QCD evolution equations for cut Mellin moments of the parton distributions: review and applications

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Abstract

We review evolution equations for truncated Mellin moments of the parton densities and discuss some applications

Main finding (D.K. and A. Kotlorz, Phys. Lett. B644, 284, 2007)
n-th truncated moment of the parton distribution

$$\underbrace{\bar{q}_n(x_{min}, Q^2) = \int_{x_{min}}^{1} dx\, x^{n-1}\, q(x, Q^2)}_{\text{truncated moment (TMM)}}$$

obeys also the DGLAP equation, but with a rescaled splitting function

$$P'(z) = z^n P(z)$$

EXPERIMENTS PROVIDE CUT MOMENTS!

Motivation

PDFs play the central role in DGLAP approach

Moments - a natural candidate in QCD analysis

- originate from OPE - basic formalism of the quantum field theory
- directly reffer to sum rules - fundamental relations in QCD
- contributions to momentum or spin of the nucleon coming from quarks and gluons

Evolution equations for truncated moments (generalization of the full moments version)

- enables directly to study physical values and their evolution
- allows to avoid uncertainties from non-available experimentally small-x region

Moment approach is a powerful tool to study SF

Contents

Operator product expansion (OPE)

OPE refers directly to the moments of the structure functions.

OPE enables to derive from QCD sum rules for the structure functions.

The product of two electromagnetic currents in the hadronic tensor $W^{\mu\nu}$ can be expanded in terms of a sum of local operators multiplied by Wilson coefficients.

The local operators in OPE for QCD are quark and gluon operators.

The contribution of any operator to $W^{\mu\nu}L_{\mu\nu}$ is of order

$$\left(\frac{1}{x}\right)^n \left(\frac{Q}{M}\right)^{2-t} \qquad t - \mathrm{twist} \qquad \mathrm{n} - \mathrm{spin}$$

OPE

The Mellin moments of the structure functions

$$\int_0^1 dx\ x^{n-1}\ F_i(x, Q^2) = \sum_A C^A_{n,i}(Q^2) M^A_n$$

$C^A_{n,i}(Q^2)$ - Fourier transforms of the Wilson coefficients
$M^A_n(Q^2)$ - parametrize the diagonal matrix elements of the composite operators between nucleon states

In DGLAP formulation

$$M^A_n(Q^2) = \int_0^1 dx\ x^{n-1}\ q^A(x, Q^2)$$

$q^A(x, Q^2)$ - parton distribution function

Evolution of SFs and PDFs

Q^2 Evolution

$$F(x, Q^2) = \sum_A \underbrace{C^A}_{\text{coefficient f.}} \otimes \underbrace{q^A}_{\text{evolved PDF}} (x)$$

$$f \otimes g\,(x) = \int_x^1 \frac{dz}{z} f(z)\, g\left(\frac{x}{z}\right)$$

C^A - Wilson coefficient functions computed in PQCD

$$C^A = \underbrace{C^{A(0)}(x)}_{\text{LO}\sim\delta(1-x)} + \alpha_s C^{A(1)}(x) + \alpha_s^2 C^{A(2)}(x) + \ldots$$

DGLAP formulation

Standard DGLAP approach

Q^2 evolution + initial parton densities
Dokshitzer, Gribov, Lipatov, Altarelli, Parisi

- operates on the parton densities $q(x, Q^2)$ (PDFs)
- describes their Q^2 evolution
- requires a knowledge of the initial PDFs at low-Q^2 scale for the wide range of x-values $x \in (0; 1)$

Standard kinematic variables

q - the virtual photon momentum $Q^2 = -q^2 > 0$,
x - the Bjorken variable $x = \frac{Q^2}{2pq}$

DGLAP evolution equations

PDFs $q(x, Q^2)$ (**unpolarised** as well as **polarised**) obey the evolution equations of type

Evolution

$$\frac{\partial q(x, Q^2)}{\partial \ln Q^2} = \frac{\alpha_s}{2\pi} \underbrace{P(x, \alpha_s(Q^2))}_{\text{SPLITTING F.}} \otimes \underbrace{q(x, Q^2)}_{\text{PDF}}$$

$$P(x, \alpha_s(Q^2)) = P^{(0)}(x) + \frac{\alpha_s(Q^2)}{2\pi} P^{(1)}(x) + \left(\frac{\alpha_s(Q^2)}{2\pi}\right)^2 P^{(2)}(x) + \ldots$$

Input - initial PDFs $q(x, Q_0^2)$ at low scale $Q_0^2 \sim 1\ \mathrm{GeV}^2$

DGLAP in moment space

Simple form of the evolution equations

$$\frac{\partial \bar{q}_n(Q^2)}{\partial \ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \gamma_n \, \bar{q}_n(Q^2)$$

Mellin transform

$$\bar{q}_n = \int\limits_0^1 dx \, x^{n-1} \, q(x) \qquad \gamma_n = \int\limits_0^1 dx \, x^{n-1} \, P(x)$$

Invert Mellin transform

$$q(x) = \frac{1}{2\pi\imath} \int\limits_{c-i\infty}^{c+i\infty} dn \, x^{-n} \, \bar{q}_n$$

Solving the DGLAP equations

Initial PDFs at Q_0^2
crucial point: small-x behavior
$\Downarrow$
Evolution from Q_0^2 to Q^2
$\swarrow$ $\searrow$

$x-$ space	$n-$ space
(polynomial expansion technique)	(Mellin transform)
$\Downarrow$	$\Downarrow$
Results: PDFs	Results: untruncated moments
$\Downarrow$	$\Downarrow$
Moments	PDFs

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Alternative / Complementary approach:

TRUNCATED MOMENTS

Truncated Mellin moments (TMM) - definitions

$$\underbrace{\bar{q}_n(Q^2) = \int\limits_0^1 dx\, x^{n-1}\, q(x, Q^2)}_{\text{untruncated moment (umm)}}$$

$$\underbrace{\bar{q}_n(x_{min}, Q^2) = \int\limits_{x_{min}}^1 dx\, x^{n-1}\, q(x, Q^2)}_{\text{truncated moment (TMM)}}$$

$$\underbrace{\bar{q}_n(x_{min}, x_{max}, Q^2) = \int\limits_{x_{min}}^{x_{max}} dx\, x^{n-1}\, q(x, Q^2)}_{\text{double truncated moment (TTMM)}}$$

