Determination mass spectrum and decay constants mesons with beauty and charm

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The relativistic corrections

- ⋆ Nonrelativistic quantum mechanics (NRQM).
- ⋆ For the description of modern experimental results
  necessary to take into account
  the relativistic correction.
- ⋆ The relativistic corrections to the interaction potential :
  The Breit potential
  The effective nonrelativistic QFT of Caswell and Lepage.
- ⋆ The relativistic correction to the kinetic part of the Hamiltonian.
- ⋆ The relativistic SE.

Bound states in the functional approach

In this section, we will present one of the alternative methods of the bound state mass determination $J(x) = \Phi^{+}(x)\Phi(x)$

$$\Pi(x-y) = \langle G_{m_1}(x,y|A)G_{m_2}(y,x|A)\rangle_A \quad (1)$$

Here the averaging over the external gauge field $A_\alpha(x)$ is performed. The Green function $G_m(y,x|A)$ for the scalar particle in the external gauge field is determined from the equation

$$\left[\left(i\frac{\partial}{\partial x_\alpha} + \frac{g}{c\hbar}\cdot A_\alpha(x)\right)^2 + \frac{c^2 m^2}{\hbar^2}\right] G_m(x,y|A) = \delta(x-y)\ , \quad (2)$$

where m is the mass of the scalar particle, and g is the coupling constant. In averaging over the external gauge field $A_\alpha(x)$, let us consider only the lowest order or only the two-point Gauss correlator

$$< \exp\left\{i\int dx A_\alpha(x)J_\alpha(x)\right\} >_A = \exp\left\{-\frac{1}{2}\int\int dxdy J_\alpha(x)D_{\alpha\beta}(x-y)J_\beta(y)\right\}$$

where $J_\alpha(x)$ is the real current. The propagator of the gauge field has the following form:

$$D_{\alpha\beta}(x-y) =< A_\alpha(x) A_\beta(y) >_A = \delta_{\alpha\beta} D(x-y) \ . \tag{4}$$

The mass of the bound state is usually defined through the loop function in the following way:

$$M = -\lim_{|x-y|\to\infty} \frac{\ln \Pi(x-y)}{|x-y|} \ . \tag{5}$$

Thus, if we know the loop function, then we can determine the bound state mass. From (1) one can see that for the determination of the loop function one needs to determine the Green function. The solution of (2) can be represented as a functional integral in the following way :

$$G_m(x,y|A) = \int_0^\infty \frac{ds}{(4s\pi)^2} \exp\left\{ -sm^2 - \frac{(x-y)^2}{4s} \right\} \tag{6}$$

$$\int d\sigma_\beta \exp\left\{ ig \int_0^1 d\xi \frac{\partial Z_\alpha(\xi)}{\partial \xi} A_\alpha(\xi) \right\} \ ,$$

where the following notation is used:

$$Z_\alpha(\xi) = (x-y)_\alpha \xi + y_\alpha - 2\sqrt{s} B_\alpha(\xi); \tag{7}$$

$$d\sigma_\beta = N \delta B_\beta \exp\left\{ -\frac{1}{2} \int_0^1 d\xi B'^2(\xi) \right\}$$

with the normalization $B_\alpha(0) = B_\alpha(1) = 0$; $\int d\sigma_\beta = 1$, where N is the normalization constant. Substituting (6) into (1) and performing averaging over the external gauge field one can obtain for the loop function

$$\Pi(x) = \int_0^\infty \int_0^\infty \frac{d\mu_1 d\mu_2}{(8\pi^2 x)^2} \cdot J(\mu_1, \mu_2) \times \tag{8}$$

$$\times \exp\left\{ -\frac{|x|}{2} \left(\frac{m_1^2}{\mu_1} + \mu_1 \right) - \frac{|x|}{2} \left(\frac{m_2^2}{\mu_2} + \mu_2 \right) \right\} .$$

