QCD inspired meson model and Swinger-Dyson equation for massless quark.

V. Shilin, V. Pervushin, A. Cherny

BLTP JINR

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Abstract

My goal was to develop methods of solution of Swinger-Dyson equation for effective models of strong interaction.
Simple model will be presented, but methods can be used in more general case.

Outline:

Effective Action of Strong Interaction
Solution of the massless Schwinger-Dyson equation
- Numerical solution
- Analytical estimations

Effective Action of Strong Interaction

We start with

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - A^a_\mu j^{a\mu} + \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu$$
$$j^{a\mu} = -g\overline{\psi}\gamma^\mu\frac{\lambda^a}{2}\psi$$

We want to derive from this lagrangian an effective action for meson-like bound state[1], under some *restrictions* and *assumptions*:

- We will work only in definite frame of reference. Below after some calculation we obtain a bound state which at whole will be at rest in this frame of reference. So only *static* problems considered.

[1] There were a lot of attempts of making this: Arbuzov, Volkov; Efimov, Ivanov, Nedelko; etc.

Gauge: $\partial_k A_k^a(x) = 0$

The gluon term in lagrangian:

$$-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} = \frac{1}{2}\dot{A}^a_i\dot{A}^a_i - \frac{1}{4}F^a_{ij}F^{aij} + \frac{1}{2}A^a_0(-\Delta + M_g^2)A^a_0 + \dots$$

where: $M_g^2 \equiv 6g^2C_gN_c$

- Let us consider dotted terms as perturbation.

Short explanation:

1. After normal ordering with vacuum 2-point correlator:

   $$\langle 0|A^a_i(x)A^b_j(x)|0\rangle = 2C_g\delta_{ij}\delta^{ab}$$

   where: $C_g \neq 0$ and $C_g < \infty$.
2. It is known from phenomenology that at small energies gluon effectively have mass (Scadron, Politzer, Zakharov, etc.).

Generating functional:

$$\mathcal{Z} = \int \mathsf{D} A^a_\mu \delta(\partial_k A^a_k) \mathsf{D}\overline{\psi} \mathsf{D}\psi \exp\Big[i \int d^4x \Big(\frac{1}{2} \dot{A}^a_i \dot{A}^a_i - \frac{1}{4} F^a_{ij} F^{aij} + \\ + \frac{1}{2} A^a_0 (-\Delta + M_g^2) A^a_0 - A^a_0 j^a_0 + A^a_i j^a_i + \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi \Big) \Big]$$

After integrating over A^a_0:

$$\mathcal{Z} = \int \mathsf{D} A^a_k \delta(\partial_k A^a_k) \mathsf{D}\overline{\psi} \mathsf{D}\psi$$

$$\exp\Big[i \int d^4x \Big(\frac{1}{2} \dot{A}^a_i \dot{A}^a_i - \frac{1}{4} F^a_{ij} F^{aij} + A^a_i j^a_i + \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi \Big) - \\ - \frac{i}{2} \int d^4x \, d^4y \; j^a_0(x) \delta(x^0 - y^0) \frac{1}{4\pi} \frac{e^{-M_g |\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} \, j^a_0(y) \Big]$$

Rewrite using $\mathcal{K}$: $-\frac{1}{2}\int d^4x\, d^4y\; j_0^a(x)\delta(x^0-y^0)\frac{1}{4\pi}\frac{e^{-M_g|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}\; j_0^a(y) =$

$$= -\frac{1}{2}\int d^4x_1\, d^4x_2\, d^3\mathbf{y}_1\, d^3\mathbf{y}_2\; \overline{\psi}^{r_1}_{\alpha_1}(x_1)\,\psi^{\alpha_2 r_2}(x_2)\delta^{r_1 s_1}\times$$

$$\underbrace{\times\gamma^{0\alpha_1}{}_{\alpha_2}\,\delta^4(x_1-x_2)\,\frac{g^2}{8\pi}\frac{e^{-M_g|\mathbf{x}_1-\mathbf{y}_2|}}{|\mathbf{x}_1-\mathbf{y}_2|}\,\delta^3(\mathbf{y}_1-\mathbf{y}_2)\,\gamma^{0\beta_2}{}_{\beta_1}\times}_{\mathcal{K}^{\alpha_1}{}_{\beta_1\alpha_2}{}^{\beta_2}(x_1,\mathbf{y}_1;x_2,\mathbf{y}_2)}$$

