Differential Gröbner bases technique and dimensional recurrences – new tools for calculating Feynman diagrams.

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Outline of Talk:

- Generalized recurrence relations
- Basics of differential Gröbner basis technique
- One-loop examples of DGB technique
- DGB for two-loop propagators
- Dimensional recurrences and nonsingular bases of Feynman integrals
- Solution of dimensional recurrences for two-loop sunrise diagram (arXiv:hep-ph/0603227)

Most frequently used technique to calculate Feynman diagrams: Integration By Parts method:

F.V. Tkachov, Phys.Lett. **100B** (1981) 65;

K.G. Chetyrkin and F.V. Tkachov, Nucl.Phys.**192** (1981) 159.

$$G^{(d)}(\{s_j\},\{m_k\}) = \int d^dk_1 \ldots \int d^dk_L \frac{1}{(\overline{k}_1^2 - m_1^2)^{\nu_1} \ldots (\overline{k}_N^2 - m_N^2)^{\nu_N}}$$

where

$$\overline{k}_{j\mu} = \sum_{n=1}^{L} \omega_{jn} k_{n,\mu} + \sum_{m=1}^{E} \eta_{jm} q_{m,\mu}$$

Use relation

$$\int d^dk_1 \ldots \int d^dk_L \frac{\partial}{\partial k_{j\mu}} \frac{k_{j\mu}}{(\overline{k}_1^2 - m_1^2)^{\nu_1} \ldots (\overline{k}_N^2 - m_N^2)^{\nu_N}} = 0$$

differentiate with respect to momentum k, represent scalar products of momenta in terms of scalar factors standing in the denominator and external momenta:

$$k_1 q_1 = \frac{1}{2}\{[(k_1+q_1)^2 - m_1^2] - [k_1^2 - m_1^2] - q_1^2\}.$$

$==>$ Problem of irreducible numerators

Another approach is possible. Use relation

$$\int d^dk_1 \ldots \int d^dk_L \frac{\partial}{\partial k_{j\mu}} \frac{k_{k\mu}}{(\overline{k}_1^2 - m_1^2)^{\nu_1} \ldots (\overline{k}_N^2 - m_N^2)^{\nu_N}} = 0$$

differentiate with respect to momentum k, represent tensor integrals emerging after differentiation in terms of integrals with shifted dimension

$$\int d^dk_1 \ldots \int d^dk_L \frac{k_{1\mu} \ldots k_{N\nu}}{(\overline{k}_1^2 - m_1^2)^{\nu_1} \ldots (\overline{k}_N^2 - m_N^2)^{\nu_N}} =$$

$$T(q, \partial, \mathbf{d}^+) \int d^dk_1 \ldots \int d^dk_L \frac{1}{(\overline{k}_1^2 - m_1^2)^{\nu_1} \ldots (\overline{k}_N^2 - m_N^2)^{\nu_N}}$$

where $\mathbf{d}^+ G^{(d)} = G^{(d+2)}$ and $\partial_j = \frac{\partial}{\partial m_j^2}$ A general formula for $T(q, \partial, \mathbf{d}^+)$ was given by O.T. ,Phys.Rev. **D54** . For any L-loop multi-leg Feynman integral a relation was derived:

$$G^{(d-2)} = D(\partial) G^{(d)}$$

where $D(\partial)$ is differential operator of order L.

$$\mathrm{Examples:} \qquad D(\partial) = \partial_1 + \partial_2, \qquad \mathrm{one-loop\ \ self-energy}$$

$$D(\partial) = \partial_1\partial_2 + \partial_1\partial_3 + \partial_2\partial_3 \quad \mathrm{two-loop\ \ sunrise}$$

Derivation of $T(q, \partial, \mathbf{d}^+)$ for one-loop integrals

$$I_{n,r}^{(d)} = \int \frac{d^d q}{\pi^{d/2}} \frac{q_{\mu_1} \ldots q_{\mu_r}}{[(q-p_1)^2 - m_1^2]^{\nu_1} \ldots [(q-p_n)^2 - m_n^2]^{\nu_n}}$$

Introduce auxiliary vector a and write

$$q_{\mu_1} \ldots q_{\mu_r} = \frac{1}{i^r} \frac{\partial}{\partial a_{\mu_1}} \ldots \frac{\partial}{\partial a_{\mu_r}} \exp[iaq]|_{a=0} \,.$$

Instead of tensor integrals we consider integral

$$I_n^{(d)}(a) = \int \frac{d^d q}{\pi^{d/2}} \frac{e^{i(aq)}}{[(q-p_1)^2 - m_1^2]^{\nu_1} \ldots [(q-p_n)^2 - m_n^2]^{\nu_n}}$$

Use

$$\frac{1}{(k^2 - m^2 + i\epsilon)^\nu} = \frac{i^{-\nu}}{\Gamma(\nu)} \int_0^\infty d\alpha \alpha^{\nu-1} \exp[i\alpha(k^2 - m^2 + i\epsilon)].$$

Perform Gaussian integration

$$\int d^d k \exp[i(Ak^2 + 2(pk))] = i \left(\frac{\pi}{iA}\right)^{\frac{d}{2}} \exp\left[-\frac{ip^2}{A}\right].$$

The final result

$$I_n^{(d)}(a) = \frac{1}{i^{(d/2-1)}} \prod_{j=1}^{n} \frac{i^{-\nu_j}}{\Gamma(\nu_j)} \int_0^\infty \ldots \int_0^\infty \frac{d\alpha_j \alpha_j^{\nu_j-1}}{D(\alpha)^{d/2}} \exp\left[i \left(\frac{Q(\alpha,\{p_s\},a)}{D(\alpha)} - \sum_{l=1}^{n} \alpha_l (m_l^2 - \imath\epsilon) \right) \right]$$

where

$$D(\alpha) = \sum_{j=1}^{n} \alpha_j,$$

$$Q(\alpha,\{p_s\},a) = Q(\alpha,\{p_s\}) + \sum_{k=1}^{n-1} (ap_k)\alpha_k - \frac{1}{4}a^2$$

$$\frac{\partial}{\partial a_\mu} ==> \frac{1}{D(\alpha)^{\frac{d}{2}}} \left[\frac{\sum_{k=1}^{n-1} (p_{k\mu})\alpha_k - \frac{1}{2}a_\mu}{D(\alpha)} \right]$$

$$d \rightarrow d+2, \quad \alpha_k \rightarrow \frac{\partial}{\partial m_k^2}$$

$$T_{\mu_1\ldots\mu_r}(q,\partial,\mathbf{d}^+) = \frac{1}{i^r} \prod_{j=1}^{r} \frac{\partial}{\partial a_{\mu_j}} \exp\left[- \left(i \sum_{k=1}^{N-1} (ap_k)\partial_k - \frac{1}{4}a^2 \right) \mathbf{d}^+ \right] \Bigg|_{a=0}$$

Examples of generalized recurrence relations

The one-loop propagator type diagram with massive particles:

$$I^{(d)}_{\nu_1\nu_2}(q^2,m_1^2,m_2^2)=\int\frac{d^dk_1}{[i\pi^{d/2}]}P^{\nu_1}_{k_1,m_1}P^{\nu_2}_{k_1-q,m_2}.$$

In this case $D(\alpha)=\alpha_1+\alpha_2$, $P_{k,m}=1/(k^2-m^2)$ and therefore

$$\begin{aligned} I^{(d-2)}_{\nu_1\nu_2}(q^2,m_1^2,m_2^2) &= -(\partial_1+\partial_2)I^{(d)}_{\nu_1\nu_2}(q^2,m_1^2,m_2^2) \\ &= -\nu_1 I^{(d)}_{\nu_1+1\ \nu_2}(q^2,m_1^2,m_2^2)-\nu_2 I^{(d)}_{\nu_1\ \nu_2+1}(q^2,m_1^2,m_2^2). \end{aligned}$$

We can get another recurrence relation connecting integrals with different d. From the identity:

$$\int d^dk_1\frac{\partial}{\partial k_{1\mu}}\left[(k_1+q)_\mu P^{\nu_1}_{k_1,m_1}P^{\nu_2}_{k_1-q,m_2}\right]\equiv 0,$$

we obtain:

$$\begin{aligned} &\nu_1\int\frac{d^dk_1}{[i\pi^{d/2}]}(qk_1)P^{\nu_1+1}_{k_1,m_1}P^{\nu_2}_{k_1-q,m_2} \\ &=\left(\frac{d}{2}-\nu_1\right)I^{(d)}_{\nu_1\nu_2}-\nu_2 I^{(d)}_{\nu_1-1\ \nu_2+1}-\nu_1 m_1^2 I^{(d)}_{\nu_1+1\ \nu_2}+\nu_2(q^2-m_1^2)I^{(d)}_{\nu_1\ \nu_2+1}. \end{aligned}$$