TMMA - history

In this approach the main role is played by the truncated moments of the quark and gluon distribution functions

History

- Non-diagonal differential DGLAP evolution equations
  (S. Forte, L. Magnea, A. Piccione, G. Ridolfi 1999-2002)
- Diagonal solutions in the double logarithmic $\ln^2 x$ approximation
  (D.K. and A. Kotlorz 2003)
- Diagonal DGLAP evolution equations for TMM and TTMM
  D.K. and A. Kotlorz, Phys. Lett. B644, 284 (2007)
- Generalization
  (S.V. Mikhailov 2012)

Non-diagonal differential evolution equations

Every n-th truncated moment couples to $(n+j)$-th ones $(j \geq 0)$

$$\frac{\partial \bar{q}_n(x_{min}, Q^2)}{\partial \ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \sum_{j=0}^{\infty} C_{jn}(x_{min})\, \bar{q}_{n+j}(x_{min}, Q^2)$$

$$\underbrace{G_n\left(\frac{x_{min}}{z}\right)}_{\text{anomalous dimension}} \equiv \int_{x_{min}/z}^{1} dx\, x^{n-1} P(x) = \underbrace{\sum_{j=0}^{\infty} C_{jn}(x_{min})\, z^k}_{\text{Taylor series around } z=1}$$

$$G_n(0) = \int_0^1 dx\, x^{n-1} P(x) = \gamma_n$$

$$C_{jn}^{(M)}(x_{min}) = \gamma_n^{(0)} \delta_{j0} - \frac{4}{3} \sum_{k=j}^{M} \frac{(-1)^j}{j!\,(k-j)!} \times \left[2 \sum_{i=n+2}^{\infty} \frac{(i+k-1)!}{i!} x_{min}^i + \frac{(n+k-1)!}{n!} \left(x_{min}^n + \frac{n+k}{n+1} x_{min}^{n+1} \right) \right]$$

Non-diagonal differential evolution equations

Closed system of $M+1$ equations:

$$\frac{d\bar{q}_{N_0}(x_{min},Q^2)}{d\ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi}[C^{(M)}_{0,N_0}(x_{min})\bar{q}_{N_0}(x_{min},Q^2)$$

$$+C^{(M)}_{1,N_0}(x_{min})\bar{q}_{N_0+1}(x_{min},Q^2)+$$

$$\ldots + C^{(M)}_{M,N_0}(x_{min})\bar{q}_{N_0+M}(x_{min},Q^2)]$$

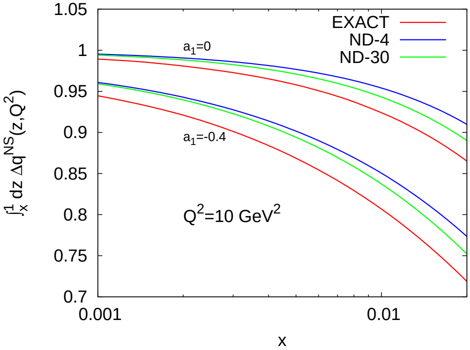
$$\frac{d\bar{q}_{N_0+1}(x_{min},Q^2)}{d\ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi}[C^{(M-1)}_{0,N_0+1}(x_{min})\bar{q}_{N_0+1}(x_{min},Q^2)$$

$$+C^{(M-1)}_{1,N_0+1}(x_{min})\bar{q}_{N_0+2}(x_{min},Q^2)+\ldots+C^{(M-1)}_{M-1,N_0+1}(x_{min})\bar{q}_{N_0+M}(x_{min},Q^2)]$$

$$\ldots$$

$$\frac{d\bar{q}_{N_0+M}(x_{min},Q^2)}{d\ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi}C^{(0)}_{0,N_0+M}(x_{min})\bar{q}_{N_0+M}(x_{min},Q^2)$$

Non-diagonal differential evolution equations



A general form of the input

$$\Delta q^{NS}(x, Q_0^2) = N(a_1, a_2, a_3)\, x^{a_1}\, (1-x)^{a_2}\, (1+a_3 x)$$

Diagonal evolution equations for single and double TMM

Main findings

$$\frac{d\bar{q}_n^{NS}(x,Q^2)}{d\ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi}\,(P'_{qq}(n)\otimes\bar{q}_n^{NS})(x,Q^2)$$

$$\frac{d}{d\ln Q^2}\begin{pmatrix}\bar{q}_n^S(x,Q^2)\\ \bar{G}_n(x,Q^2)\end{pmatrix} = \frac{\alpha_s(Q^2)}{2\pi}\begin{pmatrix}P'_{qq}(n) & P'_{qG}(n)\\ P'_{Gq}(n) & P'_{GG}(n)\end{pmatrix}\otimes\begin{pmatrix}\bar{q}_n^S(x,Q^2)\\ \bar{G}_n(x,Q^2)\end{pmatrix}$$

Rescaled splitting functions:

$$P'_{ij}(n,x) = x^n\,P_{ij}(x)$$

$$P_{ij}(x) = P_{ij}^{(0)}(x) + \frac{\alpha_s(Q^2)}{2\pi}P_{ij}^{(1)}(x) + \left(\frac{\alpha_s(Q^2)}{2\pi}\right)^2 P_{ij}^{(2)}(x) + \cdots$$

Anomalous dimension:

$$\gamma'_{s,n} \equiv \int_0^1 dx\,x^{s-1}\,P'(n,x) = \gamma_{s+n}$$

Diagonal evolution equations for single and double TMM

Main findings

$$\bar{g}_{1\,n}(x, Q^2) = \frac{1}{2}\sum_q e_q^2 \times$$

$$\times \left[\Delta\bar{q}_n(x, Q^2) + \frac{\alpha_s(Q^2)}{2\pi}\left(C'_q(n)\otimes\Delta\bar{q}_n + C'_G(n)\otimes\Delta\bar{G}_n\right)(x, Q^2)\right]$$