Here

$$J(\mu_1,\mu_2) = N_1 N_2 \int\int \delta r_1 \delta r_2 \times \tag{9}$$

$$\times \exp\left\{ -\frac{1}{2}\int_0^x d\tau \left[\mu_1 \dot{r}_1^2(\tau) + \mu_2 \dot{r}_2^2(\tau)\right] \right\} \times$$

$$\times \exp\left\{ -W_{1,1} - W_{2,2} + 2\sum_{i,j=1;i\neq j}^{2} W_{i,j} \right\} ,$$

and the following notation is introduced:

$$W_{i,j} = \frac{g^2}{2}(-1)^{i+j}\int_0^x\int_0^x d\tau_1 d\tau_2 Z'^{(i)}{}_\alpha(\tau_1) \times$$

$$\times \quad D_{\alpha\beta}\left(Z^{(i)}(\tau_1) - Z^{(j)}(\tau_2)\right) Z'^{(j)}{}_\beta(\tau_2) \,. \tag{10}$$

$$\lim_{|x|\to\infty} J(\mu_1,\mu_2) \Longrightarrow \exp\{-x\cdot E(\mu_1,\mu_2)\}\ , \tag{11}$$

From the (8)

$$M = \frac{1}{2}\ \min_{\mu_1,\mu_2}\left\{\frac{m_1^2}{\mu_1} + \mu_1 + \frac{m_2^2}{\mu_2} + \mu_2 + 2E(\mu_1,\mu_2)\right\}\ , \tag{12}$$

$$H = \frac{1}{2\mu_1}\mathbf{P}_1^2 + \frac{1}{2\mu_2}\mathbf{P}_2^2 + V(\mathbf{r}_1,\mathbf{r}_2)\ , \tag{13}$$

$$H\Psi(\mathbf{r}_1,\mathbf{r}_2) \quad = \quad E(\mu_1,\mu_2)\Psi(\mathbf{r}_1,\mathbf{r}_2)\ . \tag{14}$$

$$\mu_j - \frac{m_j^2}{\mu_j} + 2\mu_j\frac{dE(\mu_1,\mu_2)}{d\mu_j} = 0\ ; \qquad j = 1,\ 2. \tag{15}$$

We will consider the parameters μ_1, μ_2 as masses of the constituent particles in the bound state. These masses differ from m_1, m_2 which represent the masses of a free state. For the further calculation introduced the reduced mass two-body systems.

$$\mu_1 = \sqrt{m_1^2 - 2\mu^2 \frac{dE}{d\mu}} \; ; \qquad \mu_2 = \sqrt{m_2^2 - 2\mu^2 \frac{dE}{d\mu}} \; ;$$

$$\frac{1}{\mu} = \frac{1}{\mu_1} + \frac{1}{\mu_2} \, ,$$

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The energy spectrum hydrogen atom

In the framework our method we determine the mass spectrum hydrogen atom. According to(12) the energy spectrum $E(M_2)$ defined from the SE

$$\left[\frac{1}{2\mu}\cdot\vec{P}^2-\frac{Z\alpha}{r}\right]\Psi=E(\mu)\Psi\ . \tag{16}$$

and we obtained for the energy spectrum

$$E(\mu)=-\frac{Z^2\alpha^2}{2n^2}\cdot\mu\ , \tag{17}$$

Then taking into account (17) and (16) for the eigenvalues of the relativistic Hamiltonian we have in case $m=m_N=\infty$

$$\begin{aligned} E_{bin} &= \sqrt{m_e^2-2\mu^2\cdot(dE/d\mu)}+\mu(dE/d\mu)+E(\mu) \\ &= m_e\cdot\sqrt{1-\frac{Z^2\alpha^2}{n^2}}\ . \end{aligned} \tag{18}$$

Bjorken James D., Drell Sidney D.
Relativistic Quantum Theory: V.1

The Quark mass.

Beringer J. et al.: Review of Particle Physics.// Phys. Rev. **D 86** ,010001 (2012);

$$
\begin{aligned}
&m_u = 2.3^{+0.7}_{-0.5}\ MeV, \quad m_d = 4.8^{+0.7}_{-0.3}\ MeV, \\
&m_s = 95 \pm 5\ MeV, \quad m_c = 1.275 \pm 0.025\ GeV, \\
&m_b(\overline{MS}) = 4.18 \pm 0.03\ GeV, \quad m_b(1S) = 4.65 \pm 0.03\ GeV\ .
\end{aligned} \tag{19}
$$

$$
m_{u,d} = 0.33\ GeV, \quad m_s = 0.55\ GeV, \quad m_c = 1.55\ GeV, \quad m_b = 4.85\ GeV\ . \tag{20}
$$

