"FAPT": A Mathematica package for QCD calculations

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In memory of Alexander Bakulev

Work on "FAPT" package in time of AB visit to Gomel (October, 2011)

This talk based on recent publication

A.P. Bakulev and V.L. Khandramai *Comp. Phys. Comm.* **184**, Iss. 1 (2013) 183-193.

Plan of talk:

1. Theoretical framework: *from standard PT to Analytic Perturbative Theory and its generalization – Fractional APT*;
2. APT/(F)APT Applications:
   - DIS SR Analysis;
   - Renorm-group Q^2-evolution;
   - Adler D-function;
3. Package "FAPT": *description of procedures and examples of usage.*

Motivation

Analytic Perturbative Theory, **APT**, [Shirkov, Solovtsov (1996,1997)]

Fractional Analytic Perturbative Theory, **(F)APT**, [Bakulev, Mikhailov, Stefanis (2005-2010)], [Bakulev, Karanikas, Stefanis (2007)]:

Analytic PT:

- Closed theoretical scheme without Landau singularities and additional parameters;
- RG-invariance, Q^2-analyticity;
- Power PT set $\{\bar{\alpha}_{\rm s}^k(Q^2)\}$ $\Rightarrow$ a non-power APT expansion set $\{\mathcal{A}_k(Q^2), \mathfrak{A}_k(s)\}$ with all $\mathcal{A}_k(Q^2)$, $\mathfrak{A}_k(s)$ regular in the IR region.

$$\sum d_k \alpha_{\rm s}^k \to \sum d_k \mathcal{A}_k$$

Introduction

The main goal is to simplify calculations in the framework of APT&(F)APT.

For this purpose we collect all relevant formulas which are necessary for the running of $\bar{\mathcal{A}}_\nu[L], L = \ln(Q^2/\Lambda^2)$ and $\bar{\mathfrak{A}}_\nu[L_s], L_s = \ln(s/\Lambda^2)$ in the framework of APT and (F)APT.

Note,

- We provide here easy-to-use `Mathematica` system procedures collected in the package **"FAPT"** organized as

  package **"RunDec"** [Chetyrkin, Kühn, Steinhauser (2000)]

- This task has been partially realized for both APT and its massive generalization [Nesterenko, Papavassiliou (2005)] as the `Maple` package **"QCDMAPT"** and as the `Fortran` package **"QCDMAPT_F"** [Nesterenko, Simolo (2010)].

Theoretical Framework

Running coupling

Running coupling $\alpha_s(\mu^2) = (4\pi/b_0)\, a_s[L]$ with $L = \ln(\mu^2/\Lambda^2)$ obtained from RG equation

$$\frac{d\, a_s[L]}{d\, L} = -a_s^2 - c_1\, a_s^3 - c_2\, a_s^4 - c_1\, a_s^3 - \ldots, \quad c_k(n_f) \equiv \frac{b_k(n_f)}{b_0(n_f)^{k+1}}\,,$$

Exact solutions of RGE known only at LO and NLO

$$a_{(1)}[L] = \frac{1}{L} \quad \textbf{(LO)}$$

$$a_{(2)}[L; n_f] = \frac{-c_1^{-1}(n_f)}{1 + W_{-1}\left(z_W[L]\right)} \quad \text{with} \quad z_W[L] = -c_1^{-1}(n_f)\, e^{-1-L/c_1(n_f)} \quad \textbf{(NLO)}$$

The higher-loop solutions $a_{(\ell)}[L; n_f]$ can be expanded in powers of the two-loop one, $a_{(2)}[L; n_f]$, as has been suggested in [Kourashev, Magradze, (1999-2003)]:

$$a_{(\ell)}[L; n_f] = \sum_{n\geq 1} C_n^{(\ell)}\, \left(a_{(2)}[L; n_f]\right)^n .$$

Heavy quark mass thresholds

$$\begin{aligned}\alpha_s^{\text{glob};(\ell)}(Q^2,\Lambda_3) &= \alpha_s^{(\ell)}\left[L(Q^2);3\right]\theta\left(Q^2<M_4^2\right)\\ &+ \alpha_s^{(\ell)}\left[L(Q^2)+\lambda_4^{(\ell)}(\Lambda_3);4\right]\theta\left(M_4^2\leq Q^2<M_5^2\right)\\ &+ \alpha_s^{(\ell)}\left[L(Q^2)+\lambda_5^{(\ell)}(\Lambda_3);5\right]\theta\left(M_5^2\leq Q^2<M_6^2\right)\\ &+ \alpha_s^{(\ell)}\left[L(Q^2)+\lambda_6^{(\ell)}(\Lambda_3);6\right]\theta\left(M_6^2\leq Q^2\right)\end{aligned}$$

