Nucleon Form Factor within Light Cone Sum Rules up to NLO and higher twists

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- We derive light-cone sum rules for the electromagnetic nucleon form factors including the next-to-leading-order corrections for the contribution of twist-three and twist-four operators and a consistent treatment of the nucleon mass corrections.
- The soft contributions are calculated in terms of small transverse distance quantities using dispersion relations and quark-hadron duality.
- The form factors are expressed in terms of nucleon wave functions at small transverse separations (DAs), without any additional parameters.
- The distribution amplitudes can be extracted from the comparison with the experimental data on form factors and compared to the results of lattice QCD simulations. A self-consistent picture emerges, with the three valence quarks carrying 40% : 30% : 30% of the proton momentum.

The work consist of three parts:

- Calculations within LCSR;
- Factorized amplitude at LO up to twist-6 and at NLO up to twist-4. We calculated 22 coefficient functions at NLO and 20 of them are new ones. To avoid the mixture with the so-called evanescent operators, we use the renormalization procedure for operators with open Dirac indices;
- Distribution amplitudes. In particular, light-cone expansion to the twist-4 accuracy of the three-quark matrix elements with generic quark positions

LCSRs for nucleon form factors: General structure

The LCSR approach allows one to calculate the form factors in terms of the nucleon (proton) DAs. To this end we consider the correlation function

$$T_\nu(P,q) = i\int d^4x\, e^{iqx}\langle 0|\, T\,[\eta(0) j_\nu^{\mathrm{em}}(x)]\,|P\rangle$$

where T denotes time-ordering and $\eta(0)$ is the Ioffe interpolating current

$$\eta(x) = \epsilon^{ijk}\left[u^i(x) C\gamma_\mu u^j(x)\right]\,\gamma_5\gamma^\mu d^k(x)\,,$$
$$\langle 0|\eta(0)|P\rangle = \lambda_1 m_N N(P)\,.$$

The matrix element of the electromagnetic current

$$j_\mu^{\rm em}(x) = e_u \bar{u}(x)\gamma_\mu u(x) + e_d \bar{d}(x)\gamma_\mu d(x)$$

taken between nucleon states is conventionally written in terms of the Dirac and Pauli form factors $F_1(Q^2)$ and $F_2(Q^2)$:

$$\langle P'|j_\mu^{\rm em}(0)|P\rangle = \bar{N}(P')\left[\gamma_\mu F_1(Q^2) - i\frac{\sigma_{\mu\nu}q^\nu}{2m_N}F_2(Q^2)\right]N(P).$$

In terms of the electric $G_E(Q^2)$ and magnetic $G_M(Q^2)$ Sachs form factors, we have

$$\begin{aligned} G_M(Q^2) &= F_1(Q^2) + F_2(Q^2), \\ G_E(Q^2) &= F_1(Q^2) - \frac{Q^2}{4m_N^2}F_2(Q^2). \end{aligned}$$

Light-Cone Basis

We define a light-like vector n_μ by the condition

$$q \cdot n = 0\,, \qquad n^2 = 0\,, \quad q = -\frac{q_\perp^2}{2} + q_\perp\,, \quad q^2 = q_\perp^2 = -Q^2\,.$$

and introduce the second light-like vector as

$$p_\mu = P_\mu - \frac{1}{2}\, n_\mu \frac{m_N^2}{P \cdot n}\,, \qquad p^2 = 0\,,$$

and

$$g^\perp_{\mu\nu} = g_{\mu\nu} - \frac{1}{pn}(p_\mu n_\nu + p_\nu n_\mu)$$

We consider the “plus” spinor projection of the correlation function involving the “plus” component of the electromagnetic current, which can be parametrized in terms of two invariant functions

$$\Lambda_+ T_+ = p_+ \left\{ m_N \mathcal{A}(Q^2, P'^2) + \hat{q}_\perp \mathcal{B}(Q^2, P'^2) \right\} N^+(P)\,,$$

where $Q^2 = -q^2$ and $P'^2 = (P-q)^2$ and

$$N^\pm(P) = \Lambda^\pm N(P),$$
$$\Lambda^+ = \frac{\hat{p}\,\hat{n}}{2pn}\,, \quad \Lambda^- = \frac{\hat{n}\,\hat{p}}{2pn}$$

Making the Borel transformation

$$\frac{1}{s - P'^2} \longrightarrow e^{-s/M^2}$$

one obtains the sum rules

$$2\lambda_1 F_1(Q^2) = \frac{1}{\pi} \int_0^{s_0} ds\, e^{(m_N^2 - s)/M^2} \mathrm{Im}\, \mathcal{A}^{\mathrm{QCD}}(Q^2, s)\,,$$
$$\lambda_1 F_2(Q^2) = \frac{1}{\pi} \int_0^{s_0} ds\, e^{(m_N^2 - s)/M^2} \mathrm{Im}\, \mathcal{B}^{\mathrm{QCD}}(Q^2, s)\,.$$

The correlation functions $\mathcal{A}(Q^2, P'^2)$ and $\mathcal{B}(Q^2, P'^2)$ can be written as a sum:

$$\mathcal{A} = e_d\,\mathcal{A}_d + e_u\mathcal{A}_u\,, \qquad \mathcal{B} = e_d\,\mathcal{B}_d + e_u\mathcal{B}_u\,.$$

Each of the functions has a perturbative expansion which we write as

$$\mathcal{A} = \mathcal{A}^{\rm LO} + \frac{\alpha_s(\mu)}{3\pi}\mathcal{A}^{\rm NLO} + \ldots$$

and similar for $\mathcal{B}$; μ is the renormalization scale.

For consistency with our NLO calculation we rewrite LO results in a different form, expanding all kinematic factors in powers of m_N^2/Q^2: We keep all corrections $\mathcal{O}(m_N^2/Q^2)$ but neglect terms $\mathcal{O}(m_N^4/Q^4)$ etc. which is consistent with taking into account contributions of twist-three, -four, -five (and, partially, twist-six) in the OPE.

The following Feynman diagrams contribute to the NLO amplitude.

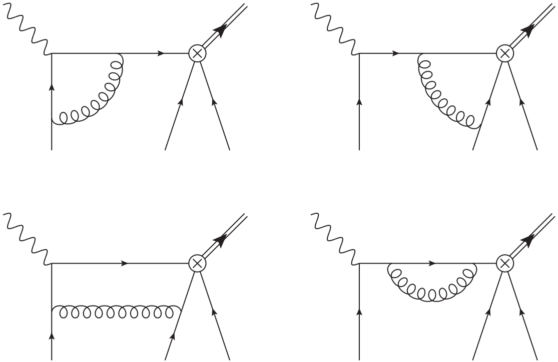


Figure: NLO corrections to the light-cone sum rule for baryon form factors.

