Final state emission radiative corrections to the process $e^+e^- \to \pi^+\pi^-(\gamma)$. Contribution to muon anomalous magnetic moment

A. I. Ahmadov, E. A. Kuraev, M. K. Volkov, O. Voskresenskaya, E. Zemlyanaya

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Talk Plan

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Motivation

Analytic calculation of the contribution to anomalous magnetic moment of muon from the channels of annihilation of an electron-positron pair to a pair of charged pi-meson with radiative correction connected with the final state, as well as corrections to the lowest order kernel are presented. The result with the point-like pi-meson assumptions is $a_{point} = a^{(1)}_{point} + \Delta a_{point}$,
$a^{(1)}_{point} = 7.0866 \cdot 10^{-9};\ \Delta a_{point} = -2.4 \cdot 10^{-10}$. Taking into account the pion form factor in the frames of the Nambu-Jona-Lasinio (NJL) approach leads to
$a_{NJL} = a^{(1)}_{NJL} + \Delta a_{NJL},\ \ a^{(1)}_{NJL} = 5.48 \cdot 10^{-8};\ \Delta a_{NJL} = -3.43 \cdot 10^{-9}$.

General formalism

It is known (M. Davier, et al., Eur.Phys.J. C 71(2011),1515; C72(2012),1874) that about seventy three per cent of contribution of hadrons to the anomalous magnetic moment of muon $a_\mu = (g-2)/2$ (B.E. Lautrup, A. Peterman and E. de Rafael, Phys. Rep. 3 (1972),4. S. Brodsky, E. de Rafael, Phys.Rev. v 168, p 1620(1968))

$$a_\mu = \frac{1}{4\pi^3}\int_{4m_\pi^3}^{\infty} ds\sigma_B^{e^+e^-\to\pi^+\pi^-}(s)K^{(1)}(\frac{s}{M^2}), \tag{1}$$

with $\sigma_B(s)$ being the total cross section in the Born approximation and

$$K^{(1)}(\frac{s}{M^2}) = \int_0^1 dx\frac{x^2(1-x)}{x^2+\frac{s}{M^2}(1-x)}, \tag{2}$$

with M being the muon mass, arises from taking into account the simplest process $e^+e^-\to\gamma^*\to\pi^+\pi^-$, whereas about sixty per cent of the error arises from the uncertainties associated with the pion pair production from the mechanisms with intermediate states of the lightest vector meson ρ,ω (G.Venanzoni,private communication).

General formalism

It seems "natural" to use the result of experimental measuring of the cross section of the process $e^+e^- \to \pi^+\pi^-$. But, unfortunately, the experimentally measured total cross section (omitting the effects of detection of the final particles) includes the emission of both virtual and real photons by the initial electron and positron (ISE) and the final state emission (FSE), and possibly, the interference of amplitudes of the emission of the initial and final particles. Assuming that the contribution of these interference terms to the total cross section is zero (charge-blind set-up), we remain with the problem of including such enhanced factors as the form factor $F_\pi(s)$ of the charged pion in the time-like region and the delicate procedure of extracting the effects of the initial state emission (both photons and charged particles). Only part of radiative corrections FSE connected with the final $\pi^+\pi^-$ can be included in the frames of one virtual photon polarization operator used above since one implies $\sigma^{e^+e^-\to hadrons}(s) = ((4\pi\alpha)^2/s) Im\Pi(s)$.

General formalism

With the polarization operator defined as a transverse part of the virtual photon self-energy tensor $\Pi_{\mu\nu}(q) = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi(q^2)$ and applying the dispersion relation (B.E. Lautrup, A. Peterman and E. de Rafael, Phys. Rep. 3 (1972)4; S. Brodsky, E. de Rafael, Phys.Rev. 168 (1968)1620; B. Krause, arXiv: hep-ph/9607259)

$$\Pi(q^2) = -\frac{q^2}{\pi} \int\limits_{4m_\pi^2}^{\infty} \frac{ds}{s} \frac{Im\Pi(s)}{q^2 - s}. \quad (3)$$

where m is the pion mass.
Replacing the Green function of the virtual photon in the one-loop vertex function by the one containing the polarization operator

$$-i\frac{g_{\mu\nu}}{q^2} \to -\frac{1}{\pi} \int\limits_{4m_\pi^2}^{\infty} \frac{ds}{s} Im\Pi(s) \frac{-ig_{\mu\nu}}{q^2 - s}, \quad (4)$$

one arrives to the known result of lowest order contribution to a_μ from the hadronic intermediate state

General formalism

$$a_\mu^{(1)} = \frac{\alpha^2}{3\pi^2} \int\limits_{4m_\pi^2}^{\infty} ds \frac{R(s)K^{(1)}(s/M^2)}{s}, \quad R(s) = \frac{3s}{4\pi\alpha^2}\sigma^{e^+e^- \to had.}(s), \tag{5}$$

and the lowest order kernel is

$$K^{(1)}(s/M^2) = \int\limits_0^1 dx \frac{x^2(1-x)}{x^2 + (1-x)(s/M^2)}. \tag{6}$$

The problem consist in removing from the experimentally measured cross section the radiative corrections associated with the initial electron-positron state, including the emission of virtual and real photon emission.