$$\times\,\delta^{r_2 s_2}\overline{\psi}^{s_2}_{\beta_2}(x_2^0,\mathbf{y}_2)\,\psi^{\beta_1 s_1}(x_1^0,\mathbf{y}_1)+\ldots$$

$$\frac{\lambda^{a r_1 r_2}}{2}\frac{\lambda^{a s_2 s_1}}{2} = \frac{1}{2}\delta^{r_1 s_1}\delta^{r_2 s_2} - \frac{1}{6}\delta^{r_1 r_2}\delta^{s_2 s_1}$$

- As we want to consider only colorless mesons, we neglect the second term. In case of 4-quark bound states it should be taken into account.

- Let's consider $\psi^{\alpha s}(x^0,\mathbf{x})\overline{\psi}^s_\beta(x^0,\mathbf{y})$ as a real bilocal field.

$$\exp\Big[-\frac{i}{2}\int d^4x_1\, d^4x_2\, d^3\mathbf{y}_1\, d^3\mathbf{y}_2\; \overline{\psi}^r_{\alpha_1}(x_1)\,\psi^{\beta_1 r}(x_1^0,\mathbf{y}_1)\times$$
$$\times\,\mathcal{K}^{\alpha_1}{}_{\beta_1\alpha_2}{}^{\beta_2}(x_1,\mathbf{y}_1;x_2,\mathbf{y}_2)\,\psi^{\alpha_2 s}(x_2)\,\overline{\psi}_{\beta_2 s}(x_2^0,\mathbf{y}_2)\Big] =$$

Introduce new bilocal field $\mathcal{M}^\alpha{}_\beta(x^0,\mathbf{x},\mathbf{y})$:

$$= \int \mathsf{D}\mathcal{M}\;\exp\Big[\frac{i}{2}\int d^4x_1\, d^4x_2\, d^3\mathbf{y}_1\, d^3\mathbf{y}_2\; \mathcal{M}^{\mathrm{T}}{}_{\alpha_1}{}^{\beta_1}(x_1^0,\mathbf{x}_1,\mathbf{y}_1)\times$$
$$\times\,\mathcal{K}^{-1\,\alpha_1}{}_{\beta_1\alpha_2}{}^{\beta_2}(x_1,\mathbf{y}_1;x_2,\mathbf{y}_2)\,\mathcal{M}^{\alpha_2}{}_{\beta_2}(x_2^0,\mathbf{x}_2,\mathbf{y}_2)\,+$$
$$+\; i\int d^4x\, d^3\mathbf{y}\;\overline{\psi}^r_\alpha(x^0,\mathbf{x})\,\psi^{\beta r}(x^0,\mathbf{y})\,\mathcal{M}^\alpha{}_\beta(x^0,\mathbf{x},\mathbf{y})\Big]$$

Finally:

$$\mathcal{Z} = \int \mathsf{D}A_k^a \delta(\partial_k A_k^a) \mathsf{D}\overline{\psi} \mathsf{D}\psi \mathsf{D}\mathcal{M}$$

$$\exp\Bigg[i \int d^4x \Big(\frac{1}{2} \dot{A}_i^a \dot{A}_i^a - \frac{1}{4} F_{ij}^a F^{aij} + A_i^a j_i^a + \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi \Big) +$$

$$+ \ \frac{i}{2} \int d^4x_1 \, d^4x_2 \, d^3\mathbf{y}_1 \, d^3\mathbf{y}_2 \ {\mathcal{M}^{\mathrm{T}}}_{\alpha_1}{}^{\beta_1}(x_1^0, \mathbf{x}_1, \mathbf{y}_1) \times$$

$$\times \, {\mathcal{K}^{-1}}^{\alpha_1}{}_{\beta_1 \alpha_2}{}^{\beta_2}(x_1, \mathbf{y}_1; x_2, \mathbf{y}_2) \, \mathcal{M}^{\alpha_2}{}_{\beta_2}(x_2^0, \mathbf{x}_2, \mathbf{y}_2) +$$

$$+ \ i \int d^4x \, d^3\mathbf{y} \ \overline{\psi}_\alpha(x^0, \mathbf{x}) \, \psi^\beta(x^0, \mathbf{y}) \, \mathcal{M}^\alpha{}_\beta(x^0, \mathbf{x}, \mathbf{y}) \Bigg]$$

For quantization of Bilocal Fields we use Stationary Phase method.

