# Quantum particle in a random media

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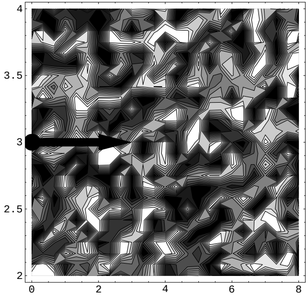
21 ноября 2014 г.

# Random media




**Localization length, or free path $\ell$ ?**

- **Functional representation**
- **Unordered media**
- **Random potential**
- **Random force**
- **Quantum field as random media**

- **Random media**

$$\mathbf{H} = \frac{\mathbf{p}^2}{2\mathbf{m}} + \mathbf{W}(\mathbf{x})$$

Potential $W(\mathbf{x}) \quad \Longrightarrow \quad$ a random media

- **Localization length** $\ell(k) = \frac{1}{\Gamma(k)}$

♣ $$\mathbf{G_k}(\mathbf{x}, \mathbf{x}'|\mathbf{W}) = \frac{1}{\frac{\mathbf{p}^2}{2\mathbf{m}} + \mathbf{W}(\mathbf{x}) - \frac{\mathbf{k}^2}{2\mathbf{m}} + \mathbf{i0}} \delta(\mathbf{x} - \mathbf{x}')$$

♣ $$\mathbf{G_k}(\mathbf{r}) = \langle \mathbf{G_k}(\mathbf{x}, \mathbf{x}'|\mathbf{W}) \rangle_{\mathbf{W}}, \qquad \mathbf{r} = |\mathbf{x} - \mathbf{x}'|$$

♣ $$\frac{1}{\ell(\mathbf{k})} = \mathbf{\Gamma}(\mathbf{k}) = -\lim_{\mathbf{r} \to \infty} \frac{\ln |\mathbf{G_k}(\mathbf{r})|}{\mathbf{r}}$$

# Functional representation

♣ $$G_k(r|W) = i\int\limits_0^\infty dt\; e^{-it\left[-\frac{1}{2m}\frac{d^2}{dx^2}+W(x)-\frac{k^2}{2m}-i0\right]}\delta(x)$$

$$= B\int\limits_0^\infty \frac{dt}{t^{\frac{3}{2}}}\; e^{\frac{i}{2}\left(\frac{tk^2}{m}+\frac{x^2m}{t}\right)}\int \frac{D\eta}{C} e^{i\int\limits_0^t d\nu\left[\frac{m}{2}\dot{\eta}^2(\nu)-W\left(x\frac{\nu}{t}+\eta(\nu)\right)\right]}$$

$$= B\int\limits_0^\infty \frac{ds}{s^{\frac{3}{2}}}\; e^{\frac{irk}{2}\left(s+\frac{1}{s}\right)}\int \frac{D\xi}{C} e^{\frac{i}{2}\int\limits_0^r d\tau\dot{\xi}^2(\tau)-im\frac{s}{k}\int\limits_0^r d\tau W\left(n\tau-\sqrt{\frac{s}{k}}\;\xi(\tau)\right)}$$

$$\xi(0) = \xi(r) = 0$$

♣ $$\left\langle e^{-i\frac{ms}{k}\int\limits_0^r d\tau W\left(n\tau+x'-\sqrt{\frac{s}{k}}\ \xi(\tau)\right)}\right\rangle_W = e^{Z_r\left[\frac{s}{k},\xi\right]}$$

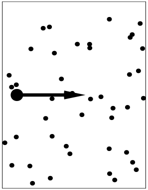
♣ $$I_r\,(k.s) = \int \frac{D\xi}{C} e^{\frac{i}{2}\int\limits_0^r d\tau\dot{\xi}^2(\tau)+Z_r\left[\frac{s}{k},\xi\right]} \to e^{r\ A\left(\frac{s}{k}\right)}$$

♣ $$G_k(r) = B\int\limits_0^\infty \frac{ds}{s^{\frac{3}{2}}}\ e^{r\left[\frac{ik}{2}\left(s+\frac{1}{s}\right)+A\left(\frac{s}{k}\right)\right]} \sim e^{r\ \Phi(k)}$$

