Applying Gröbner Bases to Solve Reduction Problems for Feynman Integrals

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- Reduction problem for Feynman integrals
- A review of algorithmic approaches
- The description of our method
- A three-loop example
- Conclusion

Work done in collaboration with V.A. Smirnov

[A.V. Smirnov & V.A. Smirnov, JHEP 0601 (2006) 001;

A.V. Smirnov, JHEP 0604 (2006) 026;

V.A. Smirnov, hep-ph/0601268; A.V. Smirnov, V.A. Smirnov, hep-ph/0606247 (short reviews);

A.G. Grozin, A.V. Smirnov and V.A. Smirnov, in preparation]

see also

http://www.srcc.msu.ru/nivc/about/lab/lab4_2/index_eng.htm

Reduction problem for Feynman integrals

A given Feynman graph $\Gamma \to$ tensor reduction $\to$ various scalar Feynman integrals that have the same structure of the integrand with various distributions of powers of propagators

$$F(a_1,\ldots,a_n) \;=\; \int \cdots \int \frac{\mathrm{d}^d k_1 \ldots \mathrm{d}^d k_h}{E_1^{a_1} \ldots E_n^{a_n}} \,.$$

$d = 4 - 2\epsilon$; the denominators E_r are either quadratic or linear with respect to the loop momenta $p_i = k_i,\; i = 1,\ldots,h$ or to the independent external momenta $p_{h+1} = q_1,\ldots,p_{h+N} = q_N$ of the graph.

Methods: analytical, numerical, semianalytical ...

An old analytical strategy:
to evaluate, by some methods, every scalar Feynman integral generated by the given graph.

A traditional strategy:

to derive, without calculation, and then apply integration by parts (IBP) relations [K.G. Chetyrkin and F.V. Tkachov'81]
between the given family of Feynman integrals as recurrence relations.

A general integral from the given family is expressed as a linear combination of some basic (master) integrals.

The whole problem of evaluation$\rightarrow$

- constructing a reduction procedure
- evaluating master integrals

No common definition of the master integrals.

After solving the reduction problem for a given family, we qualify some integrals as master integrals.

The notion of a master integral depends on the chosen relations and the ordering.

$F(a_1, \ldots, a_n)$ are functions of integer variables
$a_1, \ldots, a_n \in \mathbb{N}^n$
$\mathcal{F}$: an infinitely dimensional linear space of such functions.
The simplest basis:

$$H_{a_1,\ldots,a_n}(a'_1, \ldots, a'_n) = \delta_{a_1,a'_1} \ldots \delta_{a_n,a'_n}$$

The relations we have can be described as elements of the adjoint vector space $\mathcal{F}^*$, i.e. the linear functionals on $\mathcal{F}$:

$$r \in \mathcal{F}^*,\ f \in \mathcal{F} \rightarrow \langle r, f \rangle$$

Classes of relations:
IBP:

$$\int \ldots \int \mathrm{d}^d k_1 \mathrm{d}^d k_2 \ldots \frac{\partial}{\partial k_i} \left(p_j \frac{1}{E_1^{a_1} \ldots E_n^{a_n}} \right) = 0 \, ,$$

$\longrightarrow$

$$\sum \alpha_i F(a_1 + b_{i,1}, \ldots, a_n + b_{i,n}) = 0 \, ,$$

Lorentz-invariance (LI) identities [T. Gehrmann and E. Remiddi'00]
parity conditions, symmetry relations, e.g.,

$$F(a_1, \ldots, a_n) = (-1)^{d_1 a_1 + \ldots d_n a_n} F(a_{\sigma(1)}, \ldots, a_{\sigma(n)}),$$

Boundary conditions (for a subset of indices i_j one has):

$$F(a_1, a_2, \ldots, a_n) = 0 \text{ when } a_{i_1} \leq 0, \ldots a_{i_k} \leq 0$$

All these relations form an infinitely dimensional vector subspace $\mathcal{R} \subset \mathcal{F}^*$.
The set of solutions of all those relations:

$$\mathcal{S} = \{f \in \mathcal{F} : \langle r, f \rangle = 0 \; \forall \; r \in \mathcal{R}\}$$

The dimension of $\mathcal{S}$ might be infinite but, practically, it appears to be finite.