The interaction Hamiltonian

In this section, the mass spectrum of the charmonium, bottom and B_c mesons with spin-spin and spin-orbit interactions is determined from the SE with the constituent mass.

$$
H\Psi = E\Psi \tag{21}
$$

The total interaction Hamiltonian of quarks is represented as:

$$H = H_c + H_{spin}, \tag{22}$$

where H_c is the central part

$$H_c = \frac{1}{2\mu}\vec{P}^2 + \sigma \cdot r - \frac{4}{3}\frac{\alpha_s}{r} \ . \tag{23}$$

The second part of the Hamiltonian is defined in the standard from

$$H_{spin} = H_{SS} + H_{LS} + H_{TT}. \tag{24}$$

Here H_{SS} is the spin-spin interaction Hamiltonian:

$$H_{SS} = \frac{2}{3\mu_1\ \mu_2}\ (\mathbf{S_1 S_2})\ \Delta V_{\rm v} = -\frac{8\ \alpha_s\ (\mathbf{S_1 S_2})}{9\mu_1\mu_2} \cdot \Delta(\frac{1}{r})\ , \tag{25}$$

and H_{LS} is the spin-orbital interaction Hamiltonian:

$$\begin{aligned} H_{LS} &= \frac{1}{4\ \mu_1^2\mu_2^2}\frac{1}{r}\left\{\left[\left((\mu_1+\mu_2)^2 + 2\mu_1\mu_2\right)(\mathbf{L}\cdot\mathbf{S}_+) + (\mu_1^2-\mu_2^2)(\mathbf{L}\cdot\mathbf{S}_-)\right]\frac{\partial}{\partial r}V_{\rm v}\right. \\ &- \left.\left[(\mu_1^2+\mu_2^2)(\mathbf{L}\cdot\mathbf{S}_+) + (\mu_1^2-\mu_2^2)(\mathbf{L}\cdot\mathbf{S}_-)\right]\frac{\partial}{\partial r}V_s\right\}, \end{aligned} \tag{26}$$

the tensor interaction Hamiltonian is

$$H_{TT} = \frac{1}{12\mu_1\mu_2}\left[\frac{1}{r}\frac{\partial}{\partial r}V_{\rm v} - \frac{\partial^2}{\partial r^2}V_{\rm v}\right]\ S_{12}\ . \tag{27}$$

Here $V_{\rm v}$ is the vector potential corresponding to the one-gluon exchange:

$$V_{\rm v} = -\frac{4\alpha_s}{3}\frac{1}{r}\ ; \tag{28}$$

and V_s is the confinement potential

$$V_s = r\sigma\ ; \tag{29}$$

and also the following notation is used,

$$\mathbf{S}_+ = \mathbf{S_1} + \mathbf{S_2}\ ; \quad \mathbf{S}_- = \mathbf{S_1} - \mathbf{S_2}\ ; \tag{30}$$

$$S_{12} = \frac{4}{(2\ell+3)(2\ell-1)}\left[\mathbf{L}^2\mathbf{S}^2 - \frac{3}{2}(\mathbf{LS}) - 3(\mathbf{LS})^2\right]\ .$$

Using expressions (22)-(30) for the interaction Hamiltonian we calculate the mass spectrum of the mesons.

Determination energy spectrum

Dineykhan M. et al. Oscillator representation in quantum physics.Lecture Notes in Physics, Springer-Verlag, Berlin, 1995, **vol. 26**.

$$r = q^{2\rho}, \qquad \Psi \Rightarrow \Psi(q^2) = q^{2\rho\ell}\Phi(q^2). \tag{31}$$

For the modified SE:

$$\begin{aligned}\Big\{ \; & - \frac{1}{2}\left(\frac{\partial^2}{\partial q^2} + \frac{d-1}{q}\frac{\partial}{\partial q}\right) - 4\rho^2\ \mu E\ q^{2(2\rho-1)} + 4\rho^2\ \mu\ \sigma\ q^{2(3\rho-1)} - \\ & - \frac{16\ \rho^2\ \mu\ \alpha_s}{3}\cdot q^{2(\rho-1)} + \frac{64\ \alpha_s\ \mu\ \rho^2\ \ell}{9\ \mu_1\ \mu_2\ q^{2(\rho+1)}}\cdot(\vec{S}_1\vec{S}_2) \\ & - \frac{\sigma\rho^2\mu}{M_1^2}q^{2(\rho-1)} + \frac{4\mu\rho^2\alpha_s}{3\mu_1\mu_2}\cdot\frac{S_{12}}{q^{2(\rho+1)}} + \\ & + \frac{4\mu\rho^2\alpha_s}{3\ M_2^2\ q^{2(\rho+1)}}\Big\}\Phi(q^2) = 0\ ,\end{aligned} \tag{32}$$