Rescaled Wilson coefficients:

$$C'_i(n, x) = x^n\, C_i(x)$$

Moments of the coefficient functions:

$$C'_{s,n} \equiv \int_0^1 dx\, x^{s-1}\, C'(n, x) = C_{s+n}$$

Comparison of the standard and TMM aproaches

Input: PDFs $q(x, Q_0^2)$

$\Downarrow$

Evolution from Q_0^2 to Q^2

$$\frac{\partial q(x, Q^2)}{\partial \ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \, (P \otimes q)$$

$\Downarrow$

Results: PDFs $q(x, Q^2)$

Input: TMM $\bar{q}_n(x_{min}, Q_0^2)$

$\Downarrow$

Evolution from Q_0^2 to Q^2

$$\frac{\partial \bar{q}_n(x_{min}, Q^2)}{\partial \ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \, (P' \otimes \bar{q}_n)$$

$\Downarrow$

Results: TMM $\bar{q}_n(x_{min}, Q^2)$

$$P'(n, z) = z^n \, P(z)$$

$$(A \otimes B) \; (x) \equiv \int_x^1 \frac{dz}{z} \, A\left(\frac{x}{z}\right) \; B(z)$$

Diagonal evolution equations for single and double TMM

There is no mixing between moments of different orders

Evolution equation for TMM

$$\frac{\partial \bar{q}_n(x_{min}, Q^2)}{\partial \ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \int\limits_{x_{min}}^{1} \frac{dz}{z}\, P'(n,z)\, \bar{q}_n\left(\frac{x_{min}}{z}, Q^2\right)$$

The splitting function for TMM

$$P'(n,z) = z^n\, P(z)$$

The evolution equation for TTMM

$$\frac{\partial \bar{q}_n(x_{min}, x_{max}, Q^2)}{\partial \ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \int\limits_{x_{min}}^{1} \frac{dz}{z}\, P'(n,z)\, \bar{q}_n\left(\frac{x_{min}}{z}, \frac{x_{max}}{z}, Q^2\right)$$

Diagonal evolution equations for single and double TMM

Since the experimental data cover only a limited range of x, except very small $x \rightarrow 0$ as well as large $x \rightarrow 1$, it is very natural and convenient to deal with the double truncated moments. Truncation at large x is less important in comparison to the small-x limit because of the rapid decrease of the parton densities as $x \rightarrow 1$, nevertheless a comprehensive analysis requires an equal treatment of the both truncated limits.

TTMM is a subtraction of two TMM

$$\bar{q}^n(x_{min}, x_{max}, Q^2) = \bar{q}^n(x_{min}, Q^2) - \bar{q}^n(x_{max}, Q^2)$$

and also satisfies the DGLAP-type evolution

Diagonal evolution equations for single and double TMM

$$\frac{d\bar{q}^n(x_{min}, x_{max}, Q^2)}{d\ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \int\limits_{x_{min}}^{1} \frac{dz}{z} P'(n,z)\, \bar{q}^n\left(\frac{x_{min}}{z}, \frac{x_{max}}{z}, Q^2\right)$$

This approach is valid for the coupled DGLAP equations for quarks and gluons and for any approximation (LO, NLO, NNLO, etc.)

The evolution equations for TTMM are a generalization of those for the single truncated and untruncated ones:
$x_{max} = 1$

$$\frac{d\bar{q}^n(x_0, Q^2)}{d\ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \int\limits_{x_0}^{1} \frac{dz}{z} P'(n,z)\, \bar{q}^n\left(\frac{x_0}{z}, Q^2\right)$$

Diagonal evolution equations for single and double TMM

$$\frac{d\bar{q}^n(x_{min}, x_{max}, Q^2)}{d\ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \int\limits_{x_{min}}^{1} \frac{dz}{z}\, P'(n,z)\, \bar{q}^n\left(\frac{x_{min}}{z}, \frac{x_{max}}{z}, Q^2\right)$$

This approach is valid for the coupled DGLAP equations for quarks and gluons and for any approximation (LO, NLO, NNLO, etc.)

The evolution equations for TTMM are a generalization of those for the single truncated and untruncated ones:
$x_{min} = 0$ and $x_{max} = 1$

$$\frac{d\bar{q}^n(Q^2)}{d\ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \int\limits_{0}^{1} \frac{dz}{z}\, P'(n,z)\, \bar{q}^n(Q^2) = \frac{\alpha_s(Q^2)}{2\pi}\, \gamma_n\, \bar{q}_n(Q^2)$$

Generalization of DGLAP equations (S.V. Mikhailov 2012)

If $f(x, Q^2)$ is a solution of DGLAP equation with the kernel P(y):

$$\frac{\partial f(z, Q^2)}{\partial \ln Q^2} = (P \otimes f)(z) \equiv$$

$$\int\limits_0^1 P(y)\, f(x, Q^2)\, \delta(z - xy)\, dx\, dy,$$

then the multi-integrated function which is a generalization of the cut moments

$$f_k(z; n_1, n_2, ..., n_k, Q^2) = \int\limits_z^1 z_k^{n_k - 1} dz_k \int\limits_{z_k}^1 z_{k-1}^{n_{k-1}-1} dz_{k-1} \ldots \int\limits_{z_2}^1 z_1^{n_1 - 1}\, f(z_1, Q^2)\, dz_1$$

is also the solution of DGLAP equation

Generalization of DGLAP equations

$$\frac{\partial f_k(z; n_1, n_2, ..., n_k, Q^2)}{\partial \ln Q^2} = (P_k \otimes f_k)(z; n_1, n_2, ..., n_k, Q^2)$$

with the kernel $P_k(y)$

$$P_k(y) = P(y) \cdot y^{n_1+n_2+...+n_k}$$

For $k = 1$ one obtains evolution equation for the truncated n_1-th moment

$$f_1(z; n_1) = \int_z^1 z_1^{n_1-1} f(z_1, Q^2)\, dz_1$$

Generalization of DGLAP equations

Evolution equation at $k = 1$:

$$\frac{\partial f_1(z; n_1, Q^2)}{\partial \ln Q^2} = (P_1 \otimes f_1)(z; n_1, Q^2) \equiv$$

$$\int_0^1 P_1(y)\, f_1(x; n_1, Q^2)\, \delta(z - xy)\, dx\, dy,$$

where

$$P_1(y) = P(y) \cdot y^{n_1}$$

If one puts $z = 0$, it reduces to the well known standard renorm-group equation for the moments $f_1(0; n_1, Q^2)$:

$$\frac{\partial f_1(0; n_1, Q^2)}{\partial \ln Q^2} = \left(\int_0^1 P(y)\, y^{n_1 - 1}\, dy \right) \cdot f_1(0; n_1, Q^2) \equiv \gamma(n_1) \cdot f_1(0; n_1, Q^2)$$