The NLO corrections read (see, our paper arXiv:1310.1375 [hep-ph]).

$$Q^2 \mathcal{A}_q^{\rm NLO} =$$
$$= \int [dx_i] \Big\{ \sum_{k=1,3} \Big[\mathbb{V}_k(x_i) C_q^{\mathbb{V}_k}(x_i, W) + \mathbb{A}_k(x_i) C_q^{\mathbb{A}_k}(x_i, W) \Big]$$
$$+ \sum_{m=1,2,3} \Big[\mathbb{V}_2^{(m)}(x_i) C_q^{\mathbb{V}_2^{(m)}}(x_i, W) + \mathbb{A}_2^{(m)}(x_i) C_q^{\mathbb{A}_2^{(m)}}(x_i, W) \Big] \Big\}$$
$$+ \mathcal{O}(\text{twist-5}) \,,$$

where

$$W = \frac{Q^2 + P'^{\,2}}{Q^2} \,.$$

$$Q^2 \mathcal{B}_q^{\rm NLO} =$$
$$= \int [dx_i] \Big[\mathbb{V}_1(x_i) D_q^{\mathbb{V}_1}(x_i, W) + \mathbb{A}_1(x_i) D_q^{\mathbb{A}_1}(x_i, W) \Big]$$
$$+ \mathcal{O}(\text{twist-5}).$$

It turns out that $C_d^{\mathbb{V}_2^{(1)}}(x_i, W) = C_d^{\mathbb{A}_2^{(1)}}(x_i, W) = 0$.

$$
\begin{aligned}
&x_2 C_d^{\mathbb{V}_1}(x_i) = \\
&\quad 2x_2x_3\Big[3(L-2)g_1(x_3) + 2(L-1)g_{11}(x_3,x_3) + g_{21}(x_3,x_3)\Big] \\
&\quad +\Big[2x_2 + (4L-3)x_3\Big]h_{11}(x_3) + (3-4L)\bar{x}_1 h_{11}(\bar{x}_1) \\
&\quad + 2x_3 h_{21}(x_3) - 2\bar{x}_1 h_{21}(\bar{x}_1) - 2\Big[3(x_2/x_3)(2L-3) + 5L - 7\Big]h_{12}(x_3) \\
&\quad +2(5L-7)h_{12}(\bar{x}_1) - \Big[6(x_2/x_3) + 5\Big]h_{22}(x_3) \\
&\quad + 5h_{22}(\bar{x}_1) + (6/x_3)(L-2)h_{13}(x_3) - (6/\bar{x}_1)(L-2)h_{13}(\bar{x}_1) \\
&\quad +(3/x_3)h_{23}(x_3) - (3/\bar{x}_1)h_{23}(\bar{x}_1)\,,
\end{aligned}
$$

$$x_2 C_d^{\mathbb{A}_1}(x_i) =$$
$$3\bar{x}_1 h_{11}(\bar{x}_1) - 3x_3 h_{11}(x_3) + 2(3L-10)h_{12}(\bar{x}_1)$$
$$-2(3L-10)h_{12}(x_3) + 3h_{22}(\bar{x}_1) - 3h_{22}(x_3)$$
$$-(6/\bar{x}_1)(L-3)h_{13}(\bar{x}_1) + (6/x_3)(L-3)h_{13}(x_3)$$
$$-(3/\bar{x}_1)h_{23}(\bar{x}_1) + (3/x_3)h_{23}(x_3)\,,$$

where

$$g_{nk}(y,x;W) = \frac{\ln^n[1-yW-i\eta]}{(-1+xW+i\eta)^k},$$
$$h_{nk}(x;W) = \frac{\ln^n[1-xW-i\eta]}{(W+i\eta)^k}$$

with $n = 0, 1, 2$ and $k = 1, 2, 3$. For $n = 0$ the first argument becomes dummy,i.e

$$g_k(x;W) \equiv g_{0k}(*,x;W)\,,$$

Results

Discussion of parameters Schematically, the general structure of form factors has the following form:

$$\mathcal{F} = \mathcal{F}_0^{\text{tw-4}} + \frac{f_N}{\lambda_1}\mathcal{F}_{f_N}^{\text{tw-3}} + \sum_{i=0,1}\eta_{1i}\mathcal{F}_{\eta_{1i}}^{\text{tw-4}} + \frac{f_N}{\lambda_1}\sum_{i=1}^{2}\sum_{j=0;j\leq i}^{2}\varphi_{ij}\mathcal{F}_{\varphi_{ij}}^{\text{tw-3}}\,.$$

Or, in other words, we have

- tw-3: $\left\{\varphi_{10}, \varphi_{11}, \varphi_{20}, \varphi_{21}, \varphi_{22}\right\}, f_N;$
- tw-4: $\left\{\eta_{10}, \eta_{11}\right\}, \lambda_1;$

The other parameters that enter LCSRs are

- the interval of duality (continuum threshold) s_0 ($s_0 = 2.25$ GeV^2);
- Borel parameter M^2 ($M^2 = 1.5$ GeV^2 and $M^2 = 2$ GeV^2 and $M^2 \simeq s_0$) ;
- factorization/renormalization scale μ^2 ($\mu^2 = 2$ GeV^2 and $\mu^2 \sim (1-x)Q^2 - xP'^2$ or $\mu^2 \leq (1-x_0)Q^2 + x_0 M^2 \leq \frac{2s_0 Q^2}{s_0+Q^2} < 2s_0$).
- We use a two-loop expression for the QCD coupling with $\Lambda^{(4)}_{QCD} = 326$ MeV resulting in the value $\alpha_s(2 \text{ GeV}^2) = 0.374$.