- Feynman integrals are a point of the solution space.

An integral $F(a_1, \ldots, a_n)$ can be expressed via some other integrals $F(a_1^1, \ldots, a_n^1)$, ..., $F(a_1^k, \ldots, a_n^k)$ if there exists an element $r \in \mathcal{R}$ such that

$$\langle r, F \rangle = F(a_1, \ldots, a_n) + \sum k_{a'_1, \ldots, a'_n} F(a'_1, \ldots, a'_n) \; .$$

To define of master (irreducible) integral we also need a priority between the points $(a_1, \ldots, a_n) \to$ *ordering* (to know which integrals to get rid of and which ones to leave).

Feynman integrals are simpler, from the analytic point of view, if they have more non-positive indices.
Solving IBP relations by hand $\to$ reducing indices to zero

Sectors ('topologies'):
2^n regions labelled by subsets $\nu \subseteq \{1, \ldots, n\}$:
σ_ν = $\{(a_1, \ldots, a_n) : a_i > 0$ if $i \in \nu\,,\;\; a_i \leq 0$ if $i \notin \nu\}$

A sector σ_ν is lower than $\sigma_{\nu'}$ if $\nu \subset \nu'$

$F(a'_1, \ldots, a'_n) \succ F(a_1, \ldots, a_n)$ if the sector of $(a_1, \ldots, a_n)$ is lower than the sector of $(a'_1, \ldots, a'_n)$.

To define an ordering completely introduce it in some way inside the sectors. At least the corner point with $a_i = 1$ and $a_i = 0$ is lower than any other point in the given sector.

$F(a_1, \ldots, a_n)$ is a master integral if there is no element $r \in \mathcal{R}$ acting on F as

$$\langle r, F \rangle = F(a_1, \ldots, a_n) + \sum k_{a'_1,\ldots,a'_n} F(a'_1, \ldots, a'_n) \ , \qquad (1)$$

where all the points $(a'_1, \ldots, a'_n)$ are lower than $(a_1, \ldots, a_n)$.

A review of algorithmic approaches

‘Laporta’s algorithm’ [S. Laporta and E. Remiddi’96; S. Laporta’00; T. Gehrmann and E. Remiddi’01]

‘When increasing the total power of the denominator and numerator, the total number of IBP equations grows faster than the number of independent Feynman integrals. Therefore this system of equations sooner or later becomes overdetermined, and one obtains the possibility to perform a reduction to master integrals’

Various implementations:

- one public version `AIR` [C. Anastasiou and A. Lazopoulos’04]
- several private versions

  [T. Gehrmann and E. Remiddi, M. Czakon, P. Marquard and D. Seidel, Y. Schröder, C. Sturm, A. Onishchenko, . . .]

Reduction using Gröbner bases

- Historically, suggested by O.V. Tarasov [O.V. Tarasov'98]
  Reduce the problem to differential equations by introducing a mass for every line,
  $a_i\mathbf{i}^+ \rightarrow \frac{\partial}{\partial m_i^2}$
- The approach of Gerdt based on Gröbner bases (the use of Janet bases) [V.P. Gerdt'04, 05]
- Another approach based on Gröbner bases (the so-called sector-bases or s-bases)
  [A.V. Smirnov and V.A. Smirnov'05]

Baikov's method

[P.A. Baikov'96,'97; V.A. Smirnov and M. Steinhauser'03]

The basic ingredient:

$$\int \cdots \int \frac{\mathrm{d}x_1 \ldots \mathrm{d}x_n}{x_1^{a_1} \ldots x_n^{a_n}} \left[P(x_1, \ldots, x_n)\right]^{(d-h-1)/2} ,$$

where P is constructed for a given family of integrals according to some rules and h is the number of loops. This representation is used

- to understand which integrals are master integrals
- to construct the coefficient functions $c_i(a_1, \ldots, a_n)$

$$F(a_1, \ldots, a_n) = \sum_i k_i \, c_i(a_1, \ldots, a_n) ,$$

where k_i do not depend on indices.