Generalization of DGLAP equations

Based on this generalization different interesting partial solutions of DGLAP can be constructed and applied to analysis of the experimental data in different restricted x-regions, respectively.

in progress...

Characteristics and advantages

Truncated Mellin moments approach

- Refers directly to the physical values - moments.
- Allows direct study the evolution of moments and the scaling violation
- Allows to avoid uncertainties from the unmeasurable regions: $x \to 0$ and $x \to 1$
- Is valid in the polarised as well as in unpolarised case
- Can be used for different approximations: LO, NLO, NNLO,...
- For the diagonal TMMA one can use standard methods of solving DGLAP evolution equations
- Generalization of DGLAP equations - novel promising tool!

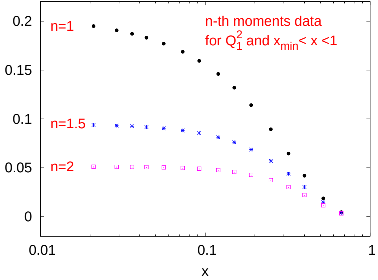
Determination of the PDFs from TMM - general idea

Reconstruction of the parton distributions

$$q(x,Q^2) = -x^{1-n}\,\frac{\partial \bar{q}_n(x,Q^2)}{\partial x}$$

1. Preparing available experimental data for TMM $\bar{q}_n(x,Q_1^2)$ as a function of x at the scale Q_1^2
2. Interpolation of the given data points into points which are Chebyshev nodes. This allows to apply the Chebyshev polynomials technique for solving the evolution equations
3. Evolution of the TMM from Q_1^2 to Q_2^2
4. Reconstruction of the parton density $q(x,Q_2^2)$ from its TMM at the scale Q_2^2 with use of the fitting procedure

Reconstruction of PDFs - a general example

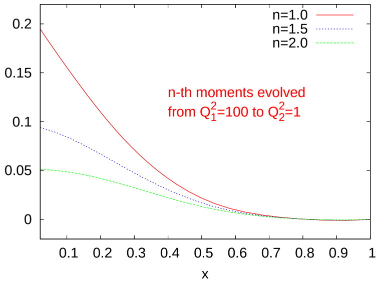


1. PREPARING EXP. DATA

$\bar{q}_n(x, Q_1^2)$ for $x_{min} \leq x \leq 1$

Experimentally moments can only be measured over some finite x-range.
The limit $x \to 0$ for the finite Q^2 requires infinite energy transfers.
Accessible for the polarised experiments: $x_{min} = 0.004$, $Q^2 \geq 1\mathrm{GeV}^2$.

Reconstruction of PDFs - a general example



2. MOMENT EVOLUTION

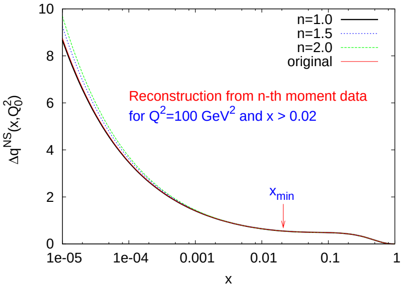
$$Q_1^2 \longrightarrow Q_2^2$$

Non-integer moments - helpful in determination of the small-x behaviour (A. Sidorov)

Second moment - sensitive in the large-x region

Can be used in final determination of γ (if necessary)

Reconstruction of PDFs - a general example



3. FITTING α, β, γ param.

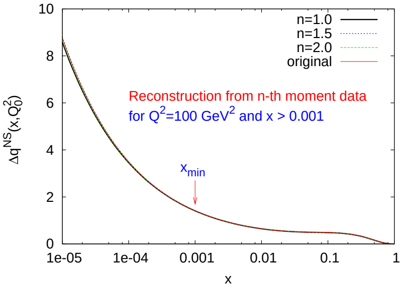
Marquardt / MINUIT minimization method

$$\Delta q^{NS}(x, Q_0^2) = N(\alpha, \beta, \gamma)\; x^{\alpha}\; (1-x)^{\beta}\; (1+\gamma x)$$

$$g_1^{NS} \sim x^{-\alpha} \left(\frac{Q^2}{Q_0^2}\right)^{\alpha/2}$$

$\alpha = -0.4$ (Bartels, Ermolaev, Manaenkov, Ryskin)

Reconstruction of PDFs - a general example



Better fit
for the extended x-range data
$x_{min} = 0.001$

$$\Delta q^{NS}(x, Q_0^2) = N(\alpha, \beta, \gamma)\ x^{\alpha}\ (1-x)^{\beta}\ (1+\gamma x)$$

$\alpha \sim\ 0,\ -1$ Satisfactory reconstruction as well

Reconstruction of PDFs - a general example

n	α	β	γ
true	0	3	20
1	-0.0143	2.987	21.31
1.5	-0.0117	2.990	21.02
2	-0.0121	2.991	21.10
2^*	0	2.998	19.90

Table: 1. Reconstruction x^0 from simulated data for $x \geq x_{min} = 0.021$. 2^*: two-steps reconstruction

n	α	β	γ
true	-0.8	3	20
1	-0.7992	2.999	19.84
1.5	-0.7998	2.999	19.95
2	-0.8000	2.999	19.99

Table: 2. Reconstruction $x^{-0.8}$ from simulated data for $x \geq x_{min} = 0.021$.