Model	Method	f_N/λ_1	φ_{10}	φ_{11}	φ_{20}	φ_{21}	φ_{22}
ABO1	LCSR (NLO)	-0.17	0.05	0.05	0.075	-0.027	0.17
ABO2	LCSR (NLO)	-0.17	0.05	0.05	0.038	-0.018	-0.13
BLW	LCSR (LO)	-0.17	0.0534	0.0664	-	-	-
BK	pQCD	-	0.0357	0.0357	-	-	-
COZ	QCDSR (LO)	-	0.163	0.194	0.41	0.06	-0.163
KS	QCDSR (LO)	-	0.144	0.169	0.56	-0.01	-0.163
	QCDSR (NLO)	-0.15	-	-	-	-	-
BS(HET)	QCDSR(LO)	-	0.152	0.205	0.65	-0.27	0.020
LAT09	LATTICE	-0.083	0.043	0.041	0.038	-0.14	-0.47
LAT13	LATTICE	-0.075	0.038	0.039	-0.050	-0.19	-0.19

Model	Method	η_{10}	η_{11}
ABO1	LCSR (NLO)	-0.039	0.140
ABO2	LCSR (NLO)	-0.027	0.092
BLW	LCSR (LO)	0.05	0.0325
BK	pQCD	-	-
COZ	QCDSR (LO)	-	-
KS	QCDSR (LO)	-	-
	QCDSR (NLO)	-	-
LAT09	LATTICE	-	-
LAT13	LATTICE	-	-

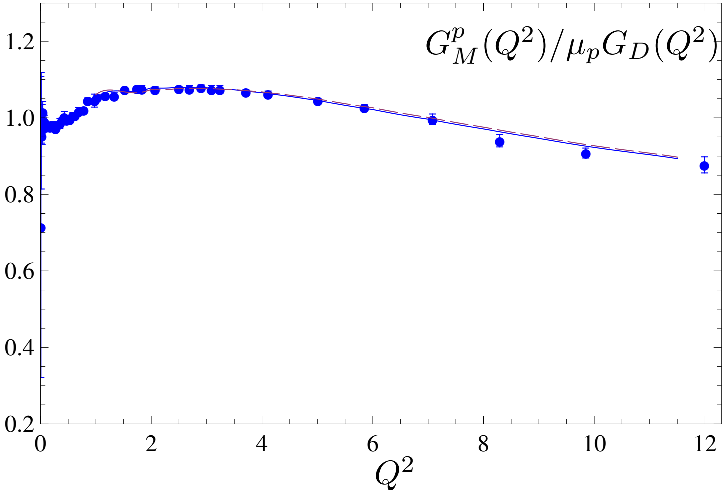
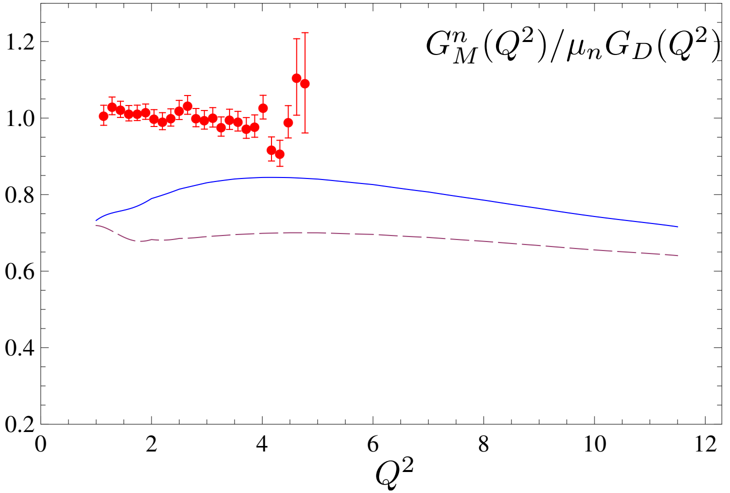


Figure: Nucleon electromagnetic form factors from LCSRs compared to the experimental data [CLAS Coll., Jeff.Lab. Hall A Coll.]. Parameters of the nucleon DAs correspond to the sets ABO1 and ABO2 in Table for the solid and dashed curves, respectively. Borel parameter $M^2 = 1.5$ GeV^2 for ABO1 and $M^2 = 2$ GeV^2 for ABO2.



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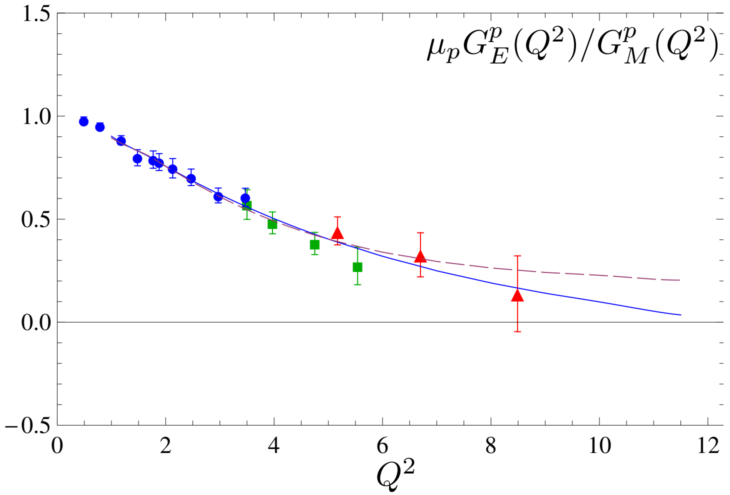

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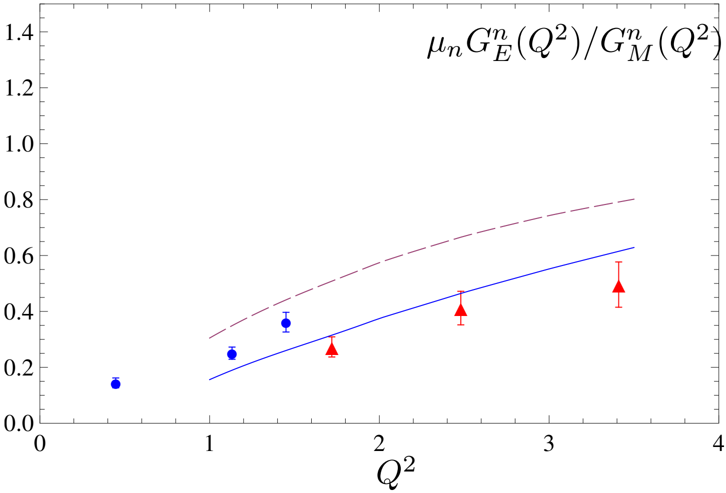