Indeed, for a candidate for a master integral with $a_1, \ldots, a_n = 0$ or 1, the basic parametric representation allows to construct a function that satisfies the relations $\mathcal{R}$ and that vanishes if $(a'_1, \ldots, a'_n)$ is lower than $(a_1, \ldots, a_n)$ (in particular, if it belongs to a lower sector).

Suppose that $(a_1, \ldots, a_n)$ is not a master integral. Then substituting this function into (1) we get 0 on the left and 1 on the right and come to a contradiction.

Now suppose that we know that the integrals with the indices $A^1 = (a^1_1, \ldots, a^1_n)$, …, $A^k = (a^k_1, \ldots, a^k_n)$ are master integrals and we constructed the corresponding solutions of this type.

These functions form a basis of the solution space $\mathcal{S}$:

$$F = \sum_i k_i C_i \ .$$

Substitute all A_j and solve the system of linear equations

$$F(A_j) = \sum_i k_i C_i(A_j) \text{ for } i \leq j \ .$$

so the coefficients k_i are expressed in terms of $F(A_i)$.

Thus, the knowledge of k_i and $C_i \rightarrow$ is enough to evaluate any Feynman integral of the given class.

The description of our method

Suppose first that we are interested in expressing any integral in the positive sector $\sigma_{\{1,\ldots,n\}}$ as a linear combination of a finite number of integrals in it.

[A.V. Smirnov and V.A. Smirnov’05, A.V. Smirnov’06]

$$\sum c_i F(a_1 + b_{i,1}, \ldots, a_n + b_{i,n}) = 0$$

The left-hand sides of IBP relations can be expressed in terms of operators of multiplication by the indices a_i and shift operators $Y_i = \mathbf{i}^+, Y_i^- = \mathbf{i}^-$, where

$$(Y_i^{\pm} F)(a_1, \ldots, a_n) = F(a_1, \ldots, a_i \pm 1, \ldots, a_n)$$

Thus, one can choose certain operators f_i corresponding to IBP relations and write

$(f_iF)(a_1,\ldots,a_n)=0$ or $(f_iF)\equiv 0$

for any element of the solution space $\mathcal{R}$. The choice is not unique, we will get rid of Y_i^-

Consider the algebra of operators $\mathcal{A}^0$ generated by shift operators Y_i^+ and multiplication operators A_i. It acts on the field of functions $\mathcal{F}$ of n integer variables.

The ideal $\mathcal{I}$ of IBP relations generated by the elements f_i.

For any element $X \in \mathcal{I}$ we have, in particular

$$(XF)(1, 1, \ldots, 1) = 0 \; .$$

Also we can write

$$F(a_1, a_2, \ldots, a_n) = (Y_1^{a_1-1} \ldots Y_n^{a_n-1} F)(1, 1, \ldots, 1)$$

The idea of the algorithm is to reduce the operator in the right-hand side of the equation using the elements of the ideal $\mathcal{I} \rightarrow$ we need an ordering on the elements of the algebra.

Orderings on the algebra of operators

We had an ordering on the points in the sector — now we need it to be compatible with a certain ordering on the algebra of operators $\rightarrow$ an additional condition:

- for any $a, b \in \sigma_\nu$ and $c \in \mathbb{Z}^n$ such that $a+c, b+c \in \sigma_\nu$ one has $a \prec b$ if and only if $a+c \prec b+c$.

Such an ordering on integrals in the positive sector $\leftrightarrow$ ordering on the algebra $\mathcal{A}^0$:
a degree of a monomial is the degree of the point, it shifts the corner of the sector to
a degree of an element of the algebra — the highest degree among its monomials

The obtained ordering on degrees satisfies the following properties:
i) for any $a \in \mathbb{N}^n$ not equal to $(0, \ldots 0)$ one has $a \prec (0, \ldots 0)$
ii) for any $a, b, c \in \mathbb{N}^n$ one has $a \prec b$ if and only if $a + c \prec b + c$.