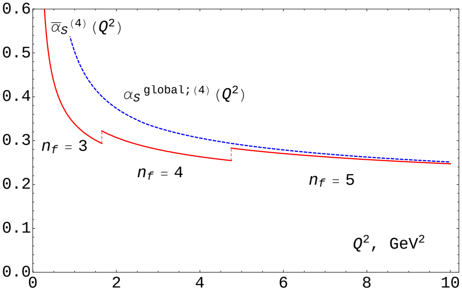


Figure: Graphical comparison: Fixed-n_f $\overline{\alpha}_s^{(4)}[Q^2,n_f]$ – Global $\overline{\alpha}_s^{glob;(4)}[Q^2]$

Problems

- **Coupling singularities**
  - LO solution generates Landau pole singularity: $a_s[L] = 1/L$
  - NLO solution generates square-root singularity: $a_s[L] \sim 1/\sqrt{L + c_1 \ln c_1}$
- **PT power-series expansion of $D(Q^2, \mu^2 = Q^2) \equiv D$ in the running coupling:**
  $D[L] = 1 + d_1 a_s[L] + d_2 a_s^2[L] + d_3 a_s^3[L] + d_4 a_s^4[L] + \ldots,$
  are not everywhere well defined
- **RG evolution:** $B(Q^2) = \left[Z(Q^2)/Z(\mu^2)\right]\ B(\mu^2)$ reduces in 1-loop approximation to $Z \sim a^{\nu}[L]\Big|_{\nu=\nu_0\equiv\gamma_0/(2b_0)}$, ν–fractional

Basics of APT

The analytic images of the strong coupling powers:

$$\bar{\mathcal{A}}_n^{(\ell)}[L;n_f]=\int_0^\infty \frac{\bar{\rho}_\nu^{(\ell)}(\sigma;n_f)}{\sigma+Q^2}\,d\sigma\,,\quad \bar{\mathfrak{A}}_n^{(\ell)}[L_s;n_f]=\int_s^\infty \frac{\bar{\rho}_n^{(\ell)}(\sigma;n_f)}{\sigma}\,d\sigma$$

define through spectral density

$$\bar{\rho}_n^{(\ell)}[L;n_f] = \frac{1}{\pi}\,\mathbf{Im}\left(\alpha_s^{(\ell)}[L-i\pi;n_f]\right)^n = \frac{\sin[n\,\varphi_{(\ell)}[L;n_f]]}{\pi\,(\beta_f\,R_{(\ell)}[L;n_f])^n}\,.$$

One-loop:

$$\bar{\rho}_1^{(1)}(\sigma) = \frac{4}{b_0}\,\mathbf{Im}\,\frac{1}{L_\sigma - i\pi} = \frac{4\pi}{b_0}\,\frac{1}{L_\sigma^2+\pi^2}\,.$$

$\mathcal{A}_1^{(1)}$ [Shirkov, Solovtsov (1996, 1997)] и $\mathfrak{A}_1^{(1)}$ [Jones, Solovtsov (1995); Jones, Solovtsov, Solovtsova (1995); Milton, Solovtsov (1996)]

$$\bar{\mathcal{A}}_1^{(1)}[L] = \frac{4\pi}{b_0}\left(\frac{1}{L}-\frac{1}{e^L-1}\right),\quad L = \ln\left(Q^2/\Lambda^2\right);$$

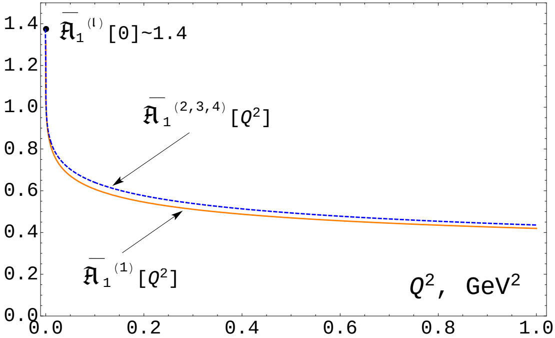
$$\bar{\mathfrak{A}}_1^{(1)}[L_s] = \frac{4}{b_0}\arccos\left(\frac{L_s}{\sqrt{L_s^2+\pi^2}}\right),\quad L_s = \ln\left(s/\Lambda^2\right).$$

IR-behavior

In the IR-region

- Universal finite IR values: $\bar{\mathcal{A}}(0) = \bar{\mathfrak{A}}(0) = 4\pi/b_0 \sim 1.4$;
- Loop stabilization at two-loop level.