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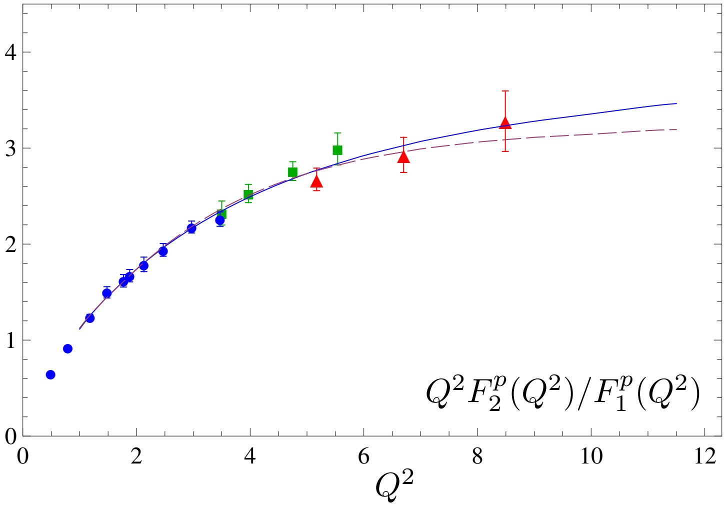

Figure: The ratio of Pauli and Dirac electromagnetic proton form factors from LCSRs compared to the experimental data [Jeff.Lab. Hall A Coll.]. Parameters of the nucleon DAs correspond to the sets ABO1 and ABO2 in Table for the solid and dashed curves, respectively. Borel parameter $M^2 = 1.5$ GeV^2 for ABO1 and $M^2 = 2$ GeV^2 for ABO2.

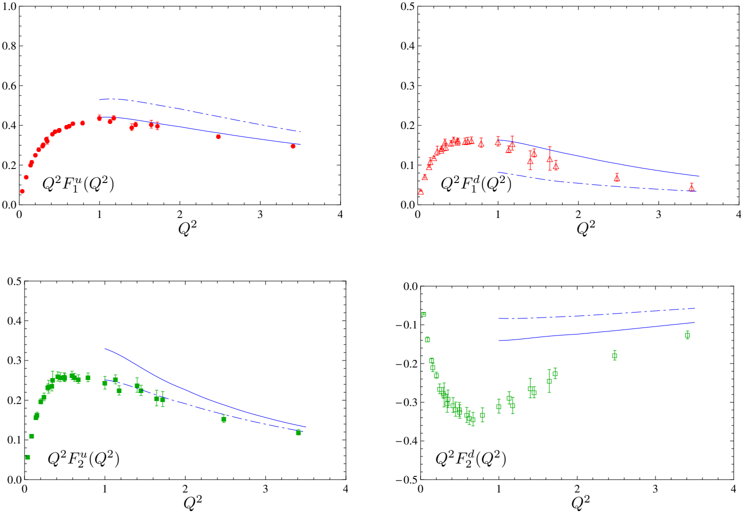

Figure: The corresponding leading-order results are shown by the dash-dotted curves for comparison. Parameters of the nucleon DAs correspond to the set ABO1 in the table.

On Light-Cone Sum Rules

The "plus" component of the electromagnetic current, which can be parametrized in terms of two invariant functions

$$\Lambda_+ T_+ = p_+ \left\{ m_N \mathcal{A}(Q^2, P'^2) + \hat{q}_\perp \mathcal{B}(Q^2, P'^2) \right\} N^+(P) \, ,$$

where $Q^2 = -q^2$ and $P'^2 = (P-q)^2$.
The correlation functions $\mathcal{A}(Q^2, P'^2)$ and $\mathcal{B}(Q^2, P'^2)$ can be calculated in QCD for sufficiently large Euclidean momenta $Q^2, -P'^2 \gtrsim 1$ GeV^2 using OPE.
The results can be presented in the form of a dispersion relation

$$\begin{aligned} \mathcal{A}^{\rm QCD}(Q^2, P'^2) &= \frac{1}{\pi} \int_0^\infty \frac{ds}{s - P'^2} \mathrm{Im}\, \mathcal{A}^{\rm QCD}(Q^2, s) + \dots \\ \mathcal{B}^{\rm QCD}(Q^2, P'^2) &= \frac{1}{\pi} \int_0^\infty \frac{ds}{s - P'^2} \mathrm{Im}\, \mathcal{B}^{\rm QCD}(Q^2, s) + \dots \end{aligned}$$

where the ellipses stand for necessary subtractions.

On the other hand, the same correlation functions can be written in terms of physical spectral densities that contain a nucleon (proton) pole at $P'^2 \to m_N^2$, the nucleon resonances and the continuum. The nucleon contribution is proportional to the e.m. form factor, whereas the contribution of higher mass states can be taken into account using quark-hadron duality:

$$\mathcal{A}^{\rm phys}(Q^2, P'^2) = \frac{2\lambda_1 F_1(Q^2)}{m_N^2 - P'^2} + \frac{1}{\pi}\int_{s_0}^{\infty} \frac{ds}{s - P'^2} \mathrm{Im}\, \mathcal{A}^{\rm QCD}(Q^2, s) + ..$$

$$\mathcal{B}^{\rm phys}(Q^2, P'^2) = \frac{\lambda_1 F_2(Q^2)}{m_N^2 - P'^2} + \frac{1}{\pi}\int_{s_0}^{\infty} \frac{ds}{s - P'^2} \mathrm{Im}\, \mathcal{B}^{\rm QCD}(Q^2, s) + ..$$

where $s_0 \simeq (1.5\ \mathrm{GeV})^2$ is the interval of duality (also called continuum threshold).

Matching the two above representations and making the Borel transformation that eliminates subtraction constants

$$\frac{1}{s - P'^2} \longrightarrow e^{-s/M^2}$$

one obtains the sum rules

$$2\lambda_1 F_1(Q^2) = \frac{1}{\pi} \int_0^{s_0} ds\, e^{(m_N^2 - s)/M^2} \mathrm{Im}\, \mathcal{A}^{\mathrm{QCD}}(Q^2, s)\,,$$

$$\lambda_1 F_2(Q^2) = \frac{1}{\pi} \int_0^{s_0} ds\, e^{(m_N^2 - s)/M^2} \mathrm{Im}\, \mathcal{B}^{\mathrm{QCD}}(Q^2, s)\,.$$

On Renormalization

By the standard way, in the d-dimensional space, the Ioffe current gets mixed under renormalization with the operators like

$$\eta_I^{(n)}(0) = [\psi^T(0)\, C\Gamma^{(n)}_{\alpha_1\ldots\alpha_n}\, \psi(0)]\, \gamma_5 \Gamma^{(n)}_{\alpha_1\ldots\alpha_n} \psi(0)\,,$$

where $\Gamma^{(n)}_{\alpha_1\ldots\alpha_n}$ is the antisymmetric product of γ-matrices. So, within the framework of the dimensional regularization, the renormalized operators take the form:

$$(\eta_I^{(n)})_R = \sum_k \mathbb{Z}_{nk}\, \eta_I^{(k)}\,.$$

here, we also have to include the mixture of the so-called evanescent operators with the physical operators.
As was shown, such a mixture can be eliminated by the finite renormalization and this finite regularization has to be taken into account in the anomalous dimension matrix.