E.g., lexicographical ordering:
A set $(i_1, \ldots, i_n)$ is higher than a set $(j_1, \ldots, j_n)$,
$(i_1, \ldots, i_n) \succ (j_1, \ldots, j_n)$
if there is $l \leq n$ such that $i_1 = j_1$, $i_2 = j_2$, ..., $i_{l-1} = j_{l-1}$ and $i_l > j_l$.

Degree-lexicographical ordering: $(i_1, \ldots, i_n) \succ (j_1, \ldots, j_n)$ if $\sum i_k > \sum j_k$, or $\sum i_k = \sum j_k$ and $(i_1, \ldots, i_n) \succ (j_1, \ldots, j_n)$ in the sense of the lexicographical ordering.

An ordering can be defined by a matrix.

Lexicographical, degree-lexicographical and reverse degree-lexicographical ordering for $n = 5$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Ordering on the positive sector $\rightarrow$
ordering of monomials of operators Y_i

For two monomials $M_1 = Y_1^{i_1-1} \dots Y_n^{i_n-1}$ and
$M_2 = Y_1^{j_1-1} \dots Y_n^{j_n-1}$
$(M_1 \cdot F)(1,\dots,1) \succ (M_2 \cdot F)(1,\dots,1)$ if and only if $M_1 \succ M_2$

The reduction problem is to reduce the monomial
$Y_1^{a_1-1} \dots Y_n^{a_n-1}$ modulo the ideal of the IBP relations

$$Y_1^{a_1-1} \dots Y_n^{a_n-1} = \sum r_i f_i + \sum c_{i_1,\dots,i_n} Y_1^{i_1-1} \dots Y_n^{i_n-1}$$

Apply to F at $a_1 = 1, \ldots, a_n = 1$ to obtain

$$F(a_1, a_2, \ldots, a_n) = \sum c_{i_1,\ldots,i_n} F(i_1, i_2, \ldots, i_n)\,,$$

where integrals on the right-hand side are “master integrals”.
The reduction procedure for an element X: similar to the division of polynomials.
Searching for an element of the basis f_i, such that the highest member of X can be obtained by the highest member of f_i multiplied by a product of Y_j (A_i treated as coefficients), taking the difference with proper coefficients, the degree of the resulting element is lower.
Irreducible monomials correspond to Feynman integrals that cannot be reduced by the use of the current basis.

But the reduction does not always lead to a reasonable number of irreducible integrals $\to$ one has to build a special basis first (for example, a Gröbner basis).

Building elements with lowest possible degrees corresponds to resulting in master integrals with minimal possible degrees.

The initial bases does not suit us — we need to build special bases. The primary idea: an algorithm based on the Buchberger algorithm — S-polynomials, reductions.

The problems:

- complicated calculations
- one is also interested in integrals not only in the positive sector.
- the problem of combining results in different sectors

Our algorithm [A.S.& V.S'05] : to build a set of special bases of the ideal of IBP relations (s-bases).
In the sector $\sigma_{\{1,\ldots,n\}}$, consider Y_i as basic operators.
In the sector σ_ν, consider Y_i for $i \in \nu$ and Y_i^- for other i as basic operators.

- Construct sector bases (s-bases), rather than Gröbner bases for all the sectors.

An s-basis for a sector σ_ν is a set of elements of a basis which provides the possibility of a reduction to master integrals in this sector *and* integrals whose indices lie in *lower* sectors, i.e. $\sigma_{\nu'}$ for $\nu' \subset \nu$.

- When performing calculations even in the positive sector, we allow Y_i in negative degrees (but the highest degree has to stay in the positive sector) — the point is that during the reduction procedure the use of such elements will lead to reducing to integrals in lower sectors.
- The construction — close to the Buchberger algorithm but it can be terminated when the 'current' basis already provides us the needed reduction.
- The basic operations are the same, i.e. calculating S-polynomials and reducing them modulo current basis, with a chosen ordering.

After constructing s-bases for all non-trivial sectors one obtains a recursive (with respect to the sectors) procedure to evaluate $F(a_1, \ldots, a_n)$ at any point and thereby reduce a given integral to master integrals.

Many sectors — seemingly the problem becomes harder. But the important simplification is that one is not trying to solve the reduction problem in each sector separately but allows to reduce the integrals in a given sector to lower sectors — similarly to the “by hand” solutions.