♣ $$\Phi(k) = \min_s\left[\frac{ik}{2}\left(s+\frac{1}{s}\right)+A\left(\frac{s}{k}\right)\right] = \min_v\left[\frac{ik^2}{2}v+\frac{i}{2v}+A\,(v)\right]$$

$$\frac{1}{\ell(k)} = \Gamma(k) = -\mathrm{Re}\ \Phi(k)$$

# Unordered media




$$\mathbf{W}(\mathbf{x}) = \mathbf{W}(\mathbf{x}|\{\mathbf{y_j}\}) = \sum_{\mathbf{j}}^{\mathbf{N}} \mathbf{V}(\mathbf{x}-\mathbf{y_j})$$

$$\mathbf{V}(\mathbf{x}) = \mathbf{g}\ \mathbf{U}\left(\frac{\mathbf{r}}{\mathbf{R}}\right), \qquad \int_0^\infty \mathbf{ds}\ \mathbf{U}(\mathbf{s}) = \mathbf{1}, \qquad \mathbf{nR^3} \ll \mathbf{1}$$

$$\mathbf{e}^{\mathbf{Z_r}\left[\frac{\mathbf{s}}{\mathbf{k}},\xi\right]} = \left\langle \mathbf{e}^{-\mathbf{im}\frac{\mathbf{s}}{\mathbf{k}}\int_0^{\mathbf{r}} \mathbf{d}\tau \mathbf{W}\left(\mathbf{n}\tau+\mathbf{x}'-\sqrt{\frac{\mathbf{s}}{\mathbf{k}}}\ \xi(\tau)|\mathbf{y_j}\}\right)} \right\rangle_{\{\mathbf{y_j}\}}$$

$$= \exp\left\{ \mathbf{nR^3} \int \frac{\mathbf{dy}}{\mathbf{R^3}} \left[ \mathbf{e}^{-\mathbf{im}\frac{\mathbf{s}}{\mathbf{k}}\int_0^{\mathbf{r}} \mathbf{d}\tau\ \mathbf{V}\left(\mathbf{y}+\mathbf{n}\tau-\sqrt{\frac{\mathbf{s}}{\mathbf{k}}}\xi(\tau)\right)} - \mathbf{1} \right] \right\}$$

$$e^{r\;A\left(\frac{s}{k}\right)}=\int\frac{D\xi}{C}e^{\frac{i}{2}\int\limits_0^r d\tau\dot{\xi}^2(\tau)+Z_r\left[\frac{s}{k},\xi\right]}$$

$$A\left(\frac{s}{k}\right)\approx n\int\frac{D\xi}{C}e^{\frac{i}{2}\int\limits_0^r d\tau\dot{\xi}^2(\tau)}\int d\mathbf{y}\left[e^{-i\frac{ms}{k}\int\limits_0^r d\tau V\left(\mathbf{y}+\mathbf{n}\tau-\sqrt{\frac{s}{k}}\xi(\tau)\right)}-1\right]$$

$$\mathbf{A}\left(\frac{\mathbf{s}}{\mathbf{k}}\right)=\mathbf{i}\frac{\mathbf{n}}{\mathbf{2}}\cdot\mathbf{4}\pi\frac{\mathbf{s}}{\mathbf{k}}\mathbf{F}\left(\frac{\mathbf{k}}{\mathbf{s}},\mathbf{0}\right)=\frac{\mathbf{n}}{\mathbf{2}}\left[\mathbf{i}\phi\left(\frac{\mathbf{k}}{\mathbf{s}}\right)-\sigma\left(\frac{\mathbf{k}}{\mathbf{s}}\right)\right]$$

$\mathbf{F}\left(\frac{\mathbf{k}}{\mathbf{s}},\mathbf{0}\right)$ - **elastic scattering amplitude for momentum** $\frac{k}{s}$ **and scattering angle** $\theta=0$.