General formalism

This procedure can be the source of errors and uncertainties.
One can include the the pion form factor in the form of the replacing

$$Im\Pi(s) \to F_\pi^2(s) Im\Pi(s). \tag{7}$$

Below we calculate the contribution to a_μ from the processes $e^+e^- \to \pi^+\pi^-$ and $e^+e^- \to \pi^+\pi^-(\gamma)$ assuming pion as a point-like particle, taking into account the emission of virtual and real photons by the charged pions only. To obtain the explicit formulae describing FSE is the motivation of our paper.

General formalism

The differential (center of mass reference frame (cmf) is implied) and total cross sections of the process

$$e_+(p_+) + e_-(p_-) \to \pi_+(q_+) + \pi_-(q_-) \tag{8}$$

in the lowest order of perturbation theory and the assumption of point like pion interaction with the virtual photon have the form

$$\frac{d\sigma}{dc} = \frac{\pi\alpha^2\beta^3}{4s}(1-c^2); \sigma(s) = \frac{\pi\alpha^2\beta^3}{3s}, \tag{9}$$

with $s = (p_+ + p_-)^2 = 4E^2$ is the square of the total energy $c = \cos\theta$, and θ is the angle between the directions of the initial electron and the negative charged pion in cmf. Inserting the explicit value of the total cross section we obtain

$$a_\mu^{(1)} = \frac{\alpha^2\rho^2}{6\pi^2}\int\limits_0^1 dx x^2(1-x)\int\limits_0^1 \frac{d\beta\beta^4}{4(1-x)+x^2\rho^2(1-\beta^2)}, \ \rho = \frac{m_\mu}{m_\pi}. \tag{10}$$

Numeric estimations give $a_\mu^{(1)} = 7.08665 \times 10^{-9}$.

General formalism

In the next order of perturbation theory we must consider the contribution arising from the correction associated with the emission of virtual and real photons (soft and hard) by the final $\pi^+\pi^-$ state. It results in the replacement $\sigma(s) \to \sigma(s)(1+\delta(s))$. Keeping in mind the correction to the kernel we obtain

$$a_\mu = \frac{1}{4\pi^3} \int\limits_{4m^2}^{\infty} ds \sigma(s)(1+\delta(s))[K^{(1)}(s/M^2) + \frac{\alpha}{\pi} K^{(2)}(s/M^2)]. \qquad (11)$$

The quantity $K^{(2)}(s/M^2)$ was computed in paper (R.Barbieri, E. Remiddi, Nucl Phys. B90 (1975)233). It is presented in Appendix. Radiative correction to the final state $\pi^+\pi^-$ will be considered below.

Emission of virtual photons

To start with virtual correction we find first the vertex function for scattering of the charged pion in the external field. Than we write it down in the annihilation channel and use to calculate the relevant virtual correction to the cross section. The vertex function of the process $\pi_-(p_1) + \gamma^*(q) \to \pi_-(p_2)$ has the form

$$\Gamma_\mu = \frac{\alpha}{4\pi}\int\frac{N_\mu dk}{(k)(1)(2)}, (k) = k^2-\lambda^2; (1) = k^2-2p_1k; (2) = k^2-2p_2k,$$
$$dk = \frac{d^4k}{i\pi^2}, N_\mu = (p_1+p_2-2k)_\mu(2p_1-k)_\lambda(2p_2-k)^\lambda. \quad (12)$$

Writing N_μ as $N_\mu = (p_1+p_2-2k)_\mu[4p_1p_2+(1)+(2)-(k)]$ and performing the loop momenta integration, we obtain for the un-renormalized vertex function

$$\Gamma_\mu^{un} = \frac{\alpha}{4\pi}(p_1+p_2)_\mu F^{un}(q^2),$$
$$F^{un}(q^2) = (2m^2-q^2)\int_1^1\frac{dx}{q_x^2}[\ln\frac{m^2}{\lambda^2}+\ln\frac{q_x^2}{m^2}-1]+3+\ln\frac{\Lambda^2}{m2},$$
$$p_1^2 = p_2^2 = m^2, q_x = p_1x+p_2(1-x), q_x^2 = m^2-x(1-x)q^2, q = p_2-p_1. \quad (13)$$