After integrating over fermions:

$$\mathcal{Z} = \int \mathsf{D}A_k^a \delta(\partial_k A_k^a) \mathsf{D}\mathcal{M} e^{iS_{eff}}$$

Swinger-Dyson (Gap) equation – fermion spectrum:

$$\left.\frac{\delta S_{eff}}{\delta \mathcal{M}}\right|_{A_k^a=0} = 0$$

Search the solution in the form

$$\mathcal{M}^{\alpha}{}_{\beta}(x^0, \mathbf{x}, \mathbf{y}) \; = \; -\Sigma^{\alpha}{}_{\beta}(x^0, \mathbf{x}, \mathbf{y}) \; + \; m\delta^{\alpha}{}_{\beta}\, \delta^3(\mathbf{x} - \mathbf{y})$$

The Swinger-Dyson equation takes form:

$$\Sigma^{\alpha_1}{}_{\beta_1}(x_1^0,\mathbf{x}_1,\mathbf{y}_1) \;=\; m\,\delta^{\alpha_1}{}_{\beta_1}\,\delta^3(\mathbf{x}_1-\mathbf{y}_1) + \\ + i\int d^4x_2\,d^4y_2\;\mathcal{K}^{\alpha_1}{}_{\beta_1\alpha_2}{}^{\beta_2}(x_1,\mathbf{y}_1;x_2,\mathbf{y}_2)\,G_\Sigma{}^{\alpha_2}{}_{\beta_2}(x_2,y_2)\,\delta(x_2^0-y_2^0)$$

where: $G_\Sigma^{-1}{}^{\alpha}{}_{\beta}(x,y) = i\gamma^{\mu\alpha}{}_{\beta}\partial_\mu\,\delta^4(x-y) \;-\; \Sigma^{\alpha}{}_{\beta}(x^0,\mathbf{x},\mathbf{y})\,\delta(x^0-y^0)$

- Try next ansatz:

$$\Sigma^{\alpha}{}_{\beta}(x^0,\mathbf{x},\mathbf{y}) = \delta^{\alpha}{}_{\beta}\frac{1}{(2\pi)^{\frac{3}{2}}}M(\mathbf{x}-\mathbf{y})$$

After Fourier-transform:

$$M(\mathbf{p})\,\delta^{\alpha_1}{}_{\beta_1} \;=\; m\,\delta^{\alpha_1}{}_{\beta_1} - \\ - i\frac{g^2}{2(2\pi)^4}\int d^4q\,\frac{1}{(\mathbf{p}-\mathbf{q})^2+M_g^2}\,\gamma^{0\alpha_1}{}_{\alpha_2}\,G_\Sigma{}^{\alpha_2}{}_{\beta_2}(q)\,\gamma^{0\beta_2}{}_{\beta_1}$$

where:

$$G_\Sigma(q) = e^{-\gamma^i\frac{q_i}{|\mathbf{q}|}\varphi(\mathbf{q})}\left(\frac{1}{q_0+E(\mathbf{q})-i\varepsilon}\cdot\frac{1+\gamma^0}{2} + \right. \\ \left. + \;\frac{1}{q_0-E(\mathbf{q})+i\varepsilon}\cdot\frac{1-\gamma^0}{2}\right)e^{\gamma^i\frac{q_i}{|\mathbf{q}|}\varphi(\mathbf{q})}\gamma^0$$

$$E(\mathbf{q}) \equiv \sqrt{M(\mathbf{q})^2+\mathbf{q}^2} \qquad\qquad \cos 2\varphi(\mathbf{q}) \equiv \frac{M(\mathbf{q})}{E(\mathbf{q})}$$

We can see that one can direct integrate over q_0.