The integral with the scalar product (qk_1) can be written in terms of scalar integrals with shifted d:

$$\int \frac{d^dk_1}{[i\pi^{d/2}]} \, (qk_1) \, P^{\nu_1+1}_{k_1,m_1} P^{\nu_2}_{k_1-q,m_2} = \nu_2 q^2 I^{(d+2)}_{\nu_1+1\;\nu_2+1}.$$

Inserting this expression into previous equation we obtain:

$$\nu_1\nu_2 q^2 I^{(d+2)}_{\nu_1+1\;\nu_2+1} - \left(\frac{d}{2} - \nu_1\right) I^{(d)}_{\nu_1\nu_2} + \nu_2 I^{(d)}_{\nu_1-1\;\nu_2+1} + \nu_1 m_1^2 I^{(d)}_{\nu_1+1\;\nu_2} - \nu_2(q^2 - m_1^2) I^{(d)}_{\nu_1\;\nu_2+1} \equiv 0.$$

In addition to the above relations two more relations can be obtained from the traditional method of integration by parts:

$$2\nu_2 m_2^2 I^{(d)}_{\nu_1\;\nu_2+1} + \nu_1 I^{(d)}_{\nu_1-1\;\nu_2+1} + \nu_1(m_1^2 + m_2^2 - q^2) I^{(d)}_{\nu_1+1\;\nu_2} - (d - 2\nu_2 - \nu_1) I^{(d)}_{\nu_1\nu_2} = 0,$$

$$\nu_1 I^{(d)}_{\nu_1+1\;\nu_2-1} - \nu_2 I^{(d)}_{\nu_1-1\;\nu_2+1} - \nu_1(m_1^2 - m_2^2 + q^2) I^{(d)}_{\nu_1+1\;\nu_2} - \nu_2(m_1^2 - m_2^2 - q^2) I^{(d)}_{\nu_1\;\nu_2+1}$$

$$+ (\nu_2 - \nu_1) I^{(d)}_{\nu_1\nu_2} = 0.$$

At $m_1 = 0,\ m_2 = m$, the relations connecting integrals $I^{(d)}_{\nu_1\nu_2}(q^2, 0, m^2)$ with different d are:

$$\nu_1\nu_2 q^2 I^{(d+2)}_{\nu_1+1\ \nu_2+1}(q^2,0,m^2) - \left(\frac{d}{2} - \nu_1\right) I^{(d)}_{\nu_1\nu_2}(q^2,0,m^2) + \nu_2 I^{(d)}_{\nu_1-1\ \nu_2+1}(q^2,0,m^2) - \nu_2 q^2 I^{(d)}_{\nu_1\ \nu_2+1}(q^2,0,m^2) \equiv 0,$$

$$I^{(d-2)}_{\nu_1\nu_2}(q^2,0,m^2) + \nu_1 I^{(d)}_{\nu_1+1\ \nu_2}(q^2,0,m^2) + \nu_2 I^{(d)}_{\nu_1\ \nu_2+1}(q^2,0,m^2) \equiv 0.$$

The integral $I^{(d)}_{\nu_1\nu_2}(q^2, 0, m^2)$ is proportional to the Gauss hypergeometric function :

$$I^{(d)}_{\nu_1\nu_2}(q^2,0,m^2) = (-1)^{\nu_1+\nu_2} \frac{\Gamma(\nu_1+\nu_2-\frac{d}{2})\Gamma(\frac{d}{2}-\nu_1)}{(m^2)^{\nu_1+\nu_2-\frac{d}{2}}\Gamma(\frac{d}{2})\Gamma(\nu_2)}\, {}_2F_1\left[\begin{array}{ll} \nu_1, \nu_1+\nu_2-\frac{d}{2}\,; & \frac{q^2}{m^2} \\ \frac{d}{2}\,; & \end{array}\right].$$

There are fifteen relations of Gauss between contiguous functions ${}_2F_1$:

$${}_2F_1\,(a \pm 1, b \pm 1, c \pm 1, x)$$

In our case $\quad a = \nu_1,\ \ b = \nu_1 + \nu_2 - d/2,\ \ c = d/2.$

Substituting explicit result into IBP relations we find they reproduce only six relations of Gauss. The reason - parameter c of ${}_2F_1$ in IBP relations does not change, therefore all corresponding relations cannot be reproduced.

Generalized recurrence relations give new relations for Feynman integrals!!!

They extend number of recurrency parameters: $\{\nu_j\} \to \{\nu_j, d\}$

Dimensionality relations can be used for:

- calculating tensor integrals
- fi nding bases of master integrals without kinematical singularities
- evaluating master integrals

Generalized recurrence relations connect integrals with different powers of propagators and also integrals with different dimensionality d. It is easy to write down a big number of integration by parts and generalized recurrence relations.

How to use these relations? What is the number of master integrals? Is there minimal number of relations which is enough to reduce all integrals to master integrals?

There is mathematical theory answering to these questions. This is

Theory of Gröbner bases.

The key element of these theory

Buchberger algorithm.

Differential Gröbner basis technique

1. **O.V. Tarasov** *“Reduction of Feynman graph amplitudes to a minimal set of basic integrals”* **,**

Acta Physica Polonica, v B29 (1998) 2655

2. **O. V. Tarasov,** *“Computation of Groebner bases for two-loop propagator type integrals,”*

Talk at ACAT-2003

Nucl. Instrum. Meth. A 534 (2004) 293 [arXiv:hep-ph/0403253].

Gröbner Basis is a nice set of recurrence relations or differential relations for Feynman integrals allowing to reduce large (in principle infinite) number of integrals in terms of finite number of integrals

Main steps of the algorithm:

- Tensor integrals express in terms of scalar ones with shifted space-time dimension

$$I_{\mu\nu\ldots} = T_{\mu\nu\ldots}(\partial, \mathbf{d}^{+}) I$$

- Scalar integrals with dots on lines represent as derivatives w.r.t. masses

$$\int \frac{d^dk_1 \ldots d^dk_L}{\ldots (k_1^2 - m_1^2)^{\nu_1}((k_1-p_1)^2 - m_2^2)^{\nu_2} \ldots}$$

$$= \frac{1}{(\nu_1-1)!(\nu_2-1)!} \frac{\partial^{\nu_1}}{\partial (m_i^2)^{\nu_1}} \frac{\partial^{\nu_2}}{\partial (m_j^2)^{\nu_2}}$$

$$\times \int \frac{d^dk_1 \ldots d^dk_L}{\ldots (k_1^2 - m_i^2)((k_1-p_1)^2 - m_j^2) \ldots} \Big|_{m_i^2 = m_1^2, m_j^2 = m_2^2},$$

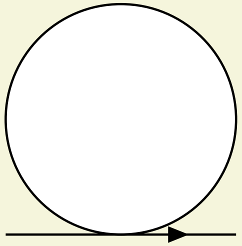
- For scalar integrals with different number of lines write down generalized recurrence relations and transform them into a system of differential equations
- Find differential Gröbner basis for this overdetermined system One can use Maple, Mathematica or other computer systems.
- Use relations from the Gröbner basis to reduce all possible integrals (i.e. higher order derivatives) in terms of fixed finite number of basic integrals (i.e. lower order derivatives)
- To reduce integrals $G^{(d+2j)}$ with shifted space-time dimension use relation:

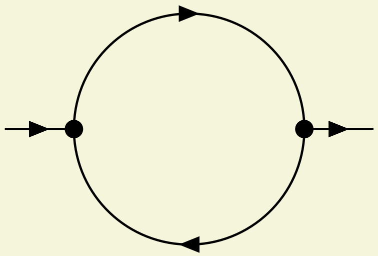
$$G^{(d-2)} = D(\partial_j)G^{(d)}$$

A very similar technique can be formulated without transformation to differential representation and introduction of different masses i.e. for integrals with different powers of propagators and with particular fixed masses.

In general case Gröbner basis for systems of recurrence relations has more parameters than differential Gröbner basis: additionally to masses, powers of propagators must be kept as parameters. Also number of terms in recurrence relations from the Gröbner basis is more than in differential Gröbner basis. It may be more effective for special kinematical configurations.