Emission of virtual photons

Here λ, λ are the ultraviolet cut-off parameter and the fictitious photon mass. The regularization consist in the construction $F(q^2) = F^{un}(q^2) - F^{un}(0)$. So we have

$$\Gamma_\mu = \frac{\alpha}{4\pi}(p_1 + p_2)_\mu F(q^2),$$

$$F(q^2) = (2m^2 - q^2)\int\limits_1^1 \frac{dx}{q_x^2}[\ln\frac{m^2}{\lambda^2} + \ln\frac{q_x^2}{m^2} - 2] + 4[1 - \ln\frac{m}{\lambda}]. \quad (14)$$

Introducing the new variable $(1-\theta)^2, \theta = -q^2/m^2$ and using

$$\int\limits_1^1 \frac{dx}{q_x^2} = \frac{2\theta}{m^2(1-\theta^2)}\ln\frac{1}{\theta},$$

$$\int\limits_1^1 \frac{dx}{q_x^2}\ln\frac{q_x^2}{m^2} = \frac{2\theta}{m^2(1-\theta^2)}[\frac{1}{2}\ln^2\theta - 2\ln\theta\ln(1+\theta) - 2Li_2(-\theta) - \frac{\pi^2}{6}], \quad (15)$$

Emission of virtual photons

we obtain

$$\Gamma_\mu(p_1,p_2) = \frac{\alpha}{\pi}(p_1+p_2)_\mu\left[(\ln\frac{m}{\lambda}-1)(\frac{1+\theta^2}{1-\theta^2}-ln\frac{1}{\theta}-1)+ \right.$$
$$\left. + \frac{1+\theta^2}{4(1-\theta^2)}[\ln^2\theta-4\ln\theta\ln(1+\theta)-4Li_2(-\theta)-\frac{\pi^2}{3}\right]. \quad (16)$$

For the crossing channel $\gamma^*(q,\mu)\to\pi_-(q_-)+\pi_+(q_+)$ we use the substitutions (R. Barbieri, J. A. Mignaco, E. Remiddi, IL Nuovo Cimento, 11A (1972)824)

$$p_2\to q_-, p_1\to -q_+, \theta\to -x+i\epsilon, 0<\epsilon<<1,$$
$$x=\frac{1-\beta}{1+\beta}, \beta=\sqrt{1-(4m^2/s)}, s=(q_++q_-)^2=4E^2. \quad (17)$$

Emission of virtual photons

This quantity acquire the imaginary part for $s > 4m^2$:

$$\Gamma_\mu = \frac{\alpha}{\pi}(q_- - q_+)_\mu F(x),$$

$$F(x) = (\ln\frac{\lambda}{m} + 1)(\frac{1+x^2}{1-x^2}\ln x + 1 + i\pi) + $$

$$+\frac{1+x^2}{4(1-x^2)}\Big[\ln^2 x - \frac{4}{3}\pi^2 -$$

$$4Li_2(x) - 4\ln x \ln(1-x) + i\pi(2\ln x - 4\ln(1-x))\Big].$$

Emission of virtual photons

Writing $ReF(x)$ as

$$ReF(x) = (-1 + \frac{1+\beta^2}{2\beta}L)\ln\frac{\lambda}{m} + F_V,$$
$$L = \ln\frac{1+\beta}{1-\beta}, \qquad (18)$$

we write down the relevant contribution to the total cross section as

$$\Delta_V\sigma(s) = \frac{2\alpha^3\beta^3}{3s}[(-1 + \frac{1+\beta^2}{2\beta}L)\ln\frac{\lambda}{m} + F_V(\beta)], \qquad (19)$$

with

$$F_V(\beta) = -1 + \frac{1+\beta^2}{2\beta}[L - \frac{1}{4}L^2 + \frac{1}{3}\pi^2 + Li_2(\frac{1-\beta}{1+\beta}) - L\ln\frac{2\beta}{1+\beta}]. \qquad (20)$$

Emission of soft photons

Consider now the contribution from the emission of the soft real photon channel. It have the form

$$\Delta_S \sigma = -\frac{\alpha}{4\pi^2}\sigma_B(s)\int' \frac{d^3k}{\omega}\left(\frac{q_-}{q_-k} - \frac{q_+}{q_+k}\right)^2, \qquad (21)$$

where the sign prime means $\omega = \sqrt{\vec{k}^2 + \lambda^2} < \Delta E$ and it is implied $\Delta E << E$. Using the relations

$$\frac{\alpha}{4\pi^2}\int' \frac{d^3k}{\omega}\frac{m^2}{(q_-k)^2} = \frac{\alpha}{\pi}[\ln\frac{2\Delta E}{\lambda} - \frac{1}{2\beta}L], \qquad L = \ln\frac{1+\beta}{1-\beta}, \qquad (22)$$

