s-bases for all non-trivial sectors one obtains a recursive (with respect to the sectors) procedure to evaluate $F(a_1, \ldots, a_n)$ at any point and thereby reduce a given integral to master integrals.

Description of the algorithm (implemented in `Mathematica`): [A.V. Smirnov, JHEP 0604 (2006) 026]

Main things you need to know about s-bases:

Main things you need to know about s-bases:

- They are a method to work with IBPs before substituting indices

Main things you need to know about s-bases:

- They are a method to work with IBPs before substituting indices
- Having as more bases as possible in nice

Main things you need to know about s-bases:

- They are a method to work with IBPs before substituting indices
- Having as more bases as possible in nice
- You can't construct them everywhere, but in many sectors they can be constructed automatically

Main things you need to know about s-bases:

- They are a method to work with IBPs before substituting indices
- Having as more bases as possible in nice
- You can't construct them everywhere, but in many sectors they can be constructed automatically
- The code `SBases` is public

Everything available at http://science.sander.su