Example of DGB for one-loop integrals





The basis for tadpole integral consists of two relations:

$$\partial_i T_i^{(d)} = \frac{d-2}{2m_i^2} T_i^{(d)}, \qquad T_i^{(d+2)} = -\frac{2m_i^2}{d} T_i^{(d)}$$

where

$$\partial_j = \frac{\partial}{\partial m_j^2} \quad \text{and} \quad T_i^{(d)} = \frac{1}{i\pi^{(d/2)}} \int \frac{d^d k_1}{k_1^2 - m_i^2}.$$

The Gröbner basis for propagator type integral consists of three relations:

$$2\lambda_{ij} \partial_i I_{2,ij}^{(d)} = (3-d)(\partial_i \lambda_{ij}) I_{2,ij}^{(d)} - \frac{\partial \lambda_{ij}}{\partial p_{ij}} \frac{(d-2)}{2m_i^2} T_i^{(d)} + 2(d-2) T_j^{(d)},$$

$$2\lambda_{ij} \partial_j I_{2,ij}^{(d)} = (3-d)(\partial_j \lambda_{ij}) I_{2,ij}^{(d)} - \frac{\partial \lambda_{ij}}{\partial p_{ij}} \frac{(d-2)}{2m_j^2} T_j^{(d)} + 2(d-2) T_i^{(d)},$$

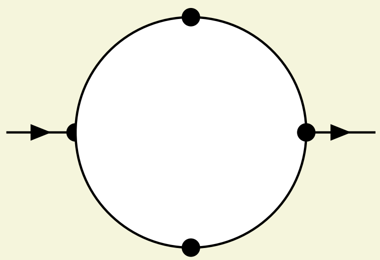
$$(d-1)g_{ij}I_{2,ij}^{(d+2)} = 2\lambda_{ij}I_{2,ij}^{(d)} + (\partial_i\lambda_{ij})T_j^{(d)} + (\partial_j\lambda_{ij})T_i^{(d)},$$

where

$$\lambda = -p_{ij}^2 - m_i^4 - m_j^4 + 2p_{ij}m_i^2 + 2p_{ij}m_j^2 + 2m_i^2m_j^2,$$

$$g_{ij} = -4p_{ij} = -4(p_i - p_j)^2.$$

Application of DGB to an integral



$$\frac{1}{i\pi^{d/2}}\int \frac{d^dk_1}{(k_1^2-m^2)^2((k_1-p_1)^2-m^2)^2} = \partial_i\partial_j \; I^{(d)}_{2,ij}\Big|_{m_i^2=m_j^2=m^2},$$

$$I^{(d)}_{2,ij} = \frac{1}{i\pi^{d/2}}\int \frac{d^dk_1}{(k_1^2-m_i^2)((k_1-p_1)^2-m_j^2)}.$$

$$\partial_i I^{(d)}_{2,ij} = f_1(m_i^2,m_j^2) I^{(d)}_{2,ij} + r_1(m_i^2,m_j^2),$$

$$\partial_j I^{(d)}_{2,ij} = f_2(m_i^2,m_j^2) I^{(d)}_{2,ij} + r_2(m_i^2,m_j^2),$$

$$\partial_j T_j^{(d)} = t_j T_j^{(d)}.$$

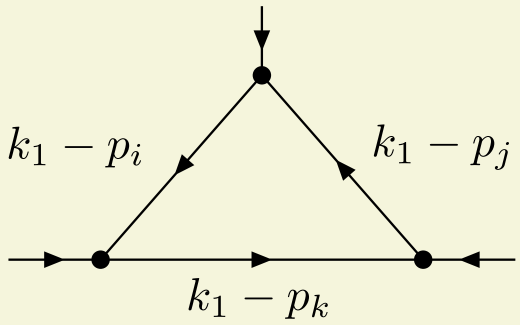
$$\begin{aligned}
\partial_j\partial_i I^{(d)}_{2,ij} &= \partial_j[f_1(m_i^2,m_j^2) I^{(d)}_{2,ij} + r_1(m_i^2,m_j^2)] \\
&= (\partial_j f_1) I^{(d)}_{2,ij} + f_1\partial_j I^{(d)}_{2,ij} + \partial_j r_1 \\
&= [\partial_j f_1 + f_1 f_2] I^{(d)}_{2,ij} + f_1 r_2 + \partial_j r_1
\end{aligned}$$

$$
\begin{aligned}
r_1 &= \frac{1}{2\lambda_{ij}}\left[-\frac{\partial \lambda_{ij}}{\partial p_{ij}}\frac{(d-2)}{2m_i^2}T_i^{(d)} + 2(d-2)T_j^{(d)}\right] = f_3 T_j^{(d)} + r_3 T_i^{(d)}, \\
r_2 &= f_4 T_i^{(d)} + r_4 T_j^{(d)}.
\end{aligned}
$$

$$
\begin{aligned}
\partial_j r_1 &= \partial_j [f_3 T_j^{(d)} + r_3 T_i^{(d)}] \\
&= (\partial_j f_3) T_j^{(d)} + f_3 \partial_j T_j^{(d)} + (\partial_j r_3) T_i^{(d)} \\
&= [(\partial_j f_3) + f_3 t_j] T_j^{(d)} + T_i^{(d)} \partial_j r_3.
\end{aligned}
$$

$$
\begin{aligned}
\partial_j \partial_i I_{2,ij}^{(d)} = [\partial_j f_1 + f_1 f_2] I_{2,ij}^{(d)} + [f_1 f_4 + \partial_j r_3] T_i^{(d)} \\
+ [r_4 f_1 + (\partial_j f_3) + f_3 t_j] T_j^{(d)}
\end{aligned}
$$

Vertex type integrals



The GB for 1-loop vertex integrals consists of 3 differential relations:

$$2\lambda_{ijk}\partial_i I^{(d)}_{3,ijk} = \frac{4-d}{2}(\partial_i\lambda_{ijk})I^{(d)}_{3,ijk} - 2(d-3)\left[\frac{p_{ij}}{\lambda_{ij}}\frac{\partial\lambda_{ijk}}{\partial y_{ik}}\,I^{(d)}_{2,ij}\right.$$
$$\left. + \frac{p_{ik}}{\lambda_{ik}}\frac{\partial\lambda_{ijk}}{\partial y_{ij}}\,I^{(d)}_{2,jk} + \frac{2p_{jk}}{\lambda_{jk}}\frac{\partial\lambda_{ijk}}{\partial y_{ii}}\,I^{(d)}_{2,jk}\right] + (d-2)\left[\frac{(\partial_j\lambda_{ijk})}{8m_k^2\lambda_{ik}}T^{(d)}_k\right.$$
$$\left. + \frac{1}{4m_i^2}\left(\frac{\partial_k\lambda_{ik}}{\lambda_{ik}}\frac{\partial\lambda_{ijk}}{\partial y_{ij}} + \frac{\partial_j\lambda_{ij}}{\lambda_{ij}}\frac{\partial\lambda_{ijk}}{\partial y_{ik}}\right)T^{(d)}_i + \frac{(\partial_k\lambda_{ijk})}{8m_j^2\lambda_{ij}}T^{(d)}_j\right],$$

$+2$ other relations by cyclic permutations

One dimensional recurrency relation

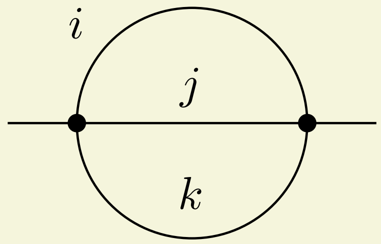
$$(d-2)g_{ijk}I^{(d+2)}_{3,ijk} = 2\lambda_{ijk}I^{(d)}_{3,ijk} + (\partial_i\lambda_{ijk})I^{(d)}_{2,jk} + (\partial_j\lambda_{ijk})I^{(d)}_{2,ik} + (\partial_k\lambda_{ijk})I^{(d)}_{2,ij}.$$

where

$$\begin{aligned}
\lambda_{ijk} = \quad & 2(p_{jk}+p_{ik}-p_{ij})(m_i^2 m_j^2 + p_{ij}m_k^2) \\
+ \quad & 2(p_{ik}+p_{ij}-p_{jk})(p_{jk}m_i^2 + m_k^2 m_j^2) \\
+ \quad & 2(p_{jk}+p_{ij}-p_{ik})(m_k^2 m_i^2 + p_{ik}m_j^2) \\
- \quad & 2m_j^4 p_{ik} - 2p_{ij}m_k^4 - 2m_i^4 p_{jk} - 2p_{ij}p_{ik}p_{jk},
\end{aligned}$$

$$g_{ijk} = \quad 2p_{ij}^2 - 4(p_{ik}+p_{jk})p_{ij} + 2(p_{ik}-p_{jk})^2.$$

Differential Gröbner bases for propagator integrals



$\equiv J_{3,ijk}^{(d)}$

Differential GB for $J_{3,ijk}^{(d)}$ integrals consists of 8 relations:

- 3 relations for $\partial_i \partial_j\,, \quad i \neq j$
- 3 relations for ∂_j^2
- 1- relation for $J_{3,ijk}^{(d+2)}$ in terms of d- dimensional integrals.
- 1- relation for $\partial_j J_{3,ijk}^{(d+2)}$ in terms of d dimensional integrals.