$$\frac{\alpha}{4\pi^2}\int' \frac{d^3k}{\omega}\frac{(q_+q_-)}{(q_+k)(q_-k)} = \frac{\alpha}{\pi}\frac{1+\beta^2}{2\beta}[\ln\frac{2\Delta E}{\lambda}L + J(\beta)], \qquad (23)$$

with

$$J(\beta) = Li_2(-\beta) - Li_2(\beta) + Li_2(\frac{1+\beta}{2}) - \frac{1}{4}L^2 + \frac{1}{2}\ln^2(\frac{1+\beta}{2}) - \frac{1}{12}\pi^2, \quad (24)$$

we obtain the contribution to the cross section

$$\Delta_S\sigma(s) = \frac{2\alpha^3\beta^3}{3s}[(-1 + \frac{1+\beta^2}{2\beta}L)\ln\frac{2\Delta E}{\lambda} + F_S(\beta)],$$
$$F_S(\beta) = -\frac{1}{2\beta L} + \frac{1+\beta^2}{2\beta}J(\beta). \quad (25)$$

Here the prime means $\omega < \Delta E$.

Hard real photon emission

Consider at least the contribution from the hard photon emission channel $\omega > \Delta E$. The matrix element of this process has the form

$$M = \frac{(4\pi\alpha)^{3/2}}{s} J^\mu T_{\mu\nu} e(k)^\nu, \quad (26)$$

with $e(k)$ is the polarization vector of the photon, $J^\mu = \bar{v}(p_+)\gamma^\mu u(p_-)$ is the current associated with the leptons, and

$$T_{\mu\nu} = \frac{1}{2(q_- k)}(2q_- + k)_\nu (Q + k)_\mu + \frac{1}{2(q_+ k)}(-2q_+ - k)_\nu (Q - k)_\mu - 2g_{\mu\nu}. \quad (27)$$

It can be checked that this expression obeys the gauge invariance conditions $T_{\mu\nu} q^\mu = T_{\mu\nu} k^\nu = 0$.

Hard real photon emission

We use as well the relation (Akhiezer A. I., Berestetskij V. B., Quantum Electrodynamics, Moscow, Science, 1981; J. D. Bjorken, S. Drell "Relativistic Quantum Fields", McGraw-Hill (1965))

$$\sum_{spin}\int |M|^2 d\Gamma_3 = -\frac{1}{3} Tr \hat{p}_+ \gamma^\mu \hat{p}_- \gamma^\nu (g_{\mu\nu} - q_\mu q_\nu / q^2) \int I d\Gamma_3,$$
$$I = T_{\rho\sigma} T^{\rho\sigma}, \quad (28)$$

with $d\Gamma_3$ being the element of the phase space of the final particles

$$d\Gamma_3 = \frac{d^3 q_-}{2E_-} \frac{d^3 q_+}{2E_+} \frac{d^3 k}{2\omega} \frac{1}{(2\pi)^5} \delta^4(q - q_- - q_+ - k). \quad (29)$$

It can be written as

$$\frac{d^3 q_-}{2E_-} \frac{d^3 q_+}{2E_+} \frac{d^3 k}{2\omega} \frac{1}{2E_+(2\pi)^5} \delta(q_0 - E_- - E_+ - \omega), \quad (30)$$

with $E_+ = \sqrt{(\vec{q}_- + \vec{k})^2 + m^2} = \sqrt{\omega^2 + E_-^2 + 2\vec{q}_- \vec{k}}$.

Hard real photon emission

Performing the integration on $\cos\theta$, where θ is the angle in cmf between 3-momenta of pion and photon, we obtain

$$d\Gamma_3 = \frac{\pi^2 s}{4(2\pi)^5} d\nu d\nu_- d\nu_+ \delta(\nu + \nu_- + \nu_+), \nu = \frac{2\omega}{q_0}, \nu_- = \frac{2E_-}{q_0}, \nu = \frac{2E_+}{q_0}, q_0 = 2E. \qquad (31)$$

In terms of energy fractions we obtain

$$I = 8 + \frac{2(1-\nu)}{1-\nu_+} + \frac{2(1-\nu)}{1-\nu_-} - \beta^2(1-\beta^2)[\frac{1}{(1-\nu_+)^2} + \frac{1}{(1-\nu_-)^2}] + + \frac{2}{\nu}(\nu - \beta^2)(\nu - 1 - \beta^2)[\frac{1}{1-\nu_+} + \frac{1}{1-\nu_-}]. \qquad (32)$$