$$\mathbf{\Phi}(\mathbf{k})=\min_{\mathbf{s}}\left[\frac{\mathbf{ik}}{\mathbf{2}}\left(\mathbf{s}+\frac{\mathbf{1}}{\mathbf{s}}\right)+\mathbf{i}\;\mathbf{n}\cdot\mathbf{2}\pi\frac{\mathbf{s}}{\mathbf{k}}\mathbf{F}\left(\frac{\mathbf{k}}{\mathbf{s}},\mathbf{0}\right)\right]\quad\Rightarrow\quad\mathbf{s}=\mathbf{1}.$$

$$\frac{\mathbf{1}}{\ell(\mathbf{k})}=\mathbf{\Gamma}(\mathbf{k})=-\mathrm{Re}\;\mathbf{\Phi}(\mathbf{k})=\mathbf{n}\sigma(\mathbf{k})\quad\Rightarrow\quad\mathbf{I}(\mathbf{r})\sim|\mathbf{G_E}(\mathbf{r})|^2\sim\mathbf{e}^{-\mathbf{r}\;\mathbf{n}\sigma(\mathbf{k})}$$

$\sigma(k)$ - **total cross section,** $\ell=\frac{1}{n\sigma(k)}$ - **free path.**

# Small momenta $k \to 0$

$$4\pi \frac{s}{k} F\left(\frac{k}{s}, 0\right) = 4\pi \frac{s}{k} \mathrm{Re}\, F\left(\frac{k}{s}, 0\right) + i\sigma\left(\frac{k}{s}\right) \;\Rightarrow\; 4\pi \frac{s}{k} a + i\sigma\left(\frac{k}{s}\right)$$

$a$ - **scattering length,** $\sigma\left(\frac{k}{s}\right) \sim a^2 \ll \frac{a}{k}$

$$\Phi(k) = \min_s \left[ i \left[ \frac{k}{2}\left(s + \frac{1}{s}\right) + \frac{2\pi n a}{k} s \right] - n\,\sigma\left(\frac{k}{s}\right) \right] \;\Rightarrow\; n\,\sigma\left(\sqrt{k^2 + 4\pi n a}\right)$$

$$\frac{1}{k^2} a = \lambda^2 a = V_{int} \text{ -"interaction" volume},$$
$$\frac{1}{n} \sim L^3 = V_{atom} \text{ -"atom" volume}$$

**multiple scattering** $k^2 \ll 4\pi n a \;\Rightarrow\; \lambda^2 a \gg L^3, \quad V_{int} \gg V_{atom}$

$$I(r) \sim e^{-n\,\sigma\left(\sqrt{k^2+4\pi n a}\right)\cdot r} = \begin{cases} e^{-n\,\sigma(k)\cdot r}, & k \to \infty, \quad V_{int} \ll V_{atom}; \\ e^{-n\,\sigma\left(\sqrt{4\pi n a}\right)\cdot r}, & k \to 0, \quad V_{int} \gg V_{atom}. \end{cases}$$

# Random potential. Linear path approximation

Potential $W(x)$ is a random function with correlator

$$\langle \mathbf{W}(\mathbf{x})\mathbf{W}(\mathbf{y})\rangle_{\mathbf{W}} = \mathbf{K}(\mathbf{x}-\mathbf{y}) = \mathbf{g}^2\mathbf{U}\left(\frac{|\mathbf{x}-\mathbf{y}|}{\mathbf{R}}\right)$$

$R$ - correlation length, $g^2$ - coupling constant

$$e^{rA\left(\frac{k}{s}\right)} = \int \frac{D\xi}{C} e^{\frac{i}{2}\int\limits_0^r d\tau \dot{\xi}^2(\tau) - \frac{m^2 s^2}{2k^2}\iint\limits_0^r d\tau d\tau' K\left(n(\tau-\tau') - \sqrt{\frac{s}{k}}(\xi(\tau)-\xi(\tau'))\right)}$$