Explicit expressions for the Gröbner bases are:

$$2D_{ijk}\partial_i\partial_j J^{(d)}_{3,ijk} = 2h_{ijk}\partial_i J^{(d)}_{3,ijk} + 2h_{jik}\partial_j J^{(d)}_{3,ijk} + 4m_k^2\sigma_{ijk}\partial_k J^{(d)}_{3,ijk}$$

$$+ \frac{(d-2)^2}{4m_i^2m_j^2}[m_i^2\phi_{jik}T_j^{(d)}T_k^{(d)} + m_j^2\phi_{ijk}T_i^{(d)}T_k^{(d)} - 2\rho_{ijk}T_i^{(d)}T_j^{(d)}],$$

$$+ \frac{1}{2}(3d-8)(d-3)\phi_{kji}J^{(d)}_{3,ijk}$$

$$2m_i^2D_{ijk}\partial_i^2 J^{(d)}_{3,ijk} = m_i^2S_{ijk}\partial_i J^{(d)}_{3,ijk} + m_j^2S_{jik}\partial_j J^{(d)}_{3,ijk} + m_k^2S_{kij}\partial_k J^{(d)}_{3,ijk}$$

$$+ (d-4)D_{ijk}\partial_i J^{(d)}_{3,ijk} - (3d-8)(d-3)\rho_{ijk}J^{(d)}_{3,ijk}$$

$$+ \frac{(d-2)^2}{4}[\phi_{ijk}T_j^{(d)}T_k^{(d)} + \phi_{jik}T_i^{(d)}T_k^{(d)} + \phi_{kij}T_i^{(d)}T_j^{(d)}],$$

and 2 dimensional recurrency relations:

$$J_{3,ijk}^{(d+2)} = w_i \partial_i J_{3,ijk}^{(d)} + w_j \partial_j J_{3,ijk}^{(d)} + w_k \partial_k J_{3,ijk}^{(d)} + w_0 J_{3,ijk}^{(d)} + t_0,$$

$$\partial_i J_{3,ijk}^{(d+2)} = w_i^{(1)} \partial_i J_{3,ijk}^{(d)} + w_j^{(1)} \partial_j J_{3,ijk}^{(d)} + w_k^{(1)} \partial_k J_{3,ijk}^{(d)} + w_0^{(1)} J_{3,ijk}^{(d)} + t_1,$$

where $t_0,\ t_1$ are tadpole contributions and $w_j^{(k)},\ w_i$ are ratios of polynomials in momentum and masses.

Polynomial coeffi cients are

$$
\begin{aligned}
D_{ijk} &= [q^2-(m_i+m_j+m_k)^2][q^2-(m_i+m_j-m_k)^2] \\
&\quad [q^2-(m_i-m_j+m_k)^2][q^2-(m_i-m_j-m_k)^2], \\
\rho_{ijk} &= -q^6+3q^4(m_i^2+m_j^2+m_k^2) \\
&\quad -q^2[3(m_i^4+m_j^4+m_k^4)+2(m_i^2m_j^2+m_i^2m_k^2+m_j^2m_k^2)]+m_i^6+m_j^6+m_k^6 \\
&\quad -m_i^2(m_j^4+m_k^4)-m_j^2(m_i^4+m_k^4)-m_k^2(m_i^4+m_j^4)+10m_i^2m_j^2m_k^2, \\
\phi_{ijk} &= 4[q^4+2q^2(m_i^2-m_j^2-m_k^2)+(m_j^2-m_k^2)^2+m_i^2(2m_j^2+2m_k^2-3m_i^2)], \\
\sigma_{ijk} &= -\frac{1}{4}(d-3)[\phi_{ijk}+\phi_{jik}+2\phi_{kij}], \\
h_{ijk} &= -\frac{1}{2}(d-3)[m_k^2\phi_{ijk}+2m_i^2\phi_{kij}-2\rho_{ijk}], \\
S_{ijk} &= -(d-3)[m_k^2\phi_{jik}+m_j^2\phi_{kij}-4\rho_{ijk}].
\end{aligned}
$$

Differential GB provides the most optimal set of relations for calculating Feynman diagrams with masses!

In future experiments on LHC, ILC certainly 4 masses : $M_{W,Z}$, M_t, M_H and M_{SUSY} should be taken into account and therefore differential GB will be useful tool for calculating electroweak radiative corrections.

Most effective strategy for calculations will be:

- For integrals with 5,6,. . . lines use recurrence relations (usually easy to derive) taking into account equal or zero masses
- Integrals with 4 and less lines reduce to bases of integrals by using differential GB

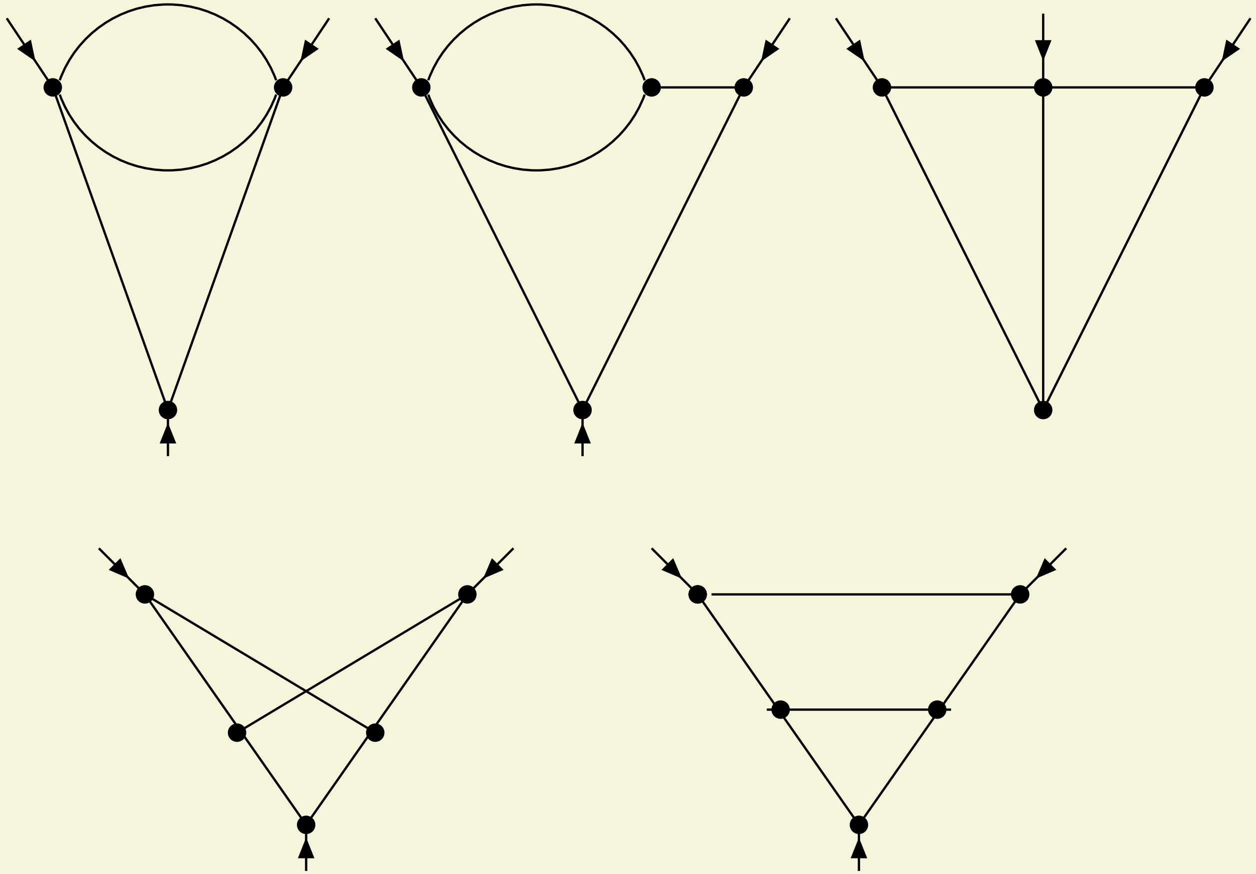
For integrals with 4 and more masses blind solving sets of IBP relations will be problematic: in 2-loop pure QED calculations progress was achieved by using specialized system Fermat!.