The domain of integration D is

$$\frac{\Delta E}{E} < \nu < \beta^2; \nu + \nu_- + \nu_+ = 2; (1-\nu_-)(1-\nu_+)(1-\nu) > \frac{m^2\nu^2}{s}, \frac{\nu}{2}(1-R) < 1 - \nu_\pm < \frac{\nu}{2}(1+R), R = \sqrt{\frac{\beta^2 - \nu}{1-\nu}}. \qquad (33)$$

Hard real photon emission

Performing the integration over $\nu_\pm$ we obtain

$$\int I d\nu_- d\nu_+ \delta(2-\nu-\nu_- -\nu_+) =$$
$$4[2R(\nu - \frac{\beta^2(1-\nu)}{\nu}) + (\frac{\beta^2(1+\beta^2)}{\nu} - 2\beta^2)\ln\frac{1+R}{1-R}]. \quad (34)$$

Performing the further integration we use the substitution $t = R, 0 < t < t_m, t_m^2 = \beta^2 - (\Delta E/E)(1-\beta^2)$. The corresponding contribution to the cross section is

$$\Delta_H\sigma(s) = \frac{2\alpha^3\beta^3}{3s}[(\frac{1+\beta^2}{2\beta}L - 1)\ln\frac{E}{\Delta E} + F_H(\beta)], \quad (35)$$

with

$$F_H(\beta) = -\frac{1+\beta^2}{\beta}G(\beta) + \ln\frac{1-\beta^2}{4\beta^2} - \frac{1}{8\beta^3}(3+\beta^2)(1-\beta^2)L + \frac{3+7\beta^2}{4\beta^2}, \quad (36)$$

Hard real photon emission

and

$$G(\beta) = \int_0^\beta \frac{dt}{1-t^2} \ln \frac{1-t^2}{\beta^2-t^2} =$$

$$Li_2(\frac{1-\beta}{2}) - Li_2(\frac{1+\beta}{2}) + Li_2(1+\beta) - Li_2(1-\beta). \quad (37)$$

The total contribution does not depend on " photon mass" λ as well as on the auxiliary parameter ΔE:

$$\Delta\sigma = \frac{2\alpha^3\beta^3}{3s}[\frac{1}{2}(\frac{1+\beta^2}{2\beta}L - 1)\ln\frac{1}{1-\beta^2} + F_V + F_S + F_H]. \quad (38)$$

After some algebra one obtain

$$\Delta\sigma^{e\bar{e}\to\pi\bar{\pi}}(s) = 2\frac{\alpha}{\pi}\sigma_B(s)\Delta(\beta), \sigma_B(s) = \frac{\pi\alpha^2\beta^3}{3s}, \beta = \sqrt{1-\frac{4m_\pi^2}{s}}, \quad (39)$$

and

$$\Delta(\beta) = -\frac{3}{2}\ln\frac{4}{1-\beta^2} - 2\ln\beta + \frac{1+\beta^2}{2\beta}[-\frac{1}{12}\pi^2 + \frac{5}{4}L + \frac{3}{2\beta}[1-\frac{1}{2\beta}L] - L\ln\beta + Li_2(\frac{1-\beta}{1+\beta}) + 3Li_2(-\beta) - Li_2(\beta) + 3Li_2(\frac{1+\beta}{2}) - 2Li_2(\frac{1-\beta}{2}) + 2\ln\beta\ln(1+\beta) - 2Li_2(1-\beta) + \frac{1}{2}\ln^2(\frac{1+\beta}{2})]. \quad (40)$$

Results for FSE in point-like pion approximation

The total contribution to a_μ can be obtained from the general formulae (see(9)) by replacement

$$\beta^4 \to \beta^4[1 + \frac{2\alpha}{\pi}\Delta(\beta)] = \beta^4[1 + \delta(s)] \tag{41}$$

Numeric estimation leads to

$$\Delta^\pi a_\mu = -6.923 \times 10^{-11}. \tag{42}$$

A total set of the lowest order RC to $a_\mu^{(1)}$ takes into account as well the correction to the kernel

$$\Delta^{ker} a_\mu = \frac{\alpha^3}{3\pi^3}\int\limits_0^1 \frac{\beta_\pi^4 d\beta_\pi}{1-\beta_\pi^2} K^{(2)}(b), \quad b = \frac{s}{M^2} = \frac{4}{\rho^2(1-\beta_\pi^2)}. \tag{43}$$