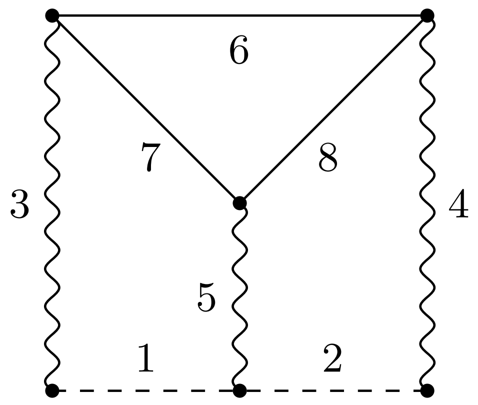
Description of the algorithm (implemented in `Mathematica`): [A.V. Smirnov’06]

Examples

- A number of diagrams with up to 6 indices;
- Reduction of a family of Feynman integrals relevant to the three-loop static quark potential (7 indices);

  [A.V. Smirnov and V.A. Smirnov'05]
- A family of Feynman integrals with 9 indices

  [A.G. Grozin, A.V. Smirnov and V.A. Smirnov, in preparation]



$$F(a_1,\ldots,a_9) = \int\int\int \frac{(2v\cdot r)^{-a_9}\mathrm{d}^d k\,\mathrm{d}^d l\,\mathrm{d}^d r}{(-2v\cdot k)^{a_1}(-2v\cdot l)^{a_2}(-k^2)^{a_3}(-l^2)^{a_4}[-(k-l)^2]^{a_5}\,(-r^2+m^2)^{a_6}[-(k+r)^2+m^2]^{a_7}[-(l+r)^2+m^2]^{a_8}}\,.$$

Symmetry:
$(1 \leftrightarrow 2, 3 \leftrightarrow 4, 7 \leftrightarrow 8)$
Boundary conditions: $F(a_1, \ldots, a_9) = 0$ if one of the following sets of lines has non-positive indices: $\{5,7\}$, $\{5,8\}$, $\{6,7\}$, $\{6,8\}$, $\{7,8\}$, $\{3,4,6\}$.
Master integrals:

$$
\begin{aligned}
I_1 &= F(1,1,0,1,1,1,1,0,0), \quad I_2 = F(1,1,1,1,0,0,1,1,0), \\
I_3 &= F(1,1,0,0,0,1,1,1,0), \\
I_4 &= F(0,1,1,0,1,1,0,1,0), \quad \bar{I}_4 = F(-1,1,1,0,1,1,0,1,0), \\
I_5 &= F(0,0,0,1,1,1,1,0,0), \quad I_6 = F(0,1,0,0,0,1,1,1,0), \\
I_7 &= F(0,1,0,0,1,1,1,0,0), \quad \bar{I}_7 = F(0,2,0,0,1,1,1,0,0), \\
I_8 &= F(0,0,0,0,0,1,1,1,0).
\end{aligned}
$$

```
In[13]:= BuildBasis[{-1, -1, -1, -1, -1, 1, 1, 1, -1},
```