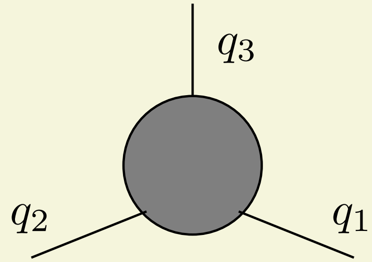
For higher order corrections with a few massive parameters recursive Gröbner bases technique will be more effective.

Differential GB technique and recursive GB technique can be complimentary.

Gröbner basis for two-loop vertex integrals with arbitrary external momenta and masses was constructed. Huge polynomial expressions were calculated once and forever. In real calculations one don't need to manipulate with these polynomials. Only at the final stage particular values of masses and external momenta should be substituted in polynomials and their derivatives.

Two − loop vertex type integrals





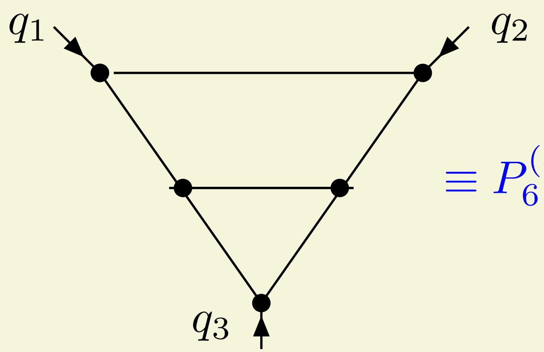
For particular external kinematics differential Gröbner bases essentailly simplifi es.

This is the case at $q_1^2 = q_2^2 = 0$, $q_3^2 \neq 0$ (needed for $H \to 2\gamma$ process) and $q_1^2 = q_2^2 = m^2$, $q_3^2 = 4m^2$ (needed in on-threshold calculations in $e^+e^- \to t\bar{t}$).

Computer implementation of differential Gröbner bases technique for these processes with arbitray masses inside loop is in progress.

Bypassing kinematical singularities via other dimensions

Let's consider the diagram:



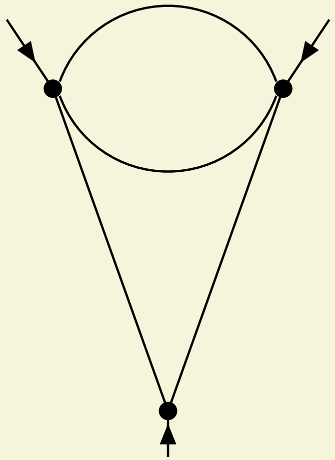
$\equiv P_6^{(d)}$

By using differential GB $P_6^{(d)}$ can be expressed in terms of other diagrams:

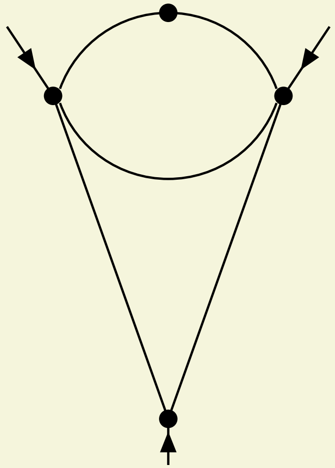
$$\frac{(4-d)}{2}P_6^{(d)} = \frac{1}{q_3^2}\Big[(d-3)I_3^{(d)}(q_1^2,q_2^2,q_3^2)I_2^{(d)}(q_3^2)$$
$$-R_2^{(d)}(q_1^2,q_2^2,q_3^2)+R_2^{(d)}(q_1^2,q_3^2,q_2^2)+R_2^{(d)}(q_2^2,q_3^2,q_1^2)$$
$$+\frac{(4-d)}{2}\left(P_5^{(d)}(q_1^2,q_3^2,q_2^2)+P_5^{(d)}(q_2^2,q_3^2,q_1^2)\right)\Big].$$

If we take the limit $q_3^2 \to 0$ we discover kinematical singularity

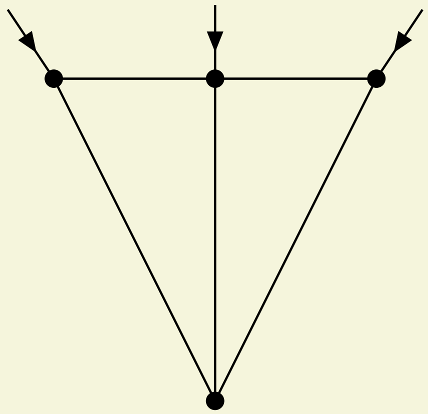
Two − loop vertex type integrals



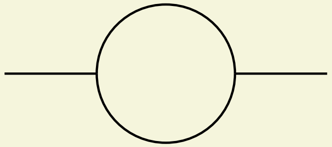
R_1



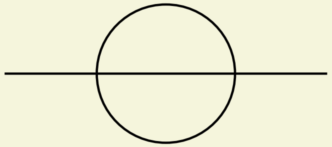
R_2



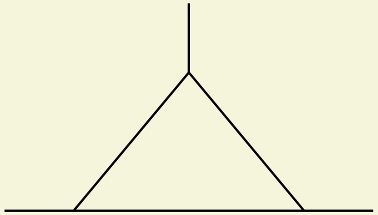
P_5



I_2



J_3



I_3

Key idea to solve the problem : choose another bases of master integrals: Transform d dimensional integrals to $d-2n$ or $d+2n$ dimensional integrals depending on the type of singularity

$$
\begin{aligned}
\Delta(d-4)^2(d-5)(3d-14)P_6^{(d)} = \\
-(3d-14)(d-6)q_3^2(q_1^4 P_5^{(d-2)}(q_1^2,q_3^2,q_2^2) + q_2^4 P_5^{(d-2)}(q_2^2,q_3^2,q_1^2)) \\
+[2q_3^2(d-5)+(q_2^2-q_1^2)(d-4)](2(d-5)q_2^2 R_1^{(d-2)}(q_1^2,q_3^2,q_2^2) + (3d-14)J_3^{(d-2)}(q_2^2)) \\
+[2q_3^2(d-5)+(q_1^2-q_2^2)(d-4)](2(d-5)q_1^2 R_1^{(d-2)}(q_2^2,q_3^2,q_1^2) + (3d-14)J_3^{(d-2)}(q_1^2)) \\
+2q_1^2(d-6)q_2^2q_3^2[R_2^{(d-2)}(q_1^2,q_2^2,q_3^2) + R_2^{(d-2)}(q_1^2,q_3^2,q_2^2) + R_2^{(d-2)}(q_2^2,q_3^2,q_1^2)] \\
+((5d-24)(q_2^2+q_2^2) - 2(d-5)q_3^2)[2(d-5)q_3^2 R_1^{(d-2)}(q_1^2,q_2^2,q_3^2) + (3d-14)J_3^{(d-2)}(q_3^2)] \\
-(3d-14)(d-5)q_3^2 I_2^{(d-2)}(q_3^2)[2q_1^2 I_2^{(d-2)}(q_1^2) + 2q_2^2 I_2^{(d-2)}(q_2^2) \\
+(q_2^2+q_1^2-q_3^2)I_2^{(d-2)}(q_3^2) - 2q_1^2q_2^2 V_3^{(d-2)}(q_1^2,q_2^2,q_3^2)],
\end{aligned}
$$

where $\Delta = q_1^4+q_2^4+q_3^4-2q_1^2q_2^2-2q_1^2q_3^2-2q_2^2q_3^2.$

Singular denominator disappear, now we can set $q_3^2=0$!!!