Large $r$ $\Rightarrow$ $\xi(\tau) \sim \tau$ $\Rightarrow$ $\xi(\tau) = b\tau + \eta(\tau)$

$$e^{rA\left(\frac{k}{s}\right)} \approx \int \frac{db}{C_b} e^{\frac{i}{2}rb^2 - \frac{m^2 s^2}{2k^2}\iint\limits_0^r d\tau d\tau' K\left(\left(n + b\sqrt{\frac{s}{k}}\right)(\tau-\tau')\right)} = \int \frac{db}{C_b} e^{r\left[\frac{i}{2}b^2 - \frac{s^2}{k^2}\frac{G}{\left|n+\sqrt{\frac{s}{k}}b\right|}\right]}$$

$$A\left(\frac{k}{s}\right) \approx \min_b \left[\frac{i}{2}b^2 - \frac{s^2}{k^2}\frac{G}{\left|n+\sqrt{\frac{s}{k}}b\right|}\right], \qquad G = \int\limits_0^\infty dv\, K(v)$$

$$\mathbf{\Phi}(\mathbf{k}) = \min_{\mathbf{s},\mathbf{b}} \left\{ \mathbf{i}\frac{\mathbf{k}}{\mathbf{2}} \left( \mathbf{s} + \frac{\mathbf{1}}{\mathbf{s}} \left( \mathbf{1} + (\mathbf{b}+\mathbf{1})^{\mathbf{2}} \right) \right) - \left( \frac{\mathbf{s}}{\mathbf{k}} \right)^{\mathbf{2}} \frac{\mathbf{G}}{|\mathbf{b}|} \right\}$$

$$= \mathbf{i}\frac{\mathbf{k}}{\mathbf{2}} \min_{\mathbf{x}>\mathbf{0}} \left\{ \sqrt{\mathbf{2}\left( \mathbf{1} + \mathbf{x}^{\mathbf{2}} + \mathbf{i}\frac{\mathbf{2G}}{\mathbf{x}\mathbf{k}^{\mathbf{3}}} \right)} - \mathbf{x} \right\}$$

$$\frac{\mathbf{1}}{\ell(\mathbf{k})} = \mathbf{\Gamma}(\mathbf{k}) = -\mathrm{Re}\ \mathbf{\Phi}(\mathbf{k}) = \begin{cases} O\left(\frac{G}{k^2}\right), & k \to \infty, \\ O\left(G^{\frac{1}{3}}\right), & k \to 0. \end{cases}$$

# Random force $\mathcal{F}(x)$. Perturbation calculation.

$$H = \frac{p^2}{2m} + (\mathcal{F}(x)x)$$

$$\langle \mathcal{F}_i(x)\mathcal{F}_j(y)\rangle = \delta_{ij}K(x-y) = \delta_{ij}g^2 U\left(\frac{|x-y|}{R}\right), \qquad \int_0^\infty ds\, U(s) < \infty.$$

$$G_k(r) = \int_0^\infty \frac{ds}{s^{\frac{3}{2}}}\, e^{\frac{irk}{2}\left(s+\frac{1}{s}\right)} \int \frac{D\xi}{C} e^{\frac{i}{2}\int_0^r d\tau \dot{\xi}^2(\tau) - \frac{1}{2}\left(\frac{ms}{k}\right)^2 \iint_0^r d\tau_1 d\tau_2 K(b(\tau_1)-b(\tau_2))(b(\tau_1)b(\tau_2))}$$

$$b(\tau) = n\tau + \sqrt{\frac{s}{k}}\xi(\tau)$$

$$\ln G_k(r) = \min_v \left[\frac{irk^2}{2}v + \frac{ir}{2v} - \frac{1}{2}v^2 r^3 \frac{m^2}{3}\int_0^{\left(\frac{2}{v}\right)^2} \frac{dz}{4\pi}\tilde{K}(z)\right] \to -\frac{3}{8}(2H)^{\frac{1}{3}} \cdot r^{\frac{5}{3}}$$

## Quantum field as a random media

1. **Quantum field $\Phi(x)$ - external gaussian random field**

$$H = \frac{\mathbf{p}^2}{2m} + h\Phi(\mathbf{x})$$

.

$$G_k(\mathbf{x}-\mathbf{y}) = \left\langle \frac{1}{\frac{\mathbf{p}^2}{2m} + h\Phi(\mathbf{x}) - \frac{\mathbf{k}^2}{2m}} \delta(\mathbf{x}-\mathbf{y}) \right\rangle_\Phi$$

2. **Quantum field and quantum particle - parts of unique system**

$$H = H_0[\Phi] + \frac{\mathbf{p}^2}{2m} + h\Phi(\mathbf{x})$$

.