After integrating over solid angles:

$$M(p) = m + \frac{g^2}{32\pi^2 p} \int_0^\infty dq \frac{qM(q)}{\sqrt{M^2(q) + q^2}} \ln \frac{M_g^2 + (p+q)^2}{M_g^2 + (p-q)^2}$$

We want to find the solution of this equation for all values of p.

Solution of the massless Schwinger-Dyson equation

- $m = 0.$
- $M(q) \to 0$ at $q \to \infty$, ($\Rightarrow$ no renormalization is need).

Introduce dimensionless variables:

$$\bar{p} \equiv \frac{p}{M_g} \quad \bar{q} \equiv \frac{q}{M_g} \quad \bar{M}(\bar{p}) \equiv \frac{M(p)}{M_g}$$

The Schwinger-Dyson equation takes form:

$$\bar{M}(\bar{p}) = \frac{g^2}{32\pi^2\bar{p}} \int\limits_0^\infty d\bar{q} \frac{\bar{q}\bar{M}(\bar{q})}{\sqrt{\bar{M}^2(\bar{q}) + \bar{q}^2}} \ln \frac{1 + (\bar{p} + \bar{q})^2}{1 + (\bar{p} - \bar{q})^2}$$

There is always a solution: $\bar{M}(\bar{p}) = 0$

Put by definition $\bar{M}(-\bar{p}) = \bar{M}(\bar{p})$, than:

$$\bar{M}(\bar{p}) = \frac{g^2}{64\pi^2\bar{p}} \int\limits_{-\infty}^{+\infty} d\bar{q} \frac{\bar{q}\bar{M}(\bar{q})}{\sqrt{\bar{M}^2(\bar{q}) + \bar{q}^2}} \ln \frac{1 + (\bar{p} + \bar{q})^2}{1 + (\bar{p} - \bar{q})^2}$$

Numerical solution

Things that should be avoided:

1. Upper limit of integration must be $+\infty$, and can not be replaced by finite quantity Λ.
2. $\bar{M}(+\infty) = 0$, otherwise integral diverge.
3. It is better to avoid replacing continuous function $\bar{M}(\bar{p})$ by a discrete table $\bar{M}(\bar{p}_i)$ with fixed numbers of points $\bar{p}_i$.

Introduce:

$$W(\bar{q}) \equiv \frac{\bar{q}\bar{M}(\bar{q})}{\sqrt{\bar{M}^2(\bar{q}) + \bar{q}^2}}$$

$W(-\bar{q}) = W(\bar{q})$

Introduce new variables:
$$\bar{p} = \lambda \tan\frac{\varphi}{2} \quad , \quad \varphi \in (-\pi, \pi)$$
$$\bar{q} = \lambda \tan\frac{\theta}{2} \quad , \quad \theta \in (-\pi, \pi)$$

where λ - some parameter.

Schwinger-Dyson equation takes form:

$$\bar{M}(\varphi) = \frac{g^2}{64\pi^2} \int\limits_{-\pi}^{+\pi} \frac{d\theta}{2\,\tan\frac{\varphi}{2}\,\cos^2\frac{\theta}{2}} \ln\Big(\frac{1+\lambda^2(\tan\frac{\varphi}{2}+\tan\frac{\theta}{2})^2}{1+\lambda^2(\tan\frac{\varphi}{2}-\tan\frac{\theta}{2})^2}\Big) W(\theta)$$

On $[-\pi, \pi]$ there is convenient system of functions – Fourier series:

$$\bar{M}(\varphi) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cdot \cos k\varphi \qquad\qquad W(\theta) = \sum_{k=1}^{\infty} b_k \cdot \sin k\theta$$