At $q_3^2 = 0$, $P_5^{(d-2)}$ terms drops out. By using relations

$$3\Delta(3d-8)(3d-4)(d-1)R_1^{(d+2)}(q_1^2,q_2^2,q_3^2) = q_3^2 Q_3[4(d-3)\Delta$$
$$+(d-4)q_1^2q_2^2]R_1^{(d)}(q_1^2,q_2^2,q_3^2) + 2q_1^2q_2^2q_3^2(\Delta+q_1^2q_2^2)\frac{(d-4)}{(d-3)}R_2^{(d)}(q_1^2,q_2^2,q_3^2)$$
$$+(3d-8)[2q_3^2(q_1^2-q_3^2)^2 - q_2^2(q_2^4+q_1^2q_3^2-q_1^2q_2^2-4q_2^2q_3^2+5q_3^4)]J_3^{(d)}(q_2^2)$$
$$+(3d-8)[2q_3^2(q_2^2-q_3^2)^2 - q_1^2(q_1^4-4q_1^2q_3^2+5q_3^4-q_1^2q_2^2+q_2^2q_3^2)]J_3^{(d)}(q_1^2)$$

$$3(3d-8)(d-2)\Delta R_2^{(d+2)}(q_1^2,q_2^2,q_3^2) = (\Delta+q_2^2Q_2)(3d-8)J_3(q_1^2)$$
$$+(\Delta+q_1^2Q_1)(3d-8)J_3(q_2^2) - 2q_3^2[(d-3)\Delta+(d-4)q_1^2q_2^2]R_1^{(d)}(q_1^2,q_2^2,q_3^2)$$
$$-q_1^2q_2^2q_3^2(d-4)Q_3/(d-3)R_2^{(d)}(q_1^2,q_2^2,q_3^2),$$

with $Q_1 = q_2^2+q_3^2-q_1^2$, $Q_2 = q_1^2+q_3^2-q_2^2$, $Q_3 = q_1^2+q_2^2-q_3^2$ we obtain:

$$(d-5)(q_1^2-q_2^2)P_6^{(d)}(q_1^2,q_2^2,0) = 3R_2^{(d)}(q_1^2,0,q_2^2) - 3R_2^{(d)}(q_2^2,0,q_1^2).$$

We assumed that dimensional regularization regularize IR and UV singularities,

$$\int \frac{d^Dk_1\ldots d^Dk_L}{D_1\ldots D_N}\bigg|_{q_i^2=0} = finite, \quad D = d, d-2, d-4, \ldots$$

IR singularities show up as additional poles in $\varepsilon = (4-d)/2$, i.e. $1/\varepsilon^4$,. . .

The algorithm was used to find on-shell value of different three point Green functions in QCD.

The general structure of the result for diagram was

$$G = \frac{Q(q_1^2, q_2^2, q_3^2)}{(q_1^2)^a (q_2^2)^b (q_3^2)^c \Delta^e}$$

In all cases after transforming to the new basis all q_j^2 disappeared from denominators.

Sometimes several shifts were needed.

No asymptotic expansions with complicated strategy of regions were needed!

The results were in complete agreement with the known calculations by A. Davydychev and P. Osland.

Since there are no factors q_j^2 in denominators

$$One\ can\ expect\ numerical\ stability\ in\ the\ vicinity\ q_j^2 = 0.$$

Similar effect one can find at the one-loop level in the algorithm by Ansgar Denner and Stefan Dittmaier. Special choice of basis integrals prevent appearance spurious singular Gram determinants.

By using the limiting procedure for arbitrary d and one zero momentum we obtained diagram by diagram agreement in calculation of two-loop vertex $< VVA >$ correlator with results by D.R.T. Jones, J.P.Leveille, Nucl. Phys. B206 (1982) 473.

For the case when $q_1^2 = q_2^2 = 0$ we obtained diagram by diagram agreement for two-loop ghost-gluon and three-gluon vertices with the ε expanded result by A.Davydychev and P.Osland, Phys.Rev. D61 (1998) 1397.

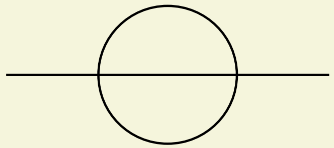
Dimensional recurrences can be used to avoid kinematical singularities directly in the differential GB. In case of $J_{3,ijk}^{(d)}$ integrals:

$$
\begin{aligned}
&12(d-1)q^2\partial_i^2 J_{3,ijk}^{(d+2)} = -2(3\Delta_{jk6} + m_i^2(6m_j^2 + 6m_k^2 - 2q^2 - m_i^2))\partial_i J_{3,ijk}^{(d)} \\
&+16m_k^2(q^2 - m_k^2)\partial_k J_{3,ijk}^{(d)} + 16m_j^2(q^2 - m_j^2)\partial_j J_{3,ijk}^{(d)} \\
&-2((7d-17)q^2 + (d-3)(m_i^2 - 5m_2^2 - 5m_k^2))J_{3,ijk}^{(d)} - 2(d-2)T_j^{(d)}T_k^{(d)} \\
&+(d-2)(1 - 3\frac{u_{jk6}}{m_i^2})T_i^{(d)}T_k^{(d)} + (d-2)(1 - 3\frac{u_{kj6}}{m_i^2})T_i^{(d)}T_j^{(d)}.
\end{aligned}
$$

$$
\begin{aligned}
&6q^2(d-2)\partial_i\partial_j J_{3,ijk}^{(d+2)} = 2m_i^2(q^2 - m_i^2 - 3m_j^2 + 3m_k^2)\partial_i J_{3,ijk}^{(d)} \\
&+2m_j^2(q^2 - 3m_i^2 - m_j^2 + 3m_k^2)\partial_j J_{3,ijk}^{(d)} - 4m_k^2(q^2 - m_k^2)\partial_k J_{3,ijk}^{(d)} \\
&+2((d-2)q^2 + (d-3)(m_i^2 + m_j^2 - 2m_k^2))J_{3,ijk}^{(d)} \\
&+2(d-2)T_i^{(d)}T_j^{(d)} - (d-2)T_k^{(d)}(T_i^{(d)} + T_j^{(d)}).
\end{aligned}
$$

- There is no Gram determinant in the denominators at the r.h.s !!!
- The number of terms at the right hand sides of the above relations decreased: - 24 in the first case and - 40 in the second case.
- Reduction to the generic dimension d can be done by using relations between $d-2$ and d dimensional integrals given at the first transparencies.
- Differential GB transforming d dimensional into $d-2$ dimensional integrals can be derived for any type of integrals!

Difference and differential equations for the sunrise integral



The generic two-loop self-energy type diagram in d dimensional Minkowski space with three equal mass propagators is given by the integral:

$$J_3^{(d)}(\nu_1,\nu_2,\nu_3) \equiv \int\int \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2}{(i\pi^{d/2})^2} \frac{1}{(k_1^2-m^2)^{\nu_1}((k_1-k_2)^2-m^2)^{\nu_2}((k_2-q)^2-m^2)^{\nu_3}}.$$

For integer values of ν_j the integrals (35) can be expressed in terms of only three basis integrals $J_3^{(d)}(1,1,1)$, $J_3^{(d)}(2,1,1)$ and $J_3^{(d)}(0,1,1) = (T_1^{(d)}(m^2))^2$ where

$$T_1^{(d)}(m^2) = \int \frac{d^d k}{[i\pi^{\frac{d}{2}}]} \frac{1}{k^2-m^2} = -\Gamma\left(1-\frac{d}{2}\right) m^{d-2}.$$

The relation connecting $d-2$ and d dimensional integrals $J_3^{(d)}(\nu_1,\nu_2,\nu_3)$:

$$\begin{aligned} J_3^{(d-2)}(\nu_1,\nu_2,\nu_3) &= \nu_1\nu_2 J_3^{(d)}(\nu_1+1,\nu_2+1,\nu_3) \\ &+ \nu_1\nu_3 J_3^{(d)}(\nu_1+1,\nu_2,\nu_3+1) + \nu_2\nu_3 J_3^{(d)}(\nu_1,\nu_2+1,\nu_3+1). \end{aligned}$$

At $\nu_1=\nu_2=\nu_3=1$ and $\nu_1=2,\ \nu_2=\nu_3=1$ we obtain two equations. Use the recurrence relations to simplify their r.h.s. Shifting $d \to d+2$ give two more relations.