Results for FSE in point-like pion approximation

Explicit form of the kernel $K^{(2)}(b)$ as well as its expansion in powers b^{-1} are presented in Appendix. Using the explicit form of $K^{(2)}$ can be successfully applied to the region $1-\beta_\pi^2 \sim 1$ and is not convenient for the region $1-\beta_\pi^2 << 1$. In this region we apply its expansion in powers M^2/s, which was obtained in (B. Krause, Phys. Lett. B **390**, 392 (1997); arXiv: hep-ph/9607259). For this aim we choice an auxiliary parameter $\beta_0 \sim 1$

$$\Delta^{ker} a_\mu = \frac{1}{4\pi^3}\left[\int\limits_0^{\beta_0} \frac{\beta_\pi^4 d\beta_\pi}{1-\beta_\pi^2} K^{(2)}(b)_{BR} + \int\limits_{\beta_0}^{1} \frac{\beta_\pi^4 d\beta_\pi}{1-\beta_\pi^2} K^{(2)}(b)_{Kr}\right]. \quad (44)$$

The result does not depend on β_0 and is

$$\Delta^{ker} a_\mu = -1.73 \cdot 10^{-10}. \quad (45)$$

The total contribution of the correction to a_μ from RC to both the final $\pi^+\pi^-$ and the kernel is

$$\Delta A_\mu = \Delta^\pi + \Delta^{ker} = -2.4 \cdot 10^{-10}. \quad (46)$$

Insertion of pion form factor. Discussion

The result, obtained in the point-like approximation about an order of magnitude lower than one measured in experiment (M. Davier, et al., Eur.Phys.J. C 71(2011),1515; C72(2012),1874) $a_\mu \approx 6.974 \cdot 10^{-8}$. The conversion of a virtual photon to the $\pi^+\pi^-(\gamma)$ state in the time-like region is realized through the intermediate state with vector mesons $\rho(775), \omega(782), \phi(1020)$ with the following decay to the two pion state. The main contribution arise from $\rho(775)$ meson state. Keeping in mind the resonance nature of this transition it can be taken into account by the replacement in (1)

$$\sigma_B(s) \to \sigma_B(s) Z, \tag{47}$$

The contribution of $\omega(782), R_\omega$ arises due to a rather small $\rho - \omega$ mixing. It has two sources- one is connected with the quark u, d mass difference $m_d - m_u = 3.75 MeV, m_u = 280 MeV$, and the other is connected with the transition $\omega \to \gamma \to \rho$ (M.K. Volkov, Sov. J, Part. Nucl.17 (1986)186 [Fiz. Elem.Chast. Atom. Yadra 17 (1986)433].)

$$B_\omega = R \frac{s}{m_\omega^2 - s - i\sqrt{s}\Gamma_\omega} \frac{s}{m_\rho^2 - s - i\sqrt{s}\Gamma_\rho}. \tag{48}$$

Insertion of pion form factor. Discussion

Adding the contribution of the photon and photon-rho meson conversion (M.K. Volkov and D. Kostunin, Phys. Rev. C 86 (2012)025202)

$$B_{\gamma\rho} = 1 + \frac{s}{m_\rho^2 - s - i\sqrt{s}\Gamma_\rho} = \frac{m_\rho^2 - i\sqrt{s}\Gamma_\rho}{m_\rho^2 - s - i\sqrt{s}\Gamma_\rho}, \quad (49)$$

we have

$$Z(x) = |B_{\gamma\rho} + B_\omega|^2 = \left[\frac{\mu_\rho^2 - \mu_\rho + \gamma_\rho^2}{(\mu_\rho^2-1)^2 + \gamma_\rho^2} + R\frac{(\mu_\omega^2-1)^2 - \gamma_\rho\gamma_\omega}{[(\mu_\omega^2-1)^2+\gamma_\omega^2][(\mu_\omega^2-1)^2+\gamma_\rho^2]}\right]^2 + \left[\frac{\gamma_\rho}{(\mu_\rho^2-1)^2+\gamma_\rho^2} + R\frac{(\mu_\omega^2-1)(\gamma_\rho+\gamma_\omega)}{[(\mu_\omega^2-1)^2+\gamma_\omega^2][(\mu_\omega^2-1)^2+\gamma_\rho^2]}\right]^2, \quad (50)$$

Insertion of pion form factor. Discussion

with

$$R = \frac{1}{3g_\rho}[\frac{g_\rho^3}{16\pi^2}\ln\frac{m_d^2}{m_u^2} - \frac{4\pi\alpha}{g_\rho}] \approx 2.27\cdot 10^{-4},$$
$$x = \frac{s}{m_\pi^2},\ \ \mu_\rho^2 = \frac{m_\rho^2}{s} = \frac{30.86}{x};\ \ \mu_\omega^2 = \frac{m_\omega^2}{s} = \frac{31.43}{x},$$
$$\gamma_\rho^2 = \frac{\Gamma_\rho^2}{s} = \frac{1.07}{x};\ \ \ \gamma_\omega^2 = \frac{\Gamma_\omega^2}{s} = \frac{6.08\cdot 10^{-2}}{x}. \tag{51}$$