The equation:

$$a_k = A_{kj} b_j$$

where: $A_{kj} \equiv \dfrac{g^2}{64\pi^3} M_{kj}$, where:

$$M_{kj} \equiv \int\limits_{-\pi}^{+\pi} d\varphi \int\limits_{-\pi}^{+\pi} d\theta \frac{\cos(k\varphi)}{2\,\tan\frac{\varphi}{2}\,\cos^2\frac{\theta}{2}} \ln\Big(1+\frac{\lambda^2 \sin\varphi\,\sin\theta}{(\cos\frac{\varphi}{2}\,\cos\frac{\theta}{2})^2+(\lambda\sin\frac{\varphi-\theta}{2})^2}\Big) \sin(j\theta)$$

$$W_0(\varphi)$$
$$\downarrow$$
$$\text{to Fourier coeff. } b_j$$
$$\downarrow$$
$$a_k = A_{kj} b_j \quad \longrightarrow \quad \bar{M}(\varphi) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cdot \cos k\varphi \quad \searrow$$
$$\uparrow \qquad\qquad\qquad \downarrow$$
$$\text{to Fourier coeff. } b_j \quad \longleftarrow \quad W(\varphi) = \frac{\lambda \sin\frac{\varphi}{2}\bar{M}(\varphi)}{\sqrt{(\cos\frac{\varphi}{2}\bar{M}(\varphi))^2 + (\lambda \sin\frac{\varphi}{2})^2}} \qquad \text{Result}$$
$$\downarrow \qquad\qquad\qquad \nearrow$$
$$W(\theta) = \sum_{k=1}^{\infty} b_k \cdot \sin k\theta \quad \longrightarrow \quad \text{Insert in exact SD equation}$$

There is only $\bar{M}(\bar{p}) = 0$ solution at $g^2 < 32$.

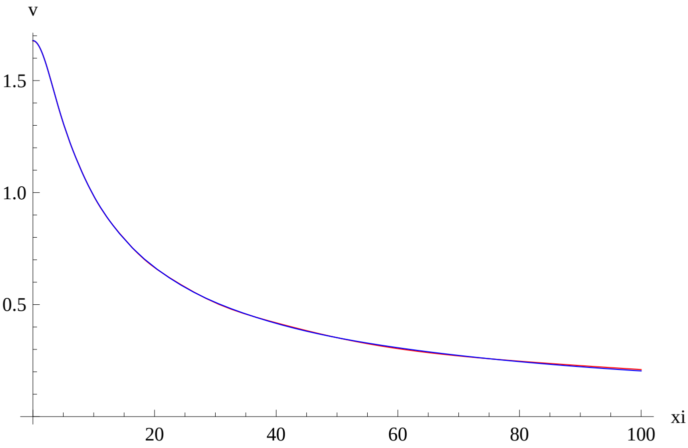

Figure: Plot $\bar{M}(\bar{p})$, at $g^2 \simeq 33.51$, $\lambda = 20$, 13 harmonics.

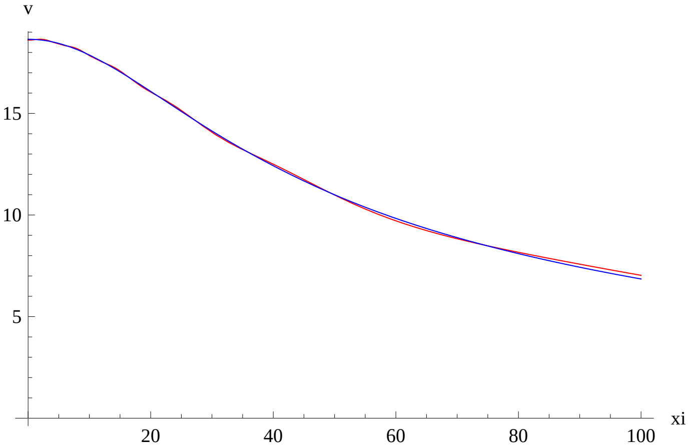

Figure: Plot $\bar{M}(\bar{p})$, at $g^2 \simeq 36.86$, $\lambda = 20$, 13 harmonics.

Analytical estimations.

Try to find asymptotical behavior of $\bar{M}(\bar{p})$ at large $\bar{p}$.