They are used to exclude $J_3^{(d)}(2,1,1)$ from one of the relations, so that we obtain a difference equation for the master integral $J_3^{(d)}(1,1,1) \equiv J_3^{(d)}$:

$$\begin{aligned} 12z^3(d+1)(d-1)(3d+4)(3d+2) \quad & J_3^{(d+4)} \\ -4m^4(1-3z)(1-42z+9z^2)z(d-1)d \quad & J_3^{(d+2)} \\ -4m^8(1-z)^2(1-9z)^2 \quad & J_3^{(d)} \\ = 3z[(z+1)(27z^2+18z-1)d^2-4z(1+9z)d-48z^2]m^{2d+2} \quad & \Gamma\left(-\frac{d}{2}\right)^2, \end{aligned}$$

where

$$z = \frac{m^2}{q^2}.$$

The integral $J_3^{(d)}$ satisfi es also a second order differential equation. Taking the second derivative of $J_3^{(d)}$ with respect to mass gives

$$\frac{d^2}{dm^2\,dm^2} J_3^{(d)}(1,1,1) = 6J_3^{(d)}(2,2,1) + 6J_3^{(d)}(3,1,1).$$

By using recurrence relations integrals on the r.h.s can be reduced to the same three basis integrals. Using

$$J_3^{(d)}(2,1,1) = \frac{1}{3}\frac{d}{dm^2} J_3^{(d)}(1,1,1)$$

we obtain:

$$2(1-z)(1-9z)z^2\frac{d^2 J_3^{(d)}}{dz^2} - z[9z^2(d-4) + 10z(d-2) + 8 - 3d]\frac{dJ_3^{(d)}}{dz}$$
$$+ (d-3)[z(d+4) + d - 4]J_3^{(d)} = 12zm^{(2d-6)}\Gamma^2\left(2 - \frac{d}{2}\right).$$

The differential equation were used to fi nd the momentum dependence of arbitrary periodic constants in the solution of the difference equation.

Solution of the dimensional recurrency

Difference equation is a second order inhomogeneous equation with polynomial coeffi cients in d. The full solution of this equation is given by:

$$J_3^{(d)} = J_{3p}^{(d)} + \widetilde{w}_a(d) J_{3a}^{(d)} + \widetilde{w}_b(d) J_{3b}^{(d)},$$

where $J_{3p}^{(d)}$ is a particular solution of the equation, $J_{3a}^{(d)}$, $J_{3b}^{(d)}$ is a fundamental system of solutions of the associated homogeneous equation and $\widetilde{w}_a(d)$, $\widetilde{w}_b(d)$ are arbitrary periodic functions of d satisfying relations:

$$\widetilde{w}_a(d+2) = \widetilde{w}_a(d), \qquad \widetilde{w}_b(d+2) = \widetilde{w}_b(d).$$

The order of the polynomials in d of the associated homogeneous difference equation can be reduced by making the substitution

$$J_3^{(d)} = \frac{\Gamma\left(\frac{d-2}{2}\right)}{\Gamma\left(\frac{3d}{2}-3\right)\Gamma\left(\frac{d-1}{2}\right)} \overline{J}_3^{(d)}.$$

The homogeneous equation for $\overline{J}_3^{(d)}$ takes the simpler form

$$\frac{16z^3}{27m^8(1-z)^2(1-9z)^2}\,\overline{J}_3^{(d+4)} - \frac{2(1-3z)(1-42z+9z^2)zd}{27m^4(1-z)^2(1-9z)^2}\,\overline{J}_3^{(d+2)} - \frac{(3d-2)(3d-4)}{36}\,\overline{J}_3^{(d)} = 0.$$

Putting

$$d = 2k - 2\varepsilon, \quad y^{(k)} = \rho^{-k}\,\overline{J}_3^{(2k-2\varepsilon)},$$

we transform equation to a standard form

$$A\rho^2 y^{(k+2)} + (B + C\,k)\rho y^{(k+1)} - (\alpha + k)(\beta + k)y^{(k)} = 0,$$

where

$$A = \frac{16z^3}{27m^8(1-z)^2(1-9z)^2}, \quad B = \frac{4\varepsilon}{27}\frac{(1-3z)(1-42z+9z^2)z}{m^4(1-z)^2(1-9z)^2},$$

$$C = -\frac{B}{\varepsilon}, \quad \alpha = -\varepsilon - \frac{1}{3}, \quad \beta = -\varepsilon - \frac{2}{3},$$

and ρ is for the time being, an arbitrary constant.

In order to get homogeneous equation into a more convenient form, we will defi ne three parameters ρ, x and γ by the equations

$$A\rho^2 = x(1-x), \;\; B\rho = \gamma - (\alpha+\beta+1)x, \;\; C\rho = 1-2x.$$

These have the solution

$$x = \frac{1-2C\rho}{2} = \frac{(1-9z)^2}{(1+3z)^3} = \frac{q^2(q^2-9m^2)^2}{(q^2+3m^2)^3},$$

$$\rho = \frac{1}{\sqrt{4A+C^2}} = \frac{27}{4}\frac{m^4(1-z)^2(1-9z)^2}{z(1+3z)^3} = \frac{27}{4}\frac{m^2(q^2-m^2)^2(q^2-9m^2)^2}{(q^2+3m^2)^3},$$

$$\gamma = B\rho + (\alpha+\beta+1)x = -\varepsilon,$$

and the equation can accordingly be written in the form

$$x(1-x)y^{(k+2)} + [(1-2x)k + \gamma - (\alpha+\beta+1)x]y^{(k+1)} - (\alpha+k)(\beta+k)y^{(k)} = 0.$$

it can be transformed to the equation with linear in k coeffi cients by rescaling $y^{(k)}$

$$y^{(k)} = \Gamma(\alpha+k)\widetilde{y}^{(k)} \quad \text{or} \quad y^{(k)} = \Gamma(\beta+k)\widetilde{y}^{(k)}.$$

The fundamental system of solutions of homogeneous equation consist of two functions. In the case when $|1-x|<1$ (large q^2) the solutions are

$$y_1^{(k)} = (-1)^k \frac{\Gamma(\alpha+k)\Gamma(\beta+k)}{\Gamma(\alpha+\beta-\gamma+k+1)} \; {}_2F_1(\alpha+k, \beta+k, \alpha+\beta-\gamma+k+1; 1-x),$$

$$y_2^{(k)} = \frac{\Gamma(\alpha+\beta-\gamma+k)}{(1-x)^k} \; {}_2F_1(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1-k; 1-x).$$

Once we know the solutions of the homogeneous equation a particular solution $J_{3p}^{(d)}$ can be obtained by using Lagrange's method of variation of parameters.

The argument of the Gauss' hypergeometric function is related to the maximum of the Kibble cubic form:

$$\Phi(s,t,u) = stu - (s+t+u)m^2(m^2+q^2) + 2m^4(m^2+3q^2),$$

provided that $s+t+u = q^2+3m^2$. The maximal value $\Phi_{\max} = \frac{1}{27}\, q^2(q^2-9m^2)^2$ occurs at $s=t=u=\frac{1}{3}\,(q^2+3m^2)$ and we see that the kinematical variable (1) can be written as

$$x = \frac{\Phi(s,t,u)}{stu}\Big|_{s=t=u=\frac{1}{3}\,(q^2+3m^2)}.$$

This observation may be useful in fi nding the characteristic variable in the general mass case.