where d is the dimension of the auxiliary space

$$d = 2 + 2\rho + 4\rho\ell\ , \tag{33}$$

and the following notation is used:

$$\frac{1}{M_1^2} = \frac{1}{\mu_1^2\ \mu_2^2}\left[(\mu_1^2+\mu_2^2)(\mathbf{L}\cdot\mathbf{S}_+)+(\mu_1^2-\mu_2^2)(\mathbf{L}\cdot\mathbf{S}_-)\right]\ ;$$
$$\frac{1}{M_2^2} = \frac{1}{\mu_1^2\ \mu_2^2}\left[\left((\mu_1+\mu_2)^2+2\mu_1\mu_2\right)(\mathbf{L}\cdot\mathbf{S}_+)+(\mu_1^2-\mu_2^2)(\mathbf{L}\cdot\mathbf{S}_-)\right] \quad (34)$$

As a result of the change of variables, we get the modified SE in the d-dimensional auxiliary space R^d. From the modified SE:

$$H\Phi(q) = \varepsilon(E)\ \Phi(q), \quad (35)$$

$$\varepsilon(E) = 0. \quad (36)$$

Following the OR method, let us represent the canonical variables in terms of the creation (a^+) and annihilation (a) operators in the R^d space;

$$q_j = \frac{a_j + a_j^+}{\sqrt{2\omega}}, \qquad P_j = \sqrt{\frac{\omega}{2}}\cdot\frac{a_j - a_j^+}{i}, \quad j = 1, ..., d, \qquad [a_i, a_j^+] = \delta_{i,j}, \quad (37)$$

where ω is the oscillator frequency which has been unknown yet.

$$H = H_0 + \varepsilon_0(E) + H_I. \quad (38)$$

Here H_0 is the Hamiltonian of free oscillators

$$H_0 = \omega(a_j^+ \ a_j) \tag{39}$$

and ε_0 is the energy of the ground state in the zeroth approximation

$$\begin{aligned}
\varepsilon_0(E) &= \frac{d\omega}{4} - \frac{4\rho^2 E \ \mu}{\omega^{2\rho-1}} \frac{\Gamma(d/2+2\rho-1)}{\Gamma(d/2)} - \frac{16\alpha_s \mu \rho^2}{3\omega^{\rho-1}} \frac{\Gamma(d/2+\rho-1)}{\Gamma(d/2)} + \\
&+ \frac{4\rho^2 \sigma \ \mu}{\omega^{3\rho-1}} \cdot \frac{\Gamma(d/2+3\rho-1)}{\Gamma(d/2)} + \frac{64 \ \alpha_s \ \mu \ \rho^2 \ell}{9 \ \mu_1 \ \mu_2} \cdot \frac{(\vec{S}_1 \vec{S}_2) \ \omega^{\rho+1} \Gamma(d/2-\rho-1)}{\Gamma(d/2)} \\
&- \frac{\rho^2 \sigma \ \mu}{M_1^2 \ \omega^{\rho-1}} \cdot \frac{\Gamma(d/2+\rho-1)}{\Gamma(d/2)} + \frac{4\alpha_s \mu \rho^2 S_{12}}{3 \ \mu_1 \mu_2} \cdot \frac{\omega^{\rho+1} \Gamma(d/2-\rho-1)}{\Gamma(d/2)} + \\
&+ \frac{4\alpha_s \mu \rho^2}{3 \ M_2^2} \cdot \frac{\omega^{\rho+1} \Gamma(d/2-\rho-1)}{\Gamma(d/2)} \ .
\end{aligned} \tag{40}$$