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \text{SearchStyle} \to 0, \text{UsingSymmetries} \to 1]$$

```
Even restrictions set

No regularized lines

Initial data protected. Use ClearBasis to clear it and the basis from memory.

Dimension = 9

Local symmetries: {{{2, 1, 4, 3, 5, 6, 8, 7, 9}, {1, 1, 1, 1, 1, 1, 1, 1, 1}}}

Using the code 100 search style (MinimizingLengthWhenSearchingQ = 0)

Evaluation limit is 200000

New element of length 6

Degree is {1, 0, 0, 0, 0, 0, 1, 0, 0}

New element of length 6

Degree is {0, 1, 0, 0, 0, 0, 0, 1, 0}

New element of length 6

Degree is {1, 0, 0, 0, 0, 0, 1, 0, 0}

New element of length 9

Degree is {0, 0, 1, 0, 0, 0, 1, 0, 0}

New element of length 9

Degree is {0, 0, 0, 1, 0, 0, 0, 1, 0}

New element of length 13

Degree is {0, 0, 0, 0, 0, 1, 0, 0, 0}

New element of length 10

Degree is {0, 0, 0, 0, 1, 0, 1, 0, 0}

New element of length 10

Degree is {0, 0, 0, 0, 1, 0, 0, 1, 0}

New element of length 11

Degree is {0, 0, 0, 0, 0, 0, 1, 0, 0}
```

```
New element of length 11

Degree is {0, 0, 0, 0, 0, 0, 0, 1, 0}

New element of length 11

Degree is {0, 0, 1, 0, 0, 1, 0, 0, 0}

New element of length 11

Degree is {0, 0, 0, 1, 0, 1, 0, 0, 0}

Saved element 6 of length 13

Saved element 7 of length 11

Saved element 8 of length 11

Test results: {False, False, False, False, False, True, True, True, False}

Sorting

Permutation = {10, 9, 6, 8, 7, 5, 12, 4, 11, 2, 1, 3}

Sorting over

Trying to reduce elements

Reducing basis element 11 of length 6

{1, 0, 0, 0, 0, 0, 1, 0, 0}

{d, 1, 7, 1, 12}

{1, 0, 0, 0, 0, 0, 0, 0, 0}

Reduction over

Degree is {1, 0, 0, 0, 0, 0, 0, 0, 0}

Element reduced. New length: 17

Saved element 1 of length 17

Test results: {True, False, False, False, False, True, True, True, False}

Sorting

Permutation = {1, 2, 3, 11, 4, 5, 6, 7, 8, 9, 10, 12}

Sorting over

Reducing basis element 12 of length 6

{1, 0, 0, 0, 0, 0, 1, 0, 0}

{d, 1, 7, 1, 12}

{0, 1, 0, 0, 0, 0, 0, 1, 0}

{d, 1, 18, 1, 12}

{1, 0, 0, 0, 0, 0, 0, 0, 0}
```

```
same degree

{d, 1, 28, 1, 18}

{0, 1, 0, 0, 0, 0, 0, 0, 0}

Reduction over

Degree is {0, 1, 0, 0, 0, 0, 0, 0, 0}

Element reduced. New length: 19

Saved element 2 of length 19

Test results: {True, True, False, False, False, True, True, True, False}

Sorting

Permutation = {1, 2, 3, 12, 4, 5, 6, 7, 8, 9, 10, 11}

Sorting over

Reducing basis element 12 of length 6

{0, 1, 0, 0, 0, 0, 0, 1, 0}

{d, 1, 7, 1, 12}

{0, 1, 0, 0, 0, 0, 0, 0, 0}

same degree

Basis element 4 replaced

Degree is {0, 1, 0, 0, 0, 0, 0, 0, 0}

{d, 1, 18, 1, 20}

{0, 0, 0, 0, 0, 1, 0, 0, 1}

{d, 1, 7, 1, 14}

{0, 0, 0, 0, 0, 0, 1, 0, 1}

{d, 1, 19, 1, 12}

{0, 0, 0, 0, 0, 0, 0, 1, 1}

Reduction over

Degree is {0, 0, 0, 0, 0, 0, 0, 1, 1}

Element reduced. New length: 20

Saved element 2 replaced, new length: 17

Test results: {True, True, False, False, False, True, True, True, False}

Sorting

Permutation = {1, 12, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}

Sorting over
```

```
Reducing basis element 12 of length 11

{0, 0, 1, 0, 0, 1, 0, 0, 0}

{d, 1, 12, 1, 14}

{0, 0, 1, 0, 0, 0, 1, 0, 0}

{d, 1, 24, 1, 12}

{0, 0, 1, 0, 0, 0, 0, 1, 0}

{1, 0, 0, 0, 0, 0, 0, 0, 0}

same degree

Basis element 6 replaced

Degree is {1, 0, 0, 0, 0, 0, 0, 0, 0}

{d, 3, 26, 7, 18}

{0, 0, 1, 0, 0, 0, 0, 0, 0}

Reduction over