Explicit analytic expression for $J_3^{(d)}$

To find the full solution we assume that q^2 is large. The solution of the associated homogeneous difference equation will be of the form

$$
\begin{aligned}
J_{3,h}^{(d)} &= w_1(z)\frac{\Gamma\left(\frac{d}{2}-\frac{1}{3}\right)\Gamma\left(\frac{d}{2}-\frac{2}{3}\right)\Gamma\left(\frac{d-2}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{3d}{2}-3\right)\Gamma\left(\frac{d-1}{2}\right)}\rho^{\frac{d}{2}}e^{i\pi\frac{d}{2}}\,{}_2F_1\left[\begin{matrix}\frac{d}{2}-\frac{1}{3},\frac{d}{2}-\frac{2}{3}\,; \\ \frac{d}{2}\,;\end{matrix}\; 1-x\right] \\
&+ w_2(z)\frac{\Gamma^2\left(\frac{d-2}{2}\right)}{\Gamma\left(\frac{3d}{2}-3\right)\Gamma\left(\frac{d-1}{2}\right)}\frac{\rho^{\frac{d}{2}}}{(1-x)^{\frac{d}{2}}}\,{}_2F_1\left[\begin{matrix}\frac{1}{3},\frac{2}{3}\,; \\ 2-\frac{d}{2}\,;\end{matrix}\; 1-x\right].
\end{aligned}
$$

The arbitrary periodic functions $w_1(z)$ and $w_2(z)$ can be determined either from the $d \to \infty$ asymptotics or using the differential equation. From differential equation we obtain two simple equations

$$
z(1-z)(1+3z)(1-9z)\frac{dw_1(z)}{dz} - 2(1+6z-39z^2)w_1(z) = 0,
$$

$$
z(1+3z)(1-9z)\frac{dw_2(z)}{dz} + 3(1-z)w_2(z) = 0.
$$

Solutions of equations

$$w_1(z) = \frac{\kappa_1 z^2(1+3z)^2}{(1-9z)^2(1-z)^2}, \quad w_2(z) = \frac{\kappa_2 z^3}{(1+3z)(1-9z)^2}.$$

Integration constants $\kappa_1, \ \kappa_2$ we fix from the first two terms of the large momentum expansion of $J_3^{(d)}$

$$J_3^{(d)} = m^{2-4\varepsilon}\Gamma^2(1+\varepsilon)\left[\frac{\Gamma(-1+2\varepsilon)\Gamma^3(1-\varepsilon)}{z\Gamma^2(1+\varepsilon)\Gamma(3-3\varepsilon)}(-z)^{2\varepsilon} + \frac{6\Gamma^2(-\varepsilon)}{\Gamma(3-2\varepsilon)}(-z)^{\varepsilon}\right] + O(z).$$

The application of Lagrange's method of finding a particular solution gives

$$J_{3p}^{(d)} = \frac{3zm^{2d-6}}{(1+\sqrt{z})^2}\Gamma^2\left(1-\frac{d}{2}\right)F_2\left(1, \frac{1}{2}, \frac{d-1}{2}, \frac{d}{2}, d-1; \sqrt{z}R, R\right),$$

where

$$R = \frac{4\sqrt{z}}{(1+\sqrt{z})^2},$$

and F_2 is the Appell function:

$$F_2(\alpha,\beta,\beta',\gamma,\gamma';x,y) = \sum_{k,l=0}^{\infty}\frac{(\alpha)_{k+l}(\beta)_k(\beta')_l}{(\gamma)_k(\gamma')_l}\frac{x^k\, y^l}{k!\, l!}, \quad |x|+|y|<1.$$

Collecting all contributions, setting $d = 4 - 2\varepsilon$, applying Euler transformation for the first ${}_2F_1$ function we obtain

$$
\begin{aligned}
J_3^{(d)} \quad = \quad & \frac{6\Gamma^2(-\varepsilon)\Gamma^2(1+\varepsilon)(-z)^{\varepsilon}(1-z)^{2-2\varepsilon}}{m^{4\varepsilon-2}\,\Gamma(3-2\varepsilon)(1+3z)}\, {}_2F_1\left[\begin{array}{cc} \frac{1}{3}, \frac{2}{3}\,; & \frac{27(1-z)^2 z}{(1+3z)^3} \\ 2-\varepsilon\,; & \end{array}\right] \\
+ \quad & \frac{\Gamma(-1+2\varepsilon)\Gamma^3(1-\varepsilon)(-z)^{2\varepsilon}(1-9z)^{2-2\varepsilon}}{m^{4\varepsilon-2}\,\Gamma(3-3\varepsilon)z(1+3z)}\, {}_2F_1\left[\begin{array}{cc} \frac{1}{3}, \frac{2}{3}\,; & \frac{27(1-z)^2 z}{(1+3z)^3} \\ \varepsilon\,; & \end{array}\right] \\
+ \quad & \frac{3zm^{2-4\varepsilon}}{(1+\sqrt{z})^2}\Gamma^2\,(-1+\varepsilon)\, F_2\left(1, \frac{1}{2}, \frac{3}{2} - \varepsilon, 2-\varepsilon, 3-2\varepsilon; \sqrt{z}R, R\right).
\end{aligned}
$$

The use of dimensional recurrences was essential to obtain this result!

Integral representation convenient for the ε expansion of ${}_2F_1$:

$$
{}_2F_1\left[\begin{array}{cc}\frac{1}{3},\frac{2}{3}; & \frac{27(1-z)^2z}{(1+3z)^3}\\ 2-\varepsilon; & \end{array}\right]
= \frac{(1+3z)}{(1-z)}\frac{\Gamma(2-\varepsilon)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}-\varepsilon\right)}\int_0^1 \frac{du}{\sqrt{u}}[(1-u)(1-wu)(1-zu)]^{\frac{1}{2}-\varepsilon}
$$

Integral representation for Appell's F_2 function

$$
F_2\left(1,\frac{3}{2}-\varepsilon,\frac{1}{2},3-2\varepsilon,2-\varepsilon,R,\sqrt{z}R\right)
= \frac{2\Gamma(3-2\varepsilon)}{\Gamma^2\left(\frac{3}{2}-\varepsilon\right)}(1+\sqrt{z})^2\int_0^1 \frac{dt\,[t(1-t)]^{\frac{1}{2}-\varepsilon}}{(4zt+1-z+L)}\,{}_2F_1\left[\begin{array}{cc}1,\varepsilon; & \frac{4tz+1-z-L}{4tz+1-z+L}\\ 2-\varepsilon; & \end{array}\right]
$$

where

$$
L=\sqrt{(4zt-1-z)^2-4z}.
$$

This integral representation can be used for the ε expansion of the F_2 function.

The imaginary part of $J_3^{(d)}$ on the cut comes from the two ${}_2F_1$ functions:

$$\mathrm{Im} J_3^{(d)} = \frac{-4z\,\pi^2\sqrt{3\pi}m^{2-4\varepsilon}}{\Gamma\left(\frac{3}{2}-\varepsilon\right)\Gamma\left(2-\varepsilon\right)\left(1+3z\right)}\left[\frac{(1-9z)^2}{108z^2}\right]^{1-\varepsilon} {}_2F_1\left[\begin{array}{c} \frac{1}{3},\frac{2}{3}\,; \\ 2-\varepsilon\,; \end{array} \frac{(1-9z)^2}{(1+3z)^3}\right].$$

At $d=4$ for the imaginary part we verify the known result.

Using explicit formula we fi nd the on-threshold value of the integral:

$$J_3^{(d)}\Big|_{q^2=9m^2} = \frac{\Gamma^2(\varepsilon)}{(1-\varepsilon)(1-2\varepsilon)}\,{}_3F_2\left[\begin{array}{c} 1,-1+2\varepsilon,\frac{3}{2}-\varepsilon\,; \\ \frac{1}{2}+\varepsilon,2-\varepsilon\,; \end{array} -\frac{1}{3}\right]$$

$$= \frac{\Gamma^2(1+\varepsilon)}{(1-\varepsilon)(1-2\varepsilon)}\left\{-\frac{3}{2\varepsilon^2}+\frac{9}{4\varepsilon}+\frac{75}{8}-\frac{8\pi}{\sqrt{3}}+O(\varepsilon)\right\}.$$

The analytic expression was not known. The fi rst several terms in the ε expansion are in agreement with the result of Davydychev and Smirnov.

Several remarks about solution of dimensional recurrency

- For the fi rst time analytic expression for the sunrise diagram was found
- The differential equation is Heun equation with four regular singular points, located at $q^2 = 0, m^2, 9m^2, \infty$. In general reduction of the Heun equation to the hypergeometric equation is a complicated mathematical problem
- The associated homogeneous difference equation for $J_3^{(d)}$ is simple, and admits reduction to a hypergeometric type of equation with linear coeffi cients.
- This is a general situation. Kinematical singularities of Feynman integrals are located on complicated manifolds. In the case when the differential equations are of the fi rst order there are no problems to solve them. However, to solve a second or higher order differential equations in general will be a problem because of complicated structure of the kinematical

singularities.

- Singularities of Feynman integrals are poles in $1/(d-p)$ with rational p. This has been used for an evident rescaling of the integral by ratios of Γ functions which allowed us to reduce the order of the polynomial coeffi cients in the difference equation.

Summary and perspectives

- The Gröbner basis technique together with dimensional recurrences provides selfcontained mathematical approach for calculating Feynman diagrams.
- The proposed technique can be used for evaluating in the Standard Model vertex and box diagrams with 4-5 mass scales.
- Dimensional recurrences can be used for analytical as well as numerical evaluation of master integrals.