The Schwinger-Dyson equation:

$$\bar{M}(\bar{p}) = \frac{g^2}{64\pi^2\bar{p}} \int\limits_{-\infty}^{+\infty} d\bar{q} \frac{\bar{q}\bar{M}(\bar{q})}{\sqrt{\bar{M}^2(\bar{q})+\bar{q}^2}} \ln\frac{1+(\bar{p}+\bar{q})^2}{1+(\bar{p}-\bar{q})^2}$$

$$\Updownarrow_{\bar{q}\mapsto-\bar{q}}$$

$$\bar{M}(\bar{p}) = -\frac{g^2}{32\pi^2\bar{p}} \int\limits_{-\infty}^{+\infty} d\bar{q} \frac{\bar{q}\bar{M}(\bar{q})}{\sqrt{\bar{M}^2(\bar{q})+\bar{q}^2}} \ln\Big(1+(\bar{p}-\bar{q})^2\Big)$$

We try to solve approximate equation, where $\bar{M}_0 \equiv \bar{M}(0)$:

$$\bar{M}(\bar{p}) = -\frac{g^2}{32\pi^2\bar{p}} \int\limits_{-\infty}^{+\infty} d\bar{q} \frac{\bar{q}\bar{M}(\bar{q})}{\sqrt{\bar{M}_0^2+\bar{q}^2}} \ln\Big(1+(\bar{p}-\bar{q})^2\Big)$$

Introduce: $\mathcal{W}(\bar{q}) \equiv \dfrac{\bar{q}\bar{M}(\bar{q})}{\sqrt{\bar{M}_0^2 + \bar{q}^2}}$

$$\sqrt{\bar{M}_0^2 + \bar{p}^2}\,\mathcal{W}(\bar{p}) \;=\; -\frac{g^2}{32\pi^2}\int\limits_{-\infty}^{+\infty} d\bar{q}\;\ln\Big(1 + (\bar{p} - \bar{q})^2\Big)\,\mathcal{W}(\bar{q})$$

After Fourier transform:

$$\sqrt{\bar{M}_0^2 + \partial^2}\,\mathcal{W}(x) \;=\; \frac{g^2}{16\pi}\,\frac{e^{-|x|}}{|x|}\,\mathcal{W}(x)$$

Consider $\bar{p} \to \infty$ asymptotics.

$$\sqrt{\bar{M}_0^2 + \bar{p}^2} \;\longrightarrow\; |\bar{p}|$$

It corresponds $x \to 0$, so Taylor expansion can be used.

Try next ansatz:

$$\bar{M}(\bar{p}) = C\frac{1}{|\bar{p}|^{\beta}}$$

The SD equation is self-consistent if:

$$\frac{1}{g^2} = \frac{1}{16\pi}\frac{\cot\left(\frac{\pi\beta}{2}\right)}{(1-\beta)}$$

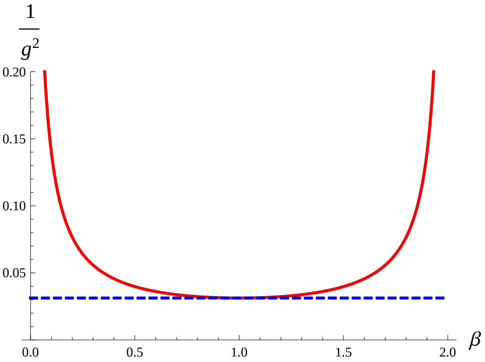

Figure: Plot $\frac{1}{16\pi}\frac{\cot(\frac{\pi\beta}{2})}{(1-\beta)}$ and $\frac{1}{32}$.

Therefore the true asymptotic (where $\gamma > 0$-some currently unknown factor):

$$\bar{M}(\bar{p}) = C\frac{\log^{\gamma}|\bar{p}|}{|\bar{p}|^{\beta}}$$

Conclusions

1. The simple model of strong interaction with massive gluon was constructed.
2. The Swinger-Dyson equation for $m = 0$ was solved numerically. Nontrivial solutions appears only if $g^2 > 32$.
3. Asymptotical behavior of $\bar{M}(p)$ at large p was analyzed analytically.

Future plans

- Pion wave-function from Bethe-Salpeter equation,
- Swinger-Dyson and Bethe-Salpeter equations for $m = 0$,
- Temperature-like factors.