The interaction Hamiltonian H_I can be represented also in the normal form of the creation a^+ and annihilation a operators and it does not contain the quadratic terms of the canonical variables

$$H_I = \int_0^\infty dx \int \left(\frac{d\eta}{\sqrt{\pi}}\right)^d \exp\left\{-\eta^2(1+x)\right\} : e_2^{-i\sqrt{x\omega}(q\eta)} : \tag{41}$$

$$\Bigg[-\frac{4\rho^2\mu}{\omega^{2\rho-1}}\frac{Ex^{-2\rho}}{\Gamma(1-2\rho)} + \frac{4\rho^2\mu}{\omega^{3\rho-1}}\frac{\sigma x^{-3\rho}}{\Gamma(1-3\rho)} - \frac{16\alpha_s\mu\rho^2}{3\omega^{\rho-1}}\frac{x^{-\rho}}{\Gamma(1-\rho)} -$$

$$-\frac{\sigma\rho^2\mu}{M_1^2\omega^{\rho-1}}\frac{x^{-\rho}}{\Gamma(1-\rho)} + \frac{4\rho^2\mu\alpha_s S_{12}}{3\mu_1\mu_2}\frac{\omega^{\rho+1}x^\rho}{\Gamma(1+\rho)} + \frac{4\rho^2\mu\alpha_s}{3M_2^2}\frac{\omega^{\rho+1}x^\rho}{\Gamma(1+\rho)} +$$

$$+\frac{64\ \rho^2\ \mu\ \alpha_s\ \ell\ (\vec{S}_1\vec{S}_2)}{9\ \mu_1\ \mu_2}\cdot\frac{x^\rho}{\Gamma(1+\rho)}\Bigg].$$

Here $:\star:$ is a symbol of normal ordering, and we also use the notation

$$e_2^{-x} = e^{-x} - 1 + x - \frac{1}{2}x^2.$$

$$\frac{\partial \varepsilon_0(E)}{\partial \omega} = 0 \; , \qquad (42)$$
$$\varepsilon(E) = 0 \; .$$

The mass determined by the system of equations

$$\mu_1 - \frac{m_1^2}{\mu_1} + 2\mu_1 \frac{dE}{d\mu_1} = 0 \; ;$$
$$\mu_2 - \frac{m_2^2}{\mu_2} + 2\mu_2 \frac{dE}{d\mu_2} = 0 \; . \qquad (43)$$

Then the mass of mesons consisting of these quarks is defined as:

$$M = \frac{1}{2}\left(\mu_1 + \frac{m_1^2}{\mu_1} + \mu_2 + \frac{m_2^2}{\mu_2}\right) + E. \qquad (44)$$

For the current quark masses use the values $m_c = 1.275 \; GeV$ and $m_b = 4.62 \; GeV$

The value of the running coupling constant of the quark-gluon interactions is determined as follows:

$$\alpha_s = \frac{4\pi}{\beta_0 \ln(\frac{\mu_{12}^2}{\Lambda^2})}; \qquad \beta_0 = 11 - \frac{2}{3} n_f; \qquad \mu_{12} = \frac{2\mu_1\mu_2}{\mu_1 + \mu_2}, \tag{45}$$

where n_f is the flavor quantum number and $\Lambda = 0.168\ GeV$ is the scale of confinement for heavy quarks.

The results of numerical calculations

Table 1. *The mass spectrum of mesons consisting of b and c quarks for the ground state.*

		$\bar{c}c$	$\bar{b}b$	$\bar{b}c$
$S=0$	m_c GeV	1.275	-	1.275
	m_b GeV	-	4.62	4.62
	α_s	0.30366	0.194679	0.248935
	σ GeV^2	0.195	0.153	0.195
	E GeV	0.413530	0.157253	0.363173
	ρ	0.526448	0.651103	0.46495
	ω^ρ GeV	0.652	1.164	0.648335
	μ_c GeV	1.42862	-	1.51306
	μ_b GeV	-	4.73493	4.68082
	M_{our} MeV	2980.05	9400.04	6277.3
	M_{exp} MeV	2981.3 ± 1.1	9390.9 ± 2.8	6277 ± 4
$S=1$	α_s	0.299085	0.194459	0.247683
	E	0.519023	0.216613	0.412532
	ρ	1.03926	1.24871	1.11493
	ω^ρ GeV	1.4311	3.4511	2.0512
	μ_c GeV	1.47617	-	1.53652
	μ_b GeV	-	4.75281	4.71302
	M_{our} MeV	3096.44	9460.3	6330.71
	M_{exp} MeV	3096.916 ± 0.011	9460.3 ± 0.26	-