Evolution Q^2 : $100 \to 1$

$$\Delta q^{NS}(x, Q_0^2) = N(\alpha, \beta, \gamma)\, x^\alpha\, (1-x)^\beta\, (1+\gamma x)$$

The best fit for $n \approx 1 - \alpha$

Reconstruction of PDFs - a general example

n	α	β	γ
true	0	3	20
1	0.0010	3.000	19.90
1.5	0.0083	3.007	19.32
2	0.1363	3.087	11.19
2^*	0	3.003	20.15

Table: 3. Reconstruction x^0 from simulated data for $x \geq x_{min} = 0.001$. 2^*: two-steps reconstruction

n	α	β	γ
true	-0.8	3	20
1	-0.7968	2.982	19.21
1.5	-0.7997	2.999	19.94
2	-0.8002	3.000	20.04

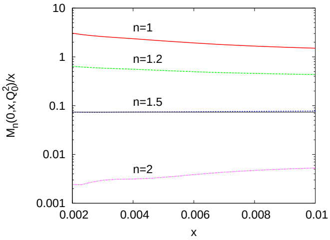
Table: 4. Reconstruction $x^{-0.8}$ from simulated data for $x \geq x_{min} = 0.001$.

Evolution Q^2 : $100 \rightarrow 1$

$$\Delta q^{NS}(x, Q_0^2) = N(\alpha, \beta, \gamma)\, x^\alpha\, (1-x)^\beta\, (1+\gamma x)$$

The best fit for $n \approx 1 - \alpha$

Reconstruction of PDFs - a general example



Determination of the small-x behaviour of the parton distributions from the evolved to $Q_0^2 = 1\ \mathrm{GeV}^2$ truncated moments

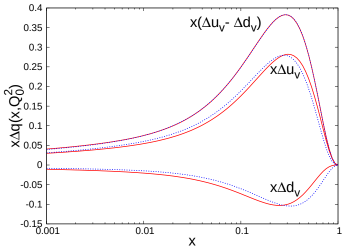
$$\bar{q}_n(0, x, Q^2) = \int_0^x dz\ z^{n-1} q(z, Q^2)$$

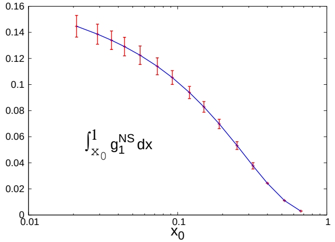
$$x \to 0: \quad \bar{q}_n(0, x) = \frac{N}{n+\alpha}\, x^{n+\alpha}$$

$$\bar{q}_n(0, x)/x \approx \frac{N}{n+\alpha}\, x^{n+\alpha-1} = N = const \quad \text{for} \quad n+\alpha = 1$$

Small-x behaviour x^α can be estimated via $\alpha = 1 - n$

Reconstruction of the PDFs - test





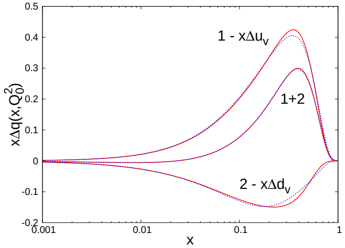
Reconstruction of LO Blümlein - Böttcher (BB) fit

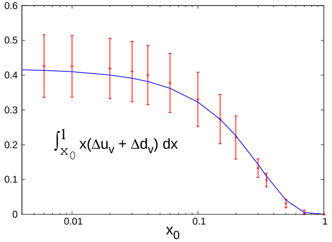
$g_1^{NS}(x, Q_0^2 = 4\ \mathrm{GeV}^2) \sim x^{-0.8}$

The first truncated moment of g_1^{NS} vs the truncation point x_0 calculated from the reconstructed fit, $Q^2 = 5\ \mathrm{GeV}^2$

$\mathrm{HERMES\ \ x-range:}\ \ 0.021\ -\ 0.7$

Reconstruction of the PDFs - test





Reconstruction of de Florian, Navarro, Sassot (DNS) fit
$\Delta u_v(x, Q_0^2 = 0.5\ \mathrm{GeV}^2) \sim x^{0.1}$
$\Delta d_v(x, Q_0^2 = 0.5\ \mathrm{GeV}^2) \sim x^{-0.1}$

The first truncated moment of the polarised function $\Delta u_v + \Delta d_v$ vs the truncation point x_0, calculated from the reconstructed fit, $Q^2 = 10\ \mathrm{GeV}^2$

$$\mathrm{COMPASS\ \ x-range:}\quad 0.006\ -\ 0.7$$

Reconstruction of the PDFs - remarks

The results of the reconstruction of the initial PDFs from their truncated moments are satisfactory.

However...

- Success of the determination of the parton densities from their truncated moments depends on the number of the fitted parameters and also on accessible x-range of the experimental data.
- For larger number of adjustable parameters the obtained fits can be not unique.
- An additional knowledge of the small-x behaviour of the parton densities, based either on the experimental data or the theoretical expectations, makes the fit procedure more reliable.

Contributions to BSR

Knowledge of the small-x behavior of $g_1^{NS} = g_1^p - g_1^n$ is necessary for determination of the Bjorken sum rule.

Bjorken sum rule (BSR)

The BSR is a fundamental rule and must be hold as a rigorous prediction of QCD in the limit of the infinite momentum transfer Q^2

$$I_{BSR} \equiv \Gamma_1^p - \Gamma_1^n = \int_0^1 dx\, (\, g_1^p(x,Q^2) - g_1^n(x,Q^2)\,) =$$

$$\underbrace{\frac{g_A}{6}\left[1 - \frac{\alpha_s}{\pi} - 3.58\,\frac{\alpha_s^2}{\pi^2} - 20.21\,\frac{\alpha_s^3}{\pi^3} + \,\ldots\right]}_{\text{leading twist}} + \underbrace{\sum_{i=2}^{\infty}\frac{\mu_{2i}(Q^2)}{Q^{2i-2}}}_{\text{higher twists}}$$

Contributions to BSR

g_A - neutron β-decay constant

$$g_A = F + D = 1.267 \pm 0.004$$

The BSR for the flavour symmetric sea quarks scenario ($\Delta\bar{u} = \Delta\bar{d}$):

$$I_{BSR}(Q^2 \to \infty) \approx 0.211$$

The small-x contribution to the BSR

$$\Delta I_{BSR}(x_1, x_2, Q^2) \equiv \int_{x_1}^{x_2} dx\, g_1^{NS}(x, Q^2)$$

$1^{\rm st}$ double truncated moment of g_1^{p-n}

HT - modifications of scaling variables (O. Teryaev, D.K.)