Degree is {0, 0, 1, 0, 0, 0, 0, 0, 0}

Element reduced. New length: 141

Saved element 3 of length 141

Test results: {True, True, True, False, False, True, True, True, False}

Sorting

Permutation = {1, 2, 3, 4, 12, 5, 6, 7, 8, 9, 10, 11}

Sorting over

Reducing basis element 12 of length 9

{0, 0, 1, 0, 0, 0, 1, 0, 0}

{d, 1, 10, 1, 12}

{0, 0, 1, 0, 0, 0, 0, 0, 0}

same degree

Basis element 5 replaced

Degree is {0, 0, 1, 0, 0, 0, 0, 0, 0}

{0, 0, 0, 0, 0, 0, 1, 0, 1}

Reduction over

Degree is {0, 0, 0, 0, 0, 0, 1, 0, 1}

Element reduced. New length: 159

Saved element 3 replaced, new length: 21
```

```
Test results: {True, True, True, False, False, True, True, True, False}

Sorting

Permutation = {1, 2, 3, 12, 4, 5, 6, 7, 8, 9, 10, 11}

Sorting over

Reducing basis element 12 of length 11

{0, 0, 0, 1, 0, 1, 0, 0, 0}

{d, 1, 12, 1, 14}

{0, 0, 0, 1, 0, 0, 1, 0, 0}

{d, 1, 24, 1, 12}

{0, 0, 0, 1, 0, 0, 0, 1, 0}

{0, 1, 0, 0, 0, 0, 0, 0, 0}

same degree

Basis element 7 replaced

Degree is {0, 1, 0, 0, 0, 0, 0, 0, 0}

{d, 3, 26, 7, 18}

{0, 0, 0, 1, 0, 0, 0, 0, 0}

Reduction over

Degree is {0, 0, 0, 1, 0, 0, 0, 0, 0}

Element reduced. New length: 141

Saved element 4 of length 141

Test results: {True, True, True, True, False, True, True, True, False}

Sorting

Permutation = {1, 2, 3, 4, 5, 12, 6, 7, 8, 9, 10, 11}

Sorting over

Reducing basis element 12 of length 9

{0, 0, 0, 1, 0, 0, 0, 1, 0}

{d, 1, 10, 1, 12}

{0, 0, 0, 1, 0, 0, 0, 0, 0}

same degree

Basis element 6 replaced

Degree is {0, 0, 0, 1, 0, 0, 0, 0, 0}

{0, 0, 0, 0, 0, 0, 0, 1, 1}
```

```
Reduction over

Degree is {0, 0, 0, 0, 0, 0, 0, 1, 1}

Element reduced. New length: 159

Saved element 4 replaced, new length: 21

Test results: {True, True, True, True, False, True, True, True, False}

Sorting

Permutation = {1, 2, 12, 3, 4, 5, 6, 7, 8, 9, 10, 11}

Sorting over

Reducing basis element 12 of length 10

{0, 0, 0, 0, 1, 0, 1, 0, 0}

{d, 1, 11, 1, 12}

{0, 0, 0, 0, 1, 0, 0, 0, 0}

Reduction over

Degree is {0, 0, 0, 0, 1, 0, 0, 0, 0}

Element reduced. New length: 22

Saved element 5 of length 22

Testing element 1

Testing element 2

Testing element 3

Testing element 4

Testing element 5

Testing element 6

Testing element 7

Testing element 8

Testing element 9

Testing element 10

Testing element 11

Testing element 12

Test results: {True, True, True, True, True, True, True, True, False}

Sorting

Permutation = {1, 2, 3, 4, 5, 6, 12, 7, 8, 9, 10, 11}

Sorting over
```

```
Reducing basis element 12 of length 10

{0, 0, 0, 0, 1, 0, 0, 1, 0}

{d, 1, 11, 1, 12}

{0, 0, 0, 0, 1, 0, 0, 0, 0}

Reduction over

Degree is {0, 0, 0, 0, 1, 0, 0, 0, 0}

Element reduced. New length: 22

Testing element 12

Test results: {True, True, True, True, True, True, True, True, False}

Sorting

Permutation = {1, 2, 3, 4, 5, 6, 7, 12, 8, 9, 10, 11}

Sorting over

Symmetry number 1 of element 1 (length 11)

Equal to basis element 4

Symmetry number 1 of element 2 (length 20)

{0, 0, 0, 0, 0, 0, 1, 0, 1}

New element of length 20

Degree is {0, 0, 0, 0, 0, 0, 1, 0, 1}

Testing element 13

Test results: {True, True, True, True, True, True, True, True, False}

Sorting

Permutation = {1, 2, 3, 4, 5, 13, 6, 7, 8, 9, 10, 11, 12}

Sorting over

Trying to reduce elements

No elements reduced

Symmetry number 1 of element 3 (length 159)

Equal to basis element 5
```