Table 2. *Mass spectrum of charmonium with the orbital excitations.*

		$J=\ell-1$ S=1	$J=\ell+1$ S=1	$J=\ell$ S=0
$\ell=1$	α_s	0.3013	0.2981	0.2978
	E GeV	0.923955	0.960388	0.945799
	ρ	0.808694	0.613677	0.230383
	ω^ρ GeV	1.14386	0.618518	0.276542
	μ_c GeV	1.45188	1.48592	1.48936
	M_{h_c} MeV	3495.5	3540.33	3526.6
$\ell=2$	α_s	0.2987	0.2944	0.2936
	E GeV	1.2229	1.22267	1.21638
	ρ	0.612313	0.595989	1.39076
	ω^ρ GeV	0.595536	0.560571	5.59744
	μ_c GeV	1.53846	1.5278	1.5371
	M_{h_c} Gev	3.81728	3.8145	3.81107

Table 3. *Mass spectrum of bottomy with the orbital excitations.*

		$J=\ell-1$ S=1	$J=\ell$ S=1	$J=\ell+1$ S=1	$J=\ell$ S=0
$\ell=1$	α_s	0.1944	0.1943	0.1943	0.1946
	E GeV	0.635856	0.6479	0.669121	0.657241
	ρ	0.628027	0.780369	0.312187	0.0915
	ω^ρ GeV	0.985258	1.36504	0.49	0.273495
	μ_c GeV	4.7567	4.76124	4.76007	4.76134
	M_χ MeV	9879.78	9892.09	9913.24	9901.44
$\ell=2$	α_s	0.1939	0.1939	0.1939	0.1939
	E GeV	0.906587	0.911645	0.916257	0.911824
	ρ	0.184697	0.177198	0.169101	0.0634526
	ω^ρ GeV	0.327369	0.321223	0.315	0.2129
	μ_c GeV	4.79492	4.79433	4.7926	4.7967
	M_χ Gev	10.153	10.158	10.1625	10.1583

Table 4. *Mass spectrum of mesons consisting of b and c quarks with radial excitation.*

		$\bar{c}c$	$\bar{b}b$	$\bar{b}c$
$S = 0$	α_s	0.2745	0.19027	0.22974
	E GeV	0.939195	0.704855	0.79797
	ρ	0.504507	0.45040495	0.537141
	ω^{ρ} GeV	0.61426	0.913661	0.732053
	μ_c GeV	1.79312	-	2.01377
	μ_b GeV	-	5.115	4.841
	M_{our} MeV	3638.9	9992.76	6833.53
	M_{exp} MeV	3638.9 ± 1.3	-	-
$S = 1$	α_s	0.27479	0.18989	0.22996
	E	1.01391	0.728737	0.836904
	ρ	0.644051	0.452765	0.571577
	ω^{ρ} GeV	0.629255	0.905397	0.73425
	μ_c GeV	1.7888	-	2.03489
	μ_b GeV	-	5.1501	4.896
	M_{our} MeV	3711.48	10023.3	6881.57
	M_{exp} MeV	3686.109 ± 0.012	10023.26 ± 3.1	-

The width of leptonic and radiative decays

The leptonic decay width of the vector mesons is determined as follows:

$$\Gamma(V \to \ell\,\bar{\ell}) = \frac{16\pi\alpha_{em}^2 e_Q^2}{M_V^2}|\Psi(0)|^2(1 - \frac{16\alpha_s}{3\pi}) \tag{46}$$

where $\alpha_{em} = 1/137$ is the electromagnetic coupling constant; e_Q is the quark charge, M_V is the vector meson mass, and $\Psi(0)$ is the values of WF at the origin.

$$|\Psi(0)|^2 = \frac{1}{4\pi}\frac{\omega^{3\rho}}{\rho\Gamma(3\rho)}. \tag{47}$$

Using $|\Psi_n(0)|^2$ let us determine the leptonic decay constant of the vector and pseudo-scalar mesons:

$$f_p^{NR} = f_v^{NR} = \sqrt{\frac{12}{M_{p,v}}}|\Psi_{p,v}(0)|, \tag{48}$$

where $M_{p,v}$ is the mass of the vector and the pseudo-scalar mesons.