SHIFT (O.Teryaev, D.K. 2013)

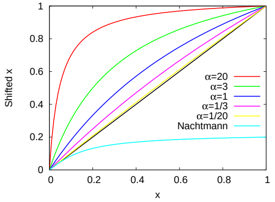
$$Q^2 \longrightarrow Q^2 + M^2$$

$$x \longrightarrow \bar{x} = \frac{Q^2 + M^2}{W^2 + Q^2 + M^2} = x\,\frac{1+\alpha}{1+\alpha x}$$

$$\alpha \equiv \frac{M^2}{Q^2}$$

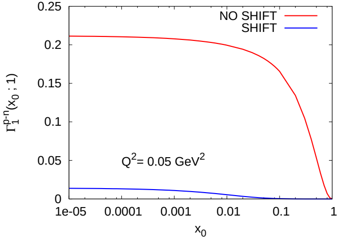
$$Q^2 \to \infty \Rightarrow \alpha = 0, \quad \bar{x} = x$$

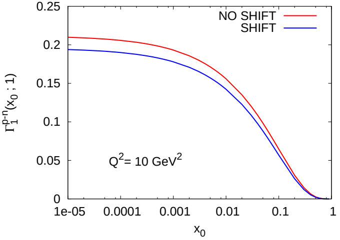
$$Q^2 \to 0 \Rightarrow \alpha \to \infty, \quad \bar{x} = 1, \quad SF \to 0$$



The shifted x ($\bar{x}$) vs x for different values of α.

Comparison of SHIFT and NO SHIFT results



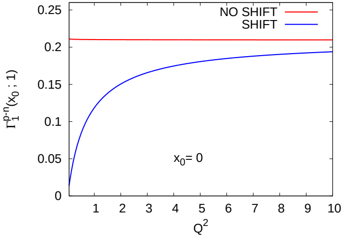


The first truncated moment of g_1^{NS} vs the truncation point x_0. $Q^2 = 0.05\ \mathrm{GeV}^2$

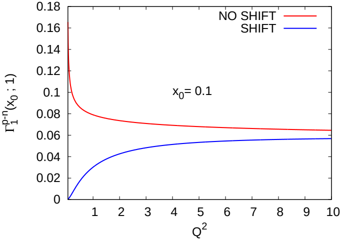
The first truncated moment of g_1^{NS} vs the truncation point x_0. $Q^2 = 10\ \mathrm{GeV}^2$

$$g_1^{NS}(x, Q_0^2 = 1\ \mathrm{GeV}^2) \sim x^{-0.5}(1-x)^3$$

Comparison of SHIFT and NO SHIFT results



The first untruncated moment of g_1^{NS} vs Q^2.



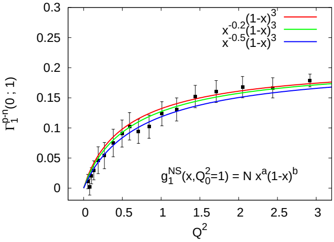
The first truncated moment of g_1^{NS} vs Q^2 for the fixed truncation point $x_0 = 0.1$.

$$g_1^{NS}(x, Q_0^2 = 1\ \mathrm{GeV}^2) \sim x^{-0.5}(1-x)^3$$

Resummed twists for $Q^2 \longrightarrow 0$

Comparison with JLab EG1b data

Phys.Rev.D78:032001,2008



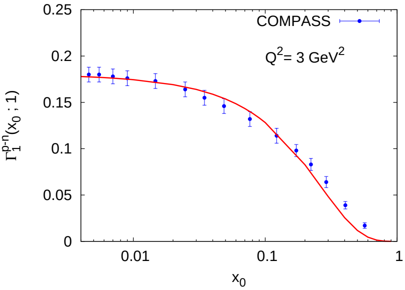
SHIFT CAN MIMIC
HT RESUMMATION

BSR vs Q^2

Contributions to BSR

Comparison with COMPASS data

arXiv:1001.4654v1,2010



Contribution to BSR:
$Q^2 = 3\ \mathrm{GeV}^2$
x range 0.004 - 0.7
Experiment
$0.175 \pm 0.009 \pm 0.015$
Evolution of TMM
0.177 (LO+SHIFT)

$g_1^{NS} \sim x^{-0.2}$

Analysis of the structure function g_2

For a complete description of the nucleon spin, one needs two polarised structure functions: g_1 and g_2. Recently, a new generation of experiments with high polarised luminosity, performed at Jefferson Lab, allows more precise study of the polarised structure functions and their moments. This is crucial in our understanding of the QCD spin sum rules, higher-twist effects and quark-hadron duality.

The function g_1 has a simple interpretation in the parton model:

$$g_1(x) = \frac{1}{2}\sum_i e_i^2\,(\Delta q_i(x) + \Delta \bar{q}_i(x)),$$

describing the distribution of quark spin in the nucleon, while function g_2 has no such physical meaning in this classic model. Due to the technical difficulties of obtaining transversely polarised targets, the structure function g_2 has not been a topic of investigations for a long time.

Analysis of the structure function g_2

Recently, new experimental data at low and intermediate momentum transfers make g_2 also a valuable and hopeful tool to study the spin structure of the nucleon.