To avoid the mixture of the evanescent operators with the physical ones, we will work with the open Dirac indices:

$$\eta_{I\,\underline{\delta}}(0) = (C\gamma_\alpha)_{\underline{\alpha\beta}} \otimes (\gamma_5\gamma_\alpha)_{\underline{\delta\gamma}}\,[\psi^T_{\underline{\alpha}}(0)\,\psi_{\underline{\beta}}(0)]\,\psi_{\underline{\gamma}}(0)$$

and the local tree quark operator with open spinor indices is renormalized as

$$\Big([\psi^T_{\underline{\alpha}}(0)\,\psi_{\underline{\beta}}(0)]\,\psi_{\underline{\gamma}}(0)\Big)_R = \mathbb{Z}_{\underline{\alpha\alpha'},\underline{\beta\beta'},\underline{\gamma\gamma'}}\,[\psi^T_{\underline{\alpha'}}(0)\,\psi_{\underline{\beta'}}(0)]\,\psi_{\underline{\gamma'}}(0)\,,$$

$$\mathbb{Z}_{\underline{\alpha\alpha'},\underline{\beta\beta'},\underline{\gamma\gamma'}} = \sum_{nmk} a_{nmk}(\epsilon)\,(\Gamma_{nmk})_{\underline{\alpha\alpha'},\underline{\beta\beta'},\underline{\gamma\gamma'}}\,,$$

where

$$a_{nmk}(\epsilon) = \sum_{p=0}^{\infty}\frac{(a^{(p)})_{nmk}}{\epsilon^p}, \qquad (\Gamma_{nmk})_{\underline{\alpha\alpha'},\underline{\beta\beta'},\underline{\gamma\gamma'}} = \gamma^{(n)}_{\underline{\alpha\alpha'}} \otimes \gamma^{(m)}_{\underline{\beta\beta'}} \otimes \gamma^{(k)}_{\underline{\gamma\gamma'}}$$

and the bare coupling constant is defined

$$\alpha_0 = \mu^{-2\epsilon}\mathbb{Z}_\alpha(\mu^2)\alpha_S(\mu^2) \quad \text{with} \quad \mathbb{Z}_\alpha(\mu^2) = 1 - \frac{\alpha_S(\mu^2)}{4\pi\,\epsilon}\beta_0\,.$$

Up to α_S-order, we have

$$\begin{aligned}
\mathbb{Z}^{-1}\, n^{\mu}\mathcal{M}_{\mu}\,\mathbb{Z}_{\mathcal{F}}^{-1} &= \Big[1-\frac{\alpha_S(\mu_1^2)}{4\pi}\frac{C_1}{\epsilon}\Big]\Big[\mathcal{M}^{\mathrm{LO}}+\frac{\alpha_S(\mu^2)}{4\pi}\mathcal{M}^{\mathrm{NLO}}\Big]\times \\
&\Big[1-\frac{\alpha_S(\mu_F^2)}{4\pi\,\epsilon}\mathbf{H}\Big] \\
&= \mathcal{M}^{\mathrm{LO}}+\frac{\alpha_S(\mu^2)}{4\pi}\mathcal{M}^{\mathrm{NLO}}_{\mathrm{fin.}}+\frac{\alpha_S(\mu^2)}{4\pi\epsilon}\mathcal{M}^{\mathrm{NLO}}_{\mathrm{sing.}}- \\
&\frac{\alpha_S(\mu_1^2)}{4\pi\epsilon}C_1\mathcal{M}^{\mathrm{LO}}-\frac{\alpha_S(\mu_F^2)}{4\pi\epsilon}\mathbf{H}*\mathcal{M}^{\mathrm{LO}}\,,
\end{aligned}$$

The singular terms have been cancelled each others, i.e.

$$\mathcal{M}^{\text{NLO}}_{\text{sing.}} - \left(\frac{\mu^2}{\mu_1^2}\right)^{\epsilon} C_1 \mathcal{M}^{\text{LO}} - \left(\frac{\mu^2}{\mu_F^2}\right)^{\epsilon} \mathbf{H} * \mathcal{M}^{\text{LO}} = 0\,.$$

Calculations of coefficient functions

In the most general form, the invariant amplitude reads

$$\begin{aligned}\mathcal{A}^\mu &= \mathcal{P}^{(\eta)}_{\underline{\alpha\beta},\underline{\delta\gamma}}\,\mathbb{Z}^{-1}_{\underline{\alpha\alpha'},\underline{\beta\beta'},\underline{\gamma\gamma'}}\int[dx_i]\,\mathcal{M}^{\mu}_{\underline{\alpha'\alpha_1},\underline{\beta'\alpha_2},\underline{\gamma'\alpha_3}}(x_i;q,P)\times \\ &\quad (\mathbb{Z}^{-1}_{\mathcal{F}})_{\underline{\alpha_1\alpha_1'},\underline{\alpha_2\alpha_2'},\underline{\alpha_3\alpha_3'}}\,\mathcal{F}^{(3q)\,R}_{\underline{\alpha_1'\alpha_2'\alpha_3'}}(x_i)\,,\end{aligned}$$

where the soft part of amplitude is defined with the open Dirac indices, *i.e.*

$$\mathcal{F}^{(3q)}_{\underline{\alpha\beta},\underline{\delta}}(x_i) \stackrel{\mathbb{F}}{=} \langle 0|\psi_{\underline{\alpha}}(z_1)\psi_{\underline{\beta}}(z_2)\psi_{\underline{\delta}}(z_3)|B(P)\rangle\,,$$

and the Ioffe current projection is defined by

$$\mathcal{P}^{(\eta)}_{\underline{\alpha\beta},\underline{\gamma\delta}} \stackrel{\text{def.}}{=} (C\gamma_\alpha)_{\underline{\alpha\beta}}(\gamma_5\gamma_\alpha)_{\underline{\gamma\delta}}\,.$$

LO calculations.