$$G_k(\mathbf{x}-\mathbf{y}) = \left\langle \frac{1}{H_0[\Phi] + \frac{\mathbf{p}^2}{2m} + h\Phi(\mathbf{x}) - \frac{\mathbf{k}^2}{2m}} \delta(\mathbf{x}-\mathbf{y}) \right\rangle_\Phi$$

# Field $\Phi(x)$

$$H_0[\Phi] = \int d\mathbf{p}\ p\ a_{\mathbf{p}}^{+} a_{\mathbf{p}}$$

$$\Phi(\mathbf{x}, t) = \int \frac{d\mathbf{p}}{(2\pi)^{\frac{3}{2}}\sqrt{2p)}} \left[ a_{\mathbf{p}} e^{-ipt+i\mathbf{px}} + a_{\mathbf{p}}^{+} e^{ipt'-i\mathbf{px}} \right], \qquad p = |\mathbf{p}|$$

$$K(x - x') = \langle 0 | \mathrm{T} \left\{ \Phi(\mathbf{x}, t)\Phi(\mathbf{x}', t') \right\} |0\rangle = D_c(\mathbf{x} - \mathbf{x}', t - t')$$
$$= \int \frac{d\mathbf{p}}{(2\pi)^3 2p} e^{-ip|t-t'|+i\mathbf{p}(\mathbf{x}-\mathbf{x}')}$$

## 1. Quantum particle and quantum field as random media

$$G_k(\mathbf{x}-\mathbf{y}) = \left\langle \frac{1}{\frac{\mathbf{p}^2}{2m} + h\Phi(\mathbf{x}) - \frac{\mathbf{k}^2}{2m}} \delta(\mathbf{x}-\mathbf{y}) \right\rangle_{\Phi} \sim e^{-r\frac{g^2}{8\pi k}}$$

## 2. Quantum field and quantum particle - parts of unique system

$$G_k(\mathbf{x}-\mathbf{y})$$

$$= \left\langle \frac{1}{H_0[\Phi] + \frac{\mathbf{p}^2}{2m} + h\Phi(\mathbf{x}) - \frac{\mathbf{k}^2}{2m}} \delta(\mathbf{x}-\mathbf{y}) \right\rangle_{\Phi} \sim \begin{cases} 1 & k < m \\ e^{-r\frac{g^2 m^2}{k^2}\frac{k-m}{8\pi^2}} & k > m \end{cases}$$

# Relativistic particle

$$\left[\Box_x + m^2 + g\Phi(x)\right] G(x - x') = \delta(x - x'), \qquad \Box = \partial_t^2 - \nabla^2, \qquad x^2 = x_0^2 - x^2$$

$$G_E(x - y) = \langle G_E(x, y|\Phi)\rangle_\Phi = \left\langle \frac{1}{\Box_x + m^2 + g\Phi(x)} \delta(x - y) \right\rangle_\Phi$$

$$\ln |G(x - y)| \to 0 \qquad \Longrightarrow \qquad \frac{1}{\ell} = \Gamma = 0$$

# Conclusion

- The path integral representation - the best method to investigate differential equations with random coefficients.
- Green function - representation in the form of the Gaussian path integral.
- Large momenta $k$ - perturbation calculations.
- Unordered media

$$\mathbf{I}(\mathbf{r}) \sim \mathbf{e}^{-\mathbf{n}\ \sigma\left(\sqrt{\mathbf{k}^2+4\pi\mathbf{na}}\right)\cdot\mathbf{r}} = \begin{cases} \mathbf{e}^{-\mathbf{n}\ \sigma(\mathbf{k})\cdot\mathbf{r}}, & \mathbf{k}\to\infty, \quad \textbf{single scatt.}; \\ \mathbf{e}^{-\mathbf{n}\ \sigma\left(\sqrt{4\pi\mathbf{na}}\right)\cdot\mathbf{r}}, & \mathbf{k}\to 0, \quad \textbf{multiple scatt.} \end{cases}$$

- Random potential - Linear path approximation for small $k$.
- Random force - stable bound states.
- Quantum field as a random media - localization is absent in the relativistic case only.