Our final results are

$$a^{(1)} = \frac{\alpha^2}{12\pi^2}\int\limits_4^\infty \frac{dx}{x^2}Z(x)(1-\frac{4}{x})^{3/2}K(x),$$

$$\Delta a = \frac{\alpha^3}{6\pi^3}\int\limits_4^\infty \frac{dx}{x^2}Z(x)(1-\frac{4}{x})^{3/2}[\Delta(\beta)K(x)+\Delta K(x)]. \tag{52}$$

Insertion of pion form factor. Discussion

The expression for $\Delta K(x)$ is presented in Appendix

$$K(x) = xK^{(1)}(x) = \int\limits_0^1 \frac{y^2(1-y)xdy}{y^2 + x(1-y)\rho^2}. \tag{53}$$

The explicit expression for $\Delta(\beta)$ is given above. The result of numerical calculations is

$$a^{(1)}_{NJL} \approx 5.48 \cdot 10^{-8}; \quad \Delta a \approx -3.43 \cdot 10^{-9}. \tag{54}$$

The contribution of the term of an order of $1/(x\rho^2)^4$ is expected to be on the level of several per cent, which determine the accuracy of our calculations. The relative wight of $\pi^+\pi^-$ hadron state is

$$\frac{a^{(1)}_{NJL}}{a_{exp}} = 0.78. \tag{55}$$

Insertion of pion form factor. Discussion

Here we do not take into account the contribution of double vacuum polarization – with the two-hadronic insertion and the QED one with the electron-positron intermediate state. Both of them were considered in the recent paper (D. Greynat and E. de Rafael, JHEP 07 (2012) 020).
In (A. Hoefer, J. Glusa, F. Jegerlehner, Eur. Phys. J. C24 (2002)59), an attempt to take into account the initial state emission of an additional pair of charged particles from the experimental data was made.
In (D. Greynat and E. de Rafael, JHEP 07 (2012) 020), a similar calculation was performed by using the duality approximation (constituent quarks and gluons – hadrons) and applying the result of (G. Kallen and A. Sabry, Danske Videnskab, 29, N 17 (1955)17) for the final state emission of a fermion-anti-fermion pair.

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The explicit form of the kernel $K^{(2)}(b)$ was obtained in paper of R. Barbieri and E. Remiddi (R.Barbieri , E. Remiddi, Nucl Phys. B 90 (1975)233). The contribution of 14 Feynman diagram was taken into account. It have a form

$$K^{(2)}(b)_{BR} = -\frac{139}{144} + \frac{115}{72}b + \left(\frac{19}{12} - \frac{7}{36}b + \frac{23}{144}b^2 + \frac{1}{b-4}\right)\log(b) + \\ (-\frac{4}{3} + \frac{127}{36}b - \frac{115}{72}b^2 + \frac{23}{144}b^3)\frac{\log y}{\sqrt{b(b-4)}} + \left(\frac{9}{4} + \frac{5}{24}b - \frac{1}{2}b^2 - \frac{2}{b}\right)\xi(2) + \\ \frac{5}{96}b^2\log^2 b + (-\frac{1}{2}b + \frac{17}{24}b^2 - \frac{7}{48}b^3)\frac{\log y}{\sqrt{b(b-4)}}\log b + \\ \left(\frac{19}{24} + \frac{53}{48}b - \frac{29}{96}b^2 - \frac{1}{3b} + \frac{2}{b-4}\right)\log^2 y + \\ (-2b + \frac{17}{6}b^2 - \frac{7}{12}b^3)\frac{1}{\sqrt{b(b-4)}}D_p(b) + \left(\frac{13}{3} - \frac{7}{6}b + \frac{1}{4}b^2 - \right. \\ \left. \frac{1}{6}b^3 - \frac{4}{b-4}\right)\frac{D_m(b)}{\sqrt{b(b-4)}} + (\frac{1}{2} - \frac{7}{6}b + \frac{1}{2}b^2)T(b), \quad (56)$$

where

$$y = \frac{\sqrt{b} - \sqrt{b-4}}{\sqrt{b} + \sqrt{b-4}},$$

$$D_p(b) = Li_2(y) + \log(y)\log(1-y) - \frac{1}{4}\log^2 y - \xi(2),$$

$$D_m(b) = Li_2(-y) + \frac{1}{4}\log^2 y + \frac{1}{2}\xi(2),$$

$$T(b) = -6Li_3(y) - 3Li_3(-y) + \log^2 y \log(1-y) +$$

$$\frac{1}{2}(\log^2 y + 6\xi(2))\log(1+y) + 2\log y(Li_2(-y) + 2Li_2(y)). \quad (57)$$