The LO amplitude, related to d-quark contribution, takes the form:

$$\mathcal{A}_\mu^{(\mathrm{LO})} = \mathcal{P}^{(\eta)}_{\underline{\alpha\beta},\underline{\delta\gamma}}\, \mathbb{Z}^{-1}_{\underline{\alpha\alpha'},\underline{\beta\beta'},\underline{\gamma\gamma'}} \times$$
$$\int [dx_i]\, \frac{(x_1 P + q)_\alpha}{(x_1 P + q)^2} \left(\mathbb{I}_{\underline{\alpha'\alpha_1}} \otimes \mathbb{I}_{\underline{\beta'\alpha_2}} \otimes (g_{\alpha\mu}\mathbb{I} + \Gamma^{(2)}_{\alpha\mu})_{\underline{\gamma'\alpha_3}} \right) \times$$
$$(\mathbb{Z}^{-1}_{\mathcal{F}})_{\underline{\alpha_1\alpha'_1},\underline{\alpha_2\alpha'_2},\underline{\alpha_3\alpha'_3}}\, \mathcal{F}^{(3q)\,R}_{\underline{\alpha'_1\alpha'_2\alpha'_3}}(x_i)\,.$$

NLO calculations.

The general structure of NLO diagrams can be presented as

$$n \cdot \mathcal{A}^{(\mathrm{NLO})} = \frac{\alpha_s(\mu^2)}{4\pi}\, \mathcal{P}^{(\eta)}_{\underline{\alpha\beta},\underline{\delta\gamma}}\, \mathbb{Z}^{-1}_{\underline{\alpha\alpha'},\underline{\beta\beta'},\underline{\gamma\gamma'}} \times$$

$$\int [dx_i] \sum_{nmk} \Gamma^{(n)}_{\underline{\alpha'\alpha_1}} \otimes \Gamma^{(m)}_{\underline{\beta'\alpha_2}} \otimes \Gamma^{(k)}_{\underline{\gamma'\alpha_3}} \sum_{p=-1}^{3} \epsilon^p b^{(p)}_{nmk}(x_i; P'^2, Q^2) \times$$

$$(\mathbb{Z}^{-1}_{\mathcal{F}})_{\underline{\alpha_1\alpha'_1},\underline{\alpha_2\alpha'_2},\underline{\alpha_3\alpha'_3}}\, \mathcal{F}^{(3q)\,R}_{\underline{\alpha'_1\alpha'_2\alpha'_3}}(x_i)\,.$$

NLO, Exchange type, d-contribution

We derive the *exchange* diagram contribution:

$$n^\mu \mathcal{A}^{(\mathrm{NLO})}_\mu(\mathrm{Exch.}) = \lim_{\epsilon\to 0}\int [dx_i]\,\mathcal{F}^{(3q)\,R}(x_i)\,\sum_{nmk}\Gamma^{(n)}\otimes\Gamma^{(m)}\otimes\Gamma^{(k)}\times$$

$$\Big[\frac{x_1}{x_2}\,f^{nmk}(\epsilon)\Big\{[Q^2]^\epsilon\big(b^{(0)}_{nmk}+\sum_{p=1}^{n}\epsilon^p\,b^{(p)}_{nmk}\big)+$$

$$[Q^2-(Q^2+P'^2)x_1]^\epsilon\big(-b^{(0)}_{nmk}+\sum_{p=1}^{n}\epsilon^p\,\bar{b}^{(p)}_{nmk}\big)\Big\}-$$

$$\frac{x_{12}}{x_2}\,f^{nmk}(\epsilon)\Big\{[Q^2]^\epsilon\big(c^{(0)}_{nmk}+\sum_{p=1}^{n}\epsilon^p\,c^{(p)}_{nmk}\big)+$$

$$[Q^2-(Q^2+P'^2)x_{12}]^\epsilon\big(-c^{(0)}_{nmk}+\sum_{p=1}^{n}\epsilon^p\,\bar{c}^{(p)}_{nmk}\big)\Big\}\Big],$$

where

$$x_{12\dots n} = \sum_{i=1}^{n} x_i \,, \qquad f^{nmk}(\epsilon) = \frac{a_{nmk}^{(-2)}}{\epsilon^2} + \frac{a_{nmk}^{(-1)}}{\epsilon} + \dots .$$

Having subtracted the singular part, the finite amplitude takes the following form:

$$n^{\mu}\mathcal{A}^{(\mathrm{NLO})}_{\mu}(\mathrm{Exch.}) = \int [dx_i]\,\mathcal{F}^{(3q)\,R}(x_i)\sum_{nmk}\Gamma^{(n)}\otimes\Gamma^{(m)}\otimes\Gamma^{(k)}\times$$
$$\Big\{\frac{x_1}{x_2}\,\mathcal{K}_{nmk}(Q^2, P'^{\,2}; x_i) - \frac{x_{12}}{x_2}\,\mathcal{L}_{nmk}(Q^2, P'^{\,2}; x_i)\Big\}\,,$$

P.S. Here, we keep in mind that

$$\ln^2\frac{Q^2}{\mu^2} - \ln^2\frac{Q'^{\,2}}{\mu^2} = -\ln^2\frac{Q'^{\,2}}{Q^2} - 2\ln\frac{Q'^{\,2}}{Q^2}\ln\frac{Q^2}{\mu^2}\,,$$
$$Q'^{\,2} = Q^2 - (P'^{\,2} + Q^2)x_{ij} \equiv Q^2(1 - Wx_{ij})$$

where

$$\mathcal{K}_{nmk}(Q^2, P'^2; x_i) = a^{(-1)}_{nmk}\left(b^{(1)}_{nmk} + \bar{b}^{(1)}_{nmk} + b^{(0)}_{nmk} \ln\frac{Q^2}{Q^2\bar{x}_1 - P'^2 x_1}\right) +$$
$$a^{(-2)}_{nmk}\left(b^{(2)}_{nmk} + \bar{b}^{(2)}_{nmk} + b^{(0)}_{nmk}\left[\frac{1}{2}\ln^2 Q^2 - \frac{1}{2}\ln^2(Q^2\bar{x}_1 - P'^2 x_1)\right]\right.$$
$$\left.+b^{(1)}_{nmk}\ln Q^2 + \bar{b}^{(1)}_{nmk}\ln(Q^2\bar{x}_1 - P'^2 x_1)\right),$$

and

$$\mathcal{L}_{nmk}(Q^2, P'^2; x_i) = a^{(-1)}_{nmk}\left(c^{(1)}_{nmk} + \bar{c}^{(1)}_{nmk} + c^{(0)}_{nmk} \ln\frac{Q^2}{Q^2\bar{x}_{12} - P'^2 x_{12}}\right) +$$
$$a^{(-2)}_{nmk}\left(c^{(2)}_{nmk} + \bar{c}^{(2)}_{nmk} + c^{(0)}_{nmk}\left[\frac{1}{2}\ln^2 Q^2 - \frac{1}{2}\ln^2(Q^2\bar{x}_{12} - P'^2 x_{12})\right]\right.$$
$$\left.+c^{(1)}_{nmk}\ln Q^2 + \bar{c}^{(1)}_{nmk}\ln(Q^2\bar{x}_{12} - P'^2 x_{12})\right).$$