Bethe,Brown and Stehn, Phys.Rev. **77**,370(1950)

Table 5. *The leptonic decay width*

	$\|\Psi_n(0)\|^2$ GeV^3	f GeV	Γ_{our} keV	Γ_{exp} keV
$J/\psi(1S)$	0.1004	0.6	6.135	5.55$\pm$ 0.14
$\Upsilon(1S)$	0.5973	0.8704	1.330	1.340$\pm$ 0.018
$J/\psi(2S)$	0.02584	0.289	1.692	2.35$\pm$0.04
$\Upsilon(2S)$	0.1557	0.432	0.605	0.612$\pm$0.011

Table 6. *Pseudoscalar and vector decay constants of the B_c meson (in Mev).*

constant	our	NR	relativ.	pheno.
f_{B_c}	417	562	433	517
$f^*_{B_C}$	544	562	503	517

Table 7. *The radiative transitions in the $c\bar{c}$ and $b\bar{b}$ systems.*

Transition $i \to f$	k MeV	I_{if} GeV^{-1}	$\Gamma_{our}(i \to f)$ keV	$\Gamma_{exp}(i \to f)$ keV
$\chi_{c_0} \to \gamma + J/\psi$	376.3	2.33	139.312	-
$\chi_{c_1} \to \gamma + J/\psi$	416.06	1.73	310.3	295.84
$\chi_{c_2} \to \gamma + J/\psi$	429.12	2.18	450.5	~ 500
$1^3D_1 \to \gamma + 1^1P_0$	308.22	1.78	267.92	~ 299
$1^3D_1 \to \gamma + 1^1P_1$	266.90	3.274	146.9	~ 99
$1^3D_1 \to \gamma + 1^1P_2$	253.13	2.751	3.54	~ 3.88
$\chi_{c_0} \to \gamma + \Upsilon$	410.57	1.422	16.81	-
$\chi_{c_1} \to \gamma + \Upsilon$	422.366	1.57	66.9	-
$\chi_{c_2} \to \gamma + \Upsilon$	442.592	0.6644	22.97	—
$1^3D_1 \to \gamma + 1^1P_0$	269.544	0.1526	0.33	—
$1^3D_1 \to \gamma + 1^1P_1$	257.56	0.135	0.06	—
$1^3D_1 \to \gamma + 1^1P_2$	236.929	0.4988	0.024	—

$$\Gamma(i \to f + \gamma) = \frac{4\alpha_{em}\ e_Q^2}{3}(2J' + 1)S_{if}^E\ k^3\ |I_{i,f}|^2 \tag{49}$$

$$S_{if}^E = \max(\ell, \ell') \begin{Bmatrix} J & 1 & J' \\ \ell' & s & \ell \end{Bmatrix}^2 \tag{50}$$

$$I_{if} = \int_0^\infty dr\ r^2 \Psi_{n'\ell'}^*(r) r \Psi_{nl}(r) \tag{51}$$

Results and discussion

- The method for the analytical determination of the relativistic corrections to the kinetic part of the interaction Hamiltonian was developed. This approach defines the analytical energy spectrum of the hadrons in the ground and excited state. In this approach the asymptotic behavior of the correlation functions for charged scalar particle in an external gauge field is investigated, and the mass of the bound state are defined. The resulting representation is similar to the Feynman path integral function which was obtained in non-relativistic quantum mechanics for the bound state consisting of a charged scalar particle with constituent mass.

- In our approach, constituent quark masses are not free parameters, are determined for each quarkonium separately and differ from the mass of a free state, i.e., from the current masses. In our approach, the relativistic corrections are taken into account by the constituent mass of the constituent particles. Our results show that the constituent mass of the bound states is greater than the mass of the free states. In this case, the constant α_s of the strong interaction differents from each other for meson. Free parameter in the our approach is the string tension σ and for quarkonium consisting of c quarks is $\sigma = 19.5\ GeV^2$ and for bottomonium consisting of b is $\sigma = 15.3\ GeV^2$.
- The mass and wave functions of the mesons are determined via the eigenvalues of nonrelativistic Hamiltonian in which the kinetic energy term is defined by the constituent mass of the bound state forming and potential energy term is determined by the contributions of every possible type of Feynman diagrams with an exchange of gauge field.
- In the framework of our approach the mass splitting between the singlet and triplet states is determined leptonic and the radiative decay widths of the $\bar{c}c$, $\bar{b}b$ and $\bar{b}c$ systems are calculated.