The function g_2 provides knowledge on higher twist effects, which are reflection of the quark-gluon correlations in the nucleon.
A particular important role in this analysis is played by moments of the spin structure functions

$$\Gamma_1^n = \int_0^1 dx\, x^{n-1} g_1(x, Q^2),$$

$$\Gamma_2^n = \int_0^1 dx\, x^{n-1} g_2(x, Q^2).$$

They are a sensitive tool for testing the QCD sum rules and determination of the higher twist contributions.

Determination of the higher twist effects

The experimental value of the function g_2, measured in the small to intermediate Q^2 region, consists of two parts: the twist-2 (leading) and the higher twist term:

$$g_2(x, Q^2) = g_2^{LT}(x, Q^2) + g_2^{HT}(x, Q^2).$$

The leading-twist term g_2^{LT} can be determined from the other structure function - g_1 via the Wandzura-Wilczek relation

$$g_2^{LT}(x, Q^2) = g_2^{WW}(x, Q^2) = -g_1(x, Q^2) + \int_x^1 \frac{dy}{y}\, g_1(y, Q^2).$$

Then, from the measurements of g_1 and g_2, one is able to extract the higher-twist term g_2^{HT}.

Wandzura-Wilczek relation in terms of TMM

We find a generalization of the Wandzura-Wilczek relation for the TMM:

$$\bar{g}_2^n(x_0, Q^2) = \frac{1-n}{n}\,\bar{g}_1^n(x_0, Q^2) - \frac{x_0^n}{n}\,\bar{g}_1^0(x_0, Q^2)$$

For $x_0 = 0$ one obtains the well known form

$$\bar{g}_2^n(Q^2) = \frac{1-n}{n}\,\bar{g}_1^n(Q^2)$$

$$\bar{g}_{1,2}^n(Q^2) = \int_0^1 dx\, x^{n-1}\, g_{1,2}(x, Q^2)$$

$$\bar{g}_{1,2}^n(x_0, Q^2) = \int_{x_0}^1 dx\, x^{n-1}\, g_{1,2}(x, Q^2)$$

Generalization of the Wandzura-Wilczek relation for TMM

The generalization of the WW relation for TMM at two different points of the truncation for $n = 1$ implies

Partial twist-2 contributions to the BC sum rule

$$\int_{x_1}^{x_2} dx\, g_2^{WW}(x, Q^2) = (x_2 - x_1) \int_{x_2}^{1} \frac{dx}{x} g_1(x, Q^2) - x_1 \int_{x_1}^{x_2} \frac{dx}{x} g_1(x, Q^2)$$

$$\int_{x_0}^{1} dx\, g_2^{WW}(x, Q^2) = -x_0 \int_{x_0}^{1} \frac{dx}{x} g_1(x, Q^2)$$

$$\int_{0}^{x_0} dx\, g_2^{WW}(x, Q^2) = x_0 \int_{x_0}^{1} \frac{dx}{x} g_1(x, Q^2)$$

Generalization if Wandzura-Wilczek relation for TMM

Determination of Higher twist (HT) effects from moments of g_2

$$g_2^{HT} \equiv g_2 - g_2^{LT}$$

Wandzura-Wilczek (WW) relation (generalization for TMMA)

$$\bar{g}_2(x,n) = \frac{1-n}{n}\,\bar{g}_1(x,n) - \frac{x^n}{n}\,\bar{g}_1(x,0)$$

$$\int\limits_{x_1}^{x_2} dx\, g_2^{WW}(x,Q^2) = (x_2 - x_1)\int\limits_{x_2}^{1} \frac{dx}{x}\, g_1(x,Q^2) - x_1 \int\limits_{x_1}^{x_2} \frac{dx}{x}\, g_1(x,Q^2)$$

HT corrections provide information on the quark-hadron duality (between short- and long-distance regions of parton interactions)

Q^2 evolution of g_2

While a general DGLAP-type equation for g_2 does not exist, for the twist-3 component of g_2 suitable evolution equations have been formulated by V. M. Braun, G. P. Korchemsky and A. N. Manashov. In the leading twist-2 approximation, the Q^2 evolution of g_2 is governed by the evolution of g_1, according to the Wandzura-Wilczek relation.

The second term on the WW relation is the $n = 0$th truncated moment of the function g_1, which evolves in the same way as g_1 itself ($P'(0, z) = P(z)$). This leads to

DGLAP evolution equation for g_2

$$\frac{dg_2^{WW}(x, Q^2)}{d \ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \int_x^1 \frac{dz}{z} P\left(\frac{x}{z}\right) g_2^{WW}(z, Q^2)$$

Q^2 evolution of g_2

Taking also into account that twist-3 parton distributions obey the DGLAP-type scale dependence (Braun, Korchemsky, Manashov), we obtain a system of evolution equations for

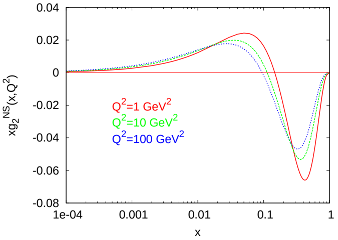
$$g_2 = g_2^{EXP} = g_2^{WW} + g_2^{twist-3}:$$

$$\frac{d\left[g_2^{EXP}(x,Q^2) - g_2^{WW}(x,Q^2)\right]}{d\ln Q^2} =$$

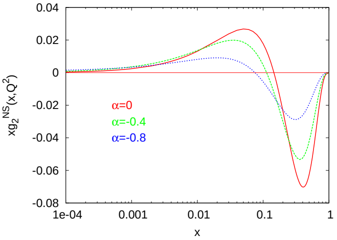
$$= \frac{\alpha_s(Q^2)}{2\pi}\int\limits_x^1 \frac{dz}{z}\, P^{twist-3}\left(\frac{x}{z}\right)\left[g_2^{EXP}(z,Q^2) - g_2^{WW}(z,Q^2)\right]$$

$$\frac{dg_2^{WW}(x,Q^2)}{d\ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi}\int\limits_x^1 \frac{dz}{z}\, P\left(\frac{x}{z}\right) g_2^{WW}(z,Q^2)$$

Q^2 evolution of g_2



The nonsinglet LO contributions to the polarised structure function $xg_2^{NS}(x, Q^2)$ for different Q^2. Parametrization of g_1 with $\alpha = -0.4$



The nonsinglet LO contributions to the polarised structure function $xg_2^{NS}(x, Q^2)$ at $Q^2 = 10\,\mathrm{GeV}^2$ for different parametrizations of g_1

Relations for MM

The evolution equations for the truncated moments are very similar to those for the parton densities. In both cases one deals with functions of two variables x and Q^2 (with additionally fixed index n for moments), which obey the differentio-integral Volterra-like equations.