On tw-three and tw-four contributions

Contributions of the leading-twist DA $\varphi_N(x_i) = V_1(x_i) - A_1(x_1)$ correspond to contributions of local (geometric) twist-three operators in the OPE of the product $T(\eta(0) j_\mu(x)$ for $x^2 \to 0$:

$$\big(D_+^{k_1} u_+\big)(0)\big(D_+^{k_2} u_+\big)(0)\big(D_+^{k_3} d_+\big)(0)\,.$$

Equivalently, the leading-twist contributions can be attributed to the single light-ray operator

$$u_+(a_1 n) u_+(a_2 n) d_+(a_3 n)\,,$$

A twist-four operator can be constructed in two different ways: either changing the "plus" projection of one of the quark fields to the "minus", or adding a transverse derivative, e.g.

$$\big(D_+^{k_1} u_-\big)(0)\big(D_+^{k_2} u_+\big)(0)\big(D_+^{k_3} d_+\big)(0)\,,$$
$$\big(D_+^{k_1} D_\perp u_+\big)(0)\big(D_+^{k_2} u_+\big)(0)\big(D_+^{k_3} d_+\big)(0)$$

Contributions of the first type correspond to the nonlocal light-ray operators $u_-(a_1 n)u_+(a_2 n)d_+(a_3 n)$ and $u_+(a_1 n)u_+(a_2 n)d_-(a_3 n)$. The contributions of operators involving a transverse derivative are more complicated and can be obtained from the light-cone expansion of the nonlocal three-quark operator

$$u_+(y_1)u_+(y_2)d_+(y_3)\,, \qquad y_i = a_i n + b_{i,\perp}$$

where $b_\perp \to 0$ is an auxiliary transverse vector. The twist-four contribution

As an example, consider the contribution of the twist-four DA $\mathbb{V}_2^{(2)}(x_i)$ to the *exchange* diagram. We have

$$\mathcal{P}^{(\eta)} \int d^4y_2 d^4y_3 \mathcal{M}^+(y_2, y_3; q, p_1)\, \mathbb{V}_2^{(2)}(\beta_i)\ (\hat{P}C)\hat{y}_2\gamma_5 N(P) =$$
$$\mathcal{P}^{(\eta)} \int d^4y_2 d^4y_3 \int (d^4p_2)(d^4p_3)\, e^{+ip_2y_2+ip_3y_3}\, \mathcal{M}^+(p_2, p_3; q, p_1) \times$$
$$\mathbb{V}_2^{(2)}(\beta_i)\ (\hat{P}C)[(P\cdot y_2)\hat{n} + \hat{y}_2^T]\gamma_5 N(P)\,,$$

where $\beta_i = P\cdot y_i$.

The longitudinal component contributes as

$$\mathcal{P}^{(\eta)} \int d^4y_2 d^4y_3 \int (d^4p_2)(d^4p_3) e^{+ip_2y_2+ip_3y_3} \mathcal{M}^+(p_2, p_3; q, p_1) \times$$
$$\int [dx_i] e^{-ix_2P\cdot y_2 - ix_3 P\cdot y_3} \mathbb{V}_2^{(2)}(x_i)\ (\hat{P}C)[\beta_2\hat{n}]\gamma_5 N(P) =$$
$$\mathcal{P}^{(\eta)} \int [dx_i] \Big[(-i)\frac{d}{dx_2}\mathbb{V}_2^{(2)}(x_i)\Big] \mathcal{M}^+(x_i)\ (\hat{P}C)\hat{n}\gamma_5 N(P)\,.$$

The transverse component contributes as

$$\mathcal{P}^{(\eta)} \int [dx_i] \mathbb{V}_2^{(2)}(x_i) \Big[i \frac{d}{dp_2^{\perp}} \mathcal{M}^{+}(p_i) \Big] \Big|_{p_i = x_i p} (\hat{P} C) \gamma^{\perp} \gamma_5 N(P) \,.$$

Moreover, owing to the Ward identity

$$\frac{\partial}{\partial p^{\perp}} \frac{\hat{p} + \hat{\ell}}{(p+\ell)^2 + i\epsilon} = - \frac{\hat{p} + \hat{\ell}}{(p+\ell)^2 + i\epsilon} \gamma^{\perp} \frac{\hat{p} + \hat{\ell}}{(p+\ell)^2 + i\epsilon}$$

a derivative is equivalent to the insertion of $\gamma^{\perp}$-matrix in the quark line.

On Distribution Amplitudes

We remind kinematics:

$$P = p + \frac{M^2}{2} n, \quad z_i = \alpha_i p + \beta_i n + z_i^T, \quad \beta_i = P.z_i\,.$$

Assuming $\alpha_i = 0$, $\beta_i \neq 0$, $z_i^T \neq 0$, the most general parametrization of the matrix element takes the form:

$$\langle 0|[\psi(z_1)\psi(z_2)]\psi(z_3)|P\rangle = \mathbb{V}_1(\hat{P}C)\gamma_5 N(P) +$$
$$M \sum_i \mathbb{V}_2^{(i)}(\hat{P}C)\hat{z}_i\gamma_5 N(P) + M\mathbb{V}_3(\gamma_\mu C)\gamma_\mu\gamma_5 N(P) +$$
$$M^2 \sum_i \mathbb{V}_4^{(i)}(\hat{z}_i C)\gamma_5 N(P) + M^2 \sum_i \mathbb{V}_5^{(i)}(\gamma_\mu C) i\sigma_{\mu z_i}\gamma_5 N(P) +$$
$$M^3 \sum_{i,j} \mathbb{V}_6^{(i,j)}(\hat{z}_i C)\hat{z}_j\gamma_5 N(P)\,.$$

Applying the following conditions:

- Condition I: matching with the longitudinal cases;
- Condition II: use of the equations of motion for each fermion:
- Condition III: translation invariance;

we derive

$$
\begin{aligned}
\mathbb{V}_2^{(1)}(x_i) &= \frac{1}{4}\Big[x_3 V_2(x_i) + (x_2 - x_1) V_3(x_i) - A_3(x_i) \\
&\quad + x_3 A_3(x_i) + x_3 A_2(x_i)\Big], \\
\mathbb{V}_2^{(2)}(x_i) &= \frac{1}{4}\Big[x_3 V_2(x_i) + (x_1 - x_2) V_3(x_i) + A_3(x_i) \\
&\quad - x_3 A_3(x_i) - x_3 A_2(x_i)\Big], \\
\mathbb{V}_2^{(3)}(x_i) &= -\frac{1}{2} x_3 V_2(x_i)\,,
\end{aligned}
$$

and, similarly,

$$
\begin{aligned}
\mathbb{A}_2^{(1)}(x_i) &= \frac{1}{4}\Big[- x_3 A_2(x_i) + (x_2 - x_1) A_3(x_i) - V_3(x_i) \\
&\quad + x_3 V_3(x_i) - x_3 V_2(x_i) \Big], \\
\mathbb{A}_2^{(2)}(x_i) &= \frac{1}{4}\Big[- x_3 A_2(x_i) + (x_1 - x_2) A_3(x_i) + V_3(x_i) \\
&\quad - x_3 V_3(x_i) + x_3 V_2(x_i) \Big], \\
\mathbb{A}_2^{(3)}(x_i) &= \frac{1}{2} x_3 A_2(x_i) \, .
\end{aligned}
$$

On Auxiliary Functions

The momentum dependence of the NLO corrections to the amplitude can conveniently be written in terms of the following functions:

$$g_{nk}(y,x;W) = \frac{\ln^n[1 - yW - i\eta]}{(-1 + xW + i\eta)^k},$$
$$h_{nk}(x;W) = \frac{\ln^n[1 - xW - i\eta]}{(W + i\eta)^k}$$

with $n = 0, 1, 2$ and $k = 1, 2, 3$. For the particular case $n = 0$ the first argument becomes dummy; for simplicity of notation we write the corresponding entries as

$$g_k(x;W) \equiv g_{0k}(*, x;W)\,.$$

Going over to the Borel parameter space and subtracting the continuum corresponds to the substitutions

$$g_{nk} \rightarrow G_{nk}(y,x;M^2) = \frac{1}{\pi}\int_0^{s_0} \frac{ds}{Q^2}\, e^{-s/M^2} \mathrm{Im}\, g_{nk}(y,x,W)\,,$$
$$h_{nk} \rightarrow H_{nk}(x;M^2) = \frac{1}{\pi}\int_0^{s_0} \frac{ds}{Q^2}\, e^{-s/M^2} \mathrm{Im}\, h_{nk}(x,W)\,,$$

where $s = P'^2$ and $W = 1 + s/Q^2$, M^2 is the Borel parameter and s_0 the continuum threshold.

LCSRs involve integrals of the type

$$\mathbf{G}_{nk} = \int [dx]\, \mathcal{F}(\underline{x})\, G_{nk}(x_i + x_j, x_i; M^2)\,,$$

$$\widetilde{\mathbf{G}}_{nk} = \int [dx]\, \mathcal{F}(\underline{x})\, G_{nk}(x_i, x_i; M^2)\,,$$

$$\mathbf{H}_{nk} = \int [dx]\, \mathcal{F}(\underline{x})\, H_{nk}(x_i + x_j; M^2)\,,$$

where $\mathcal{F}(\underline{x}) = \mathcal{F}(x_i, x_j, 1 - x_i - x_j)$ is a function of quark momentum fractions and $x_i, x_j \in \{x_1, x_2, x_3\}$. In addition one needs

$$\widehat{\mathbf{G}}_{01} = \int [dx]\, \mathcal{F}(\underline{x})\, G_{01}(*, x_i + x_j; M^2)$$

(only this special case).

Using the following notations:

$$x_{ij} = x_i + x_j\,, \qquad \bar{x} = 1 - x\,, \qquad x_0 = \frac{Q^2}{s_0 + Q^2}\,,$$

$$E(x) = \exp\left[-\frac{\bar{x}Q^2}{xM^2}\right],$$

$$\Big[\mathcal{F}(x_i, x_j)\Big]_+ = \mathcal{F}(x_i, x_j) - \mathcal{F}(x_0, x_j)$$

and

$$\mathcal{F} \otimes \mathcal{G} = \int_{x_0}^{1} dx_i \int_0^{1-x_i} dx_j\, \mathcal{F}(\underline{x})\, \mathcal{G}(\underline{x})\,,$$

$$\mathcal{F} \circledast \mathcal{G} = \int_0^{x_0} dx_i \int_{x_0 - x_i}^{1-x_i} dx_j\, \mathcal{F}(\underline{x})\, \mathcal{G}(\underline{x})\,.$$

We obtain:

$$\begin{aligned} \mathbf{G}_{01} &= -\mathcal{F} \otimes \frac{E(x_i)}{x_i}, \\ \widehat{\mathbf{G}}_{01} &= -\mathcal{F}\Big[\otimes + \circledast\Big]\frac{E(x_{ij})}{x_{ij}}, \end{aligned}$$

and so on.

Our results for $\mathbf{G}_{11}, \mathbf{G}_{21}, \mathbf{H}_{11}, \mathbf{H}_{12}, \mathbf{H}_{21}, \mathbf{H}_{22}$ differ from the corresponding expressions $g_7 - g_{12}$ in K. Passek-Kumericki and G. Peters, Phys. Rev. D **78**, 033009 (2008) by extra terms from the $\circledast$ integration region; in addition our expression for $\mathbf{G}_{21}$ does not contain a contribution $\sim \pi^2(1 - \delta(x_j))$.

Conclusions

Our calculation incorporates the following new elements as compared to previous studies:

- Next-to-leading order QCD corrections to the contributions of twist-three and twist-four DAs;
- Exact account of “kinematic” contributions to the nucleon DAs of twist-four and twist-five induced by lower geometric twist operators (Wandzura-Wilczek terms);
- Light-cone expansion to the twist-four accuracy of the three-quark matrix elements with generic quark positions;
- A new calculation of twist-five off-light cone contributions;
- A more general model for the leading-twist DA, including contributions of second-order polynomials.