$$\begin{pmatrix} 0 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 0 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 0 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 1 & 0 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 1 & 1 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 0 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 1 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 0 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 \end{pmatrix}$$

```
{1, 2}

{1, 0, 0, 0, 0, 0, 0, 0, 0}

New element of length 20

Degree is {1, 0, 0, 0, 0, 0, 0, 0, 0}

Testing element 14

Saved element 9 of length 20

All tests done

New element of length 17

Degree is {1, 0, 0, 0, 0, 0, 0, 0, 0}

New element of length 17

Degree is {0, 1, 0, 0, 0, 0, 0, 0, 0}

Sorting

Permutation = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 16, 13, 14, 15}

Sorting over

Evaluation time: 6.27005
```

In[14]:= **F[{-2, -2, -1, 0, -1, 2, 3, 4, -2}]**

In[15]:= **%**

Out[15]= $$\frac{2\,(-2+d)\,(576+1864\,d+1010\,d^2+463\,d^3-319\,d^4-5\,d^5+11\,d^6)\,G[\{0, 0, 0, 0, 0, 1, 1, 1, 0\}]}{3\,d^2\,(2+d)\,(4+d)}$$

Examples of reduction:

$$F(1,\ldots,1,0) = -\frac{3(d-4)(3d-10)}{8(d-5)(2d-9)} I_1 - \frac{3(d-4)(3d-10)}{16(d-5)(2d-9)} I_2$$
$$-\frac{(d-3)(3d-10)(3d-8)}{8(d-5)(3d-13)(3d-11)} I_3 - \frac{3(d-2)(3d-11)(3d-10)(3d-8)}{64(d-5)(2d-9)(2d-7)(3d-13)} \bar{I}_4$$
$$+\frac{9(d-4)(d-2)(3d-10)(3d-8)}{64(d-5)(2d-9)(2d-7)(3d-13)} I_5 - \frac{3(3d-10)(3d-8)}{32(d-5)(2d-9)(2d-7)} \bar{I}_7 \;,$$

$$F(1,\ldots,1,-1) = \frac{3(d-3)(3d-11)}{16(d-5)(d-4)(2d-9)} I_4$$
$$-\frac{(d-2)(2d-7)(2d-5)}{8(d-3)(2d-9)(3d-13)} I_6 - \frac{3(2d-7)^2(2d-5)(3d-11)(3d-7)}{256(d-4)^2(d-3)(2d-9)} I_7 \;.$$

Conclusion

- Other examples can be found in [A.V. Smirnov & V.A. Smirnov'05; http://www.srcc.msu.ru/nivc/about/lab/lab4_2/index_eng.htm]
- The algorithm consists of two parts: constructing the bases and the reduction (that needs the bases files)
- The reduction part of the algorithm is also available at my site (not updated since January, but I am planning to add a number of bases files and the updated reduction algorithm)
- The construction of the bases in problems with up to five indices is often almost automatic. In complicates cases one has to choose the ordering and some options carefully to have the bases constructed.

- The algorithm can work successfully at the level of modern calculations, e.g., in problems with up to 12 indices.
- There are various interesting practical and mathematical problems. Which orderings are optimal for a given sector? What is the order of CPU time needed for the construction of the corresponding s-basis? Will the algorithm work for a given problem?
- Further improvements are necessary for more sophisticated calculations. Probably, implementing the ideas used in other algorithms.
- A purely mathematical problem: to prove that the dimension of $\mathcal{S}$ is finite.