The only difference lies in the splitting function, which for moments has the rescaled form $P' = x^n P$. This similarity allows one to solve the equations for truncated moments with use of standard methods of solving the DGLAP equations.

Analysis of the evolution, performed in moment space, when applying to the truncated moments, implies dealing with such an exotic structure like 'Moment of Moment'.

Relations for MM

truncated moment

$$\bar{q}_n(x) = \int_x^1 dz\, z^{n-1}\, q(z)$$

untruncated moment

$$\bar{q}_n \equiv \bar{q}_n(0) = \int_0^1 dz\, z^{n-1}\, q(z)$$

untruncated $s-$th moment of truncated $n-$th moment

$$\mathcal{M}(s,n) = \int_0^1 dx\, x^{s-1}\, \bar{q}_n(x)$$

Relations for MM

Relations between un- and truncated MM

$$\mathcal{M}(s,n) = \frac{1}{s}\,\bar{q}_{s+n}$$

$$\bar{q}_n(x) = \frac{1}{2\pi\imath}\int_{c-\imath\infty}^{c+\imath\infty}\frac{ds}{s}\,x^{-s}\,\bar{q}_{s+n}$$

$$\bar{q}_n = (n-s)\,\mathcal{M}(n-s,s) = (n-s)\int\limits_0^1 \frac{dx}{x}\,x^{n-s}\,\bar{q}_s(x)$$

Relations between the truncated and untruncated moments have a large practical meaning and could be applied when the untruncated moments are known e.g. from lattice calculations.

PDF $q(x, Q^2)$	function	TMM $\bar{q}_n(x, Q^2)$
$\frac{\partial q(x,Q^2)}{\partial \ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi}\,(P \otimes q)$	x-space evol.	$\frac{\partial \bar{q}_n(x,Q^2)}{\partial \ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi}\,(P' \otimes \bar{q}_n)$
$\frac{\partial \bar{q}_n}{\partial \ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi}\,\gamma_n\,\bar{q}_n$	moment-space evol.	$\frac{\partial \mathcal{M}(s,n)}{\partial \ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi}\,\gamma_{s+n}\,\mathcal{M}(s,n)$
$q(x,Q^2) = \frac{1}{2\pi\imath}\int dn\, x^{-n}\,\bar{q}_n$	solution	$\bar{q}_n(x,Q^2) = \frac{1}{2\pi\imath}\int ds\, x^{-s}\,\mathcal{M}(s,n)$

Replacement: $\mathcal{M}(s,n)(Q_0^2) = \frac{1}{s}\,\bar{q}_{s+n}(Q_0^2)$

Perspectives

Possible future applications of TMMA

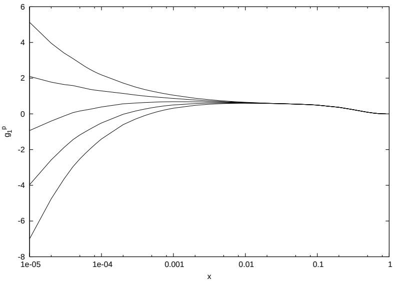
Studying the fundamental properties of nucleon structure

- momentum fraction carried by quarks (moments of F_1, F_2)
- quark helicities contributions to the spin of nucleon (moments of g_1) DIS and SIDIS experimental data
- particularly important:
  estimation of the polarised gluon contribution ΔG from more precise COMPASS and RHIC data and resolving the spin puzzle:

$$\frac{1}{2}\Delta\Sigma + \Delta G + L_q + L_G = \frac{1}{2}$$

Perspectives

The polarised structure function g_1 for the proton at $Q^2 = 10\ \mathrm{GeV}^2$



Results for different contributions of ΔG to the proton's spin.
From up to bottom:
-0.25, 0, 0.25, 0.5, 0.75

RHIC - knowledge of the small-x behaviour of $\Delta G(x, Q^2)$
A limit on the gluon spin contribution from PHENIX:
$-0.7 < \Delta G < 0.5$

Perspectives

Determination of Higher twist (HT) effects from moments of g_2

$$g_2^{HT} \equiv g_2 - g_2^{LT}$$

- Wandzura-Wilczek (WW) relation (generalization for TMMA)

$$\bar{g}_2(x,n) = \frac{1-n}{n}\,\bar{g}_1(x,n) - \frac{x^n}{n}\,\bar{g}_1(x,0)$$

- Q^2 evolution equations for g_2
- test of sum rules: Burkhardt-Cottingham (BC) Efremov-Leader-Teryaev (ELT)

HT corrections provide information on the quark-hadron duality (between short- and long-distance regions of parton interactions)

Perspectives

TMMA for generalized parton distributions (GPDs)

Moments of GPDs can be related to the total angular momentum (spin and orbital) carried by various quark flavors

Measurements sensitive to Generalized Parton Distributions -
- Deeply Virtual Compton Scattering (DVCS) (Jefferson Lab)

An important step towards a full accounting of the nucleon spin

In light of the recent progress in experimental program, the comprehensive theoretical analysis of the structure functions and their moments is of a great importance

Summary

TMMA enables to study fundamental properties of the nucleon in a restricted experimentally range of Bjorken-x

EXPERIMENTS PROVIDE CUT MOMENTS!

No uncertainties from the unmeasurable regions!

Evolution

TMM obey the same DGLAP-like evolution equations as PDFs!

$$P'(z, n) = z^n P(z)$$

https://maps.google.de/maps?f=d&source=s_d&sa...

Opole $50^\circ\ 40'$ N, $17^\circ\ 56'$ E