The function $Li_2(y), Li_3(y)$ are the dilogarithm and trilogarithm defined through

$$Li_2(y) = -\int\limits_0^y \frac{dt}{t}\ln(1-t) = -\int\limits_0^1 \frac{dt}{t}\ln(1-ty),$$

$$Li_2(-y) = -\int\limits_0^y \frac{dt}{t}\ln(1+t) = -\int\limits_0^1 \frac{dt}{t}\ln(1+ty);$$

$$Li_3(y) = \int\limits_0^y \frac{dt}{t}[\ln t - \ln y]\ln(1-t) = \int\limits_0^1 \frac{dt}{t}\ln t\ln(1-ty),$$

$$Li_3(-y) = \int\limits_0^y \frac{dt}{t}[\ln t - \ln y]\ln(1+t) = \int\limits_0^1 \frac{dt}{t}\ln t\ln(1+ty). \qquad (58)$$

Appendix

In paper of B.Krause (B. Krause, Phys. Lett. B **390**, 392 (1997); arXiv: hep-ph/9607259) the expansion on $b = s/m_\mu^2$ was obtained

$$K^{(2)}(b)_{Kr} = \frac{1}{b}\Big\{[\frac{223}{54} - 2\xi_2 - \frac{23}{36}L] + \frac{1}{b}[\frac{8785}{1152} - \frac{37}{8}\xi_2 - \frac{367}{216}L + \frac{19}{144}L^2] + \frac{1}{b^2}[\frac{13072841}{432000} - \frac{883}{40}\xi_2 - \frac{10079}{3600}L + \frac{141}{80}L^2] + \frac{1}{b^3}[\frac{2034703}{16000} - \frac{3903}{40}\xi_2 - \frac{6517}{1800}L + \frac{961}{80}L^2]\Big\}. \quad (59)$$

with

$$\xi_2 = \frac{\pi^2}{6}, L = \ln b. \quad (60)$$

In the text above we use

$$\Delta K(x) = \frac{1}{\rho^2}[c_0 + d_0 L + \frac{1}{x\rho^2}[c_1 + d_1 L + d_2 L^2] + \frac{1}{(x\rho^2)^2}[c_2 + d_2 L + e_2 L^2] + \frac{1}{(x\rho^2)^3}[c_3 + c_3 L + d_3 L^2]], \; Ł = \ln(x\rho^2) \quad (61)$$

Appendix

The numeric values are

$$c_0 = -0.843;\ \ d_0 = -0.639;\ \ e_0 = 0;$$
$$c_1 = 0.027;\ \ d_1 = -2.8;\ \ e_1 = 0.132;$$
$$c_2 = -6.01;\ \ d_2 = -2.8;\ \ e_2 = 1.76;$$
$$c_3 = -33.1;\ \ d_3 = -3.62;\ \ e_3 = 12.01. \tag{62}$$

In Figure 1 the β_π dependence of the exact integrand $F_{BR}(\beta_\pi) = \beta_\pi^4 K_{BR}^{(2)}/(1-\beta_\pi^2)$ and its expansion in powers of M^2/s $F_{Kr}(\beta_\pi) = \beta_\pi^4 K_{Kr}^{(2)}/(1-\beta_\pi^2)$ are presented. One see the large compensations in F_{BR} take place for $\beta \to 1$.

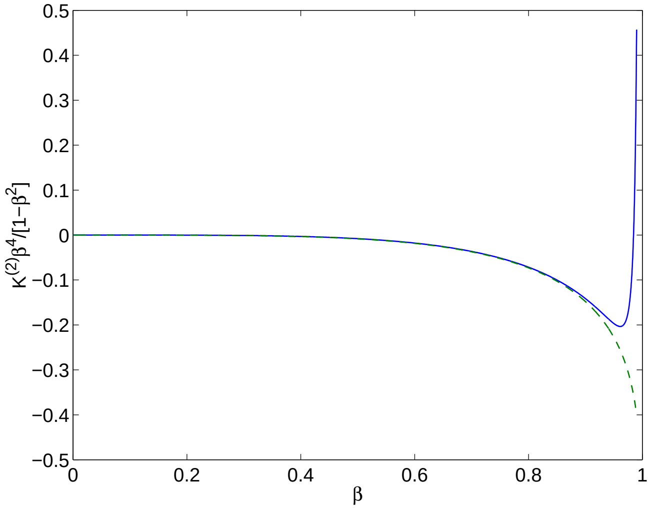

Figure: Dependence of $K^{(2)}(\beta)$: solid line - exact formulae, dashed line - power M^2/s expansion.