This yields practical **weak loop dependence** of $\bar{\mathcal{A}}(Q^2)$, $\bar{\mathfrak{A}}(s)$, and higher expansion functions:



Why we need (F)APT?

In standard QCD PT we have not only power series

$$F[L] = \sum_m f_m\, a_s^m[L],$$

but also:

- RG-improvement to account for higher-orders $\to$

$$Z[L] = \mathbf{exp}\left\{\int^{a_s[L]} \frac{\gamma(a)}{\beta(a)}\, da\right\} \xrightarrow{\text{1-loop}} [a_s[L]]^{\gamma_0/(2\beta_0)}$$

- Factorization $\to (a_s[L])^n\, L^m$
- Two-loop case $\to (a_s)^\nu\, \mathbf{ln}(a_s)$

New functions:

- $(a_s)^\nu$ **(done in ''FAPT'' package)**
- $(a_s)^\nu\, \mathbf{ln}(a_s), (a_s)^\nu\, L^m$, **(in preparation)**

(F)APT: one-loop Euclidian $\bar{\mathcal{A}}_\nu[L]$

Euclidean coupling ($L = \ln(Q^2/\Lambda^2)$):

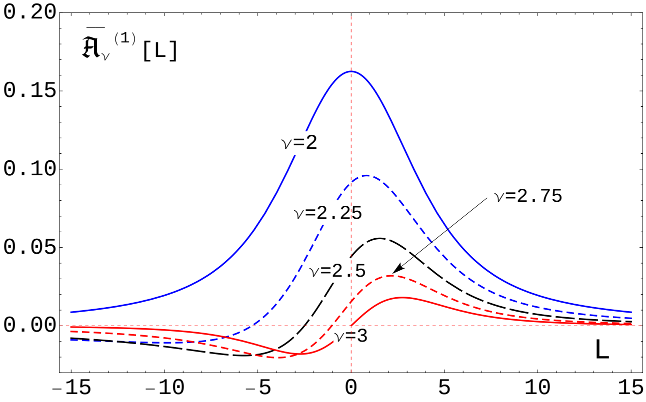
$$\bar{\mathcal{A}}_\nu[L] = \frac{4\pi}{b_0}\left(\frac{1}{L^\nu} - \frac{F(e^{-L}, 1-\nu)}{\Gamma(\nu)}\right)$$

Here $F(z, \nu)$ is reduced Lerch transcendent function (analytic function in ν).

Properties:

- $\bar{\mathcal{A}}_0[L] = 1$;
- $\bar{\mathcal{A}}_{-m}[L] = L^m$ for $m \in \mathbb{N}$;
- $\bar{\mathcal{A}}_m[L] = (-1)^m \bar{\mathcal{A}}_m[-L]$ for $m \geq 2\,,\ m \in \mathbb{N}$;
- $\bar{\mathcal{A}}_m[\pm\infty] = 0$ for $m \geq 2\,,\ m \in \mathbb{N}$;

(F)APT: one-loop Euclidian $\bar{\mathcal{A}}_\nu[L]$



(F)APT: one-loop Minkowskian $\bar{\mathfrak{A}}_\nu[L]$

Minkowskian coupling ($L = \ln(s/\Lambda^2)$):

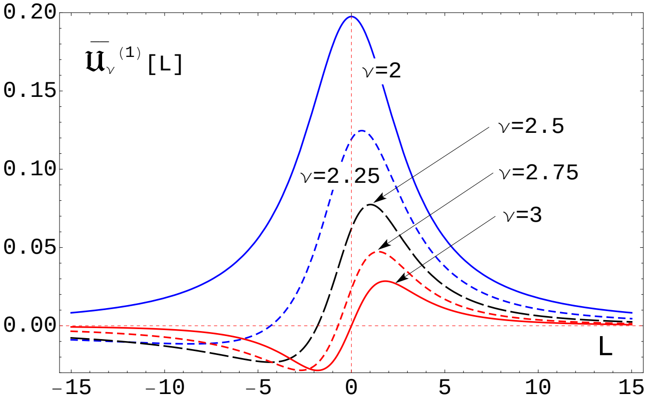
$$\bar{\mathfrak{A}}_\nu[L] = \frac{4}{b_0} \frac{\sin\left[(\nu-1)\arccos\left(L/\sqrt{\pi^2+L^2}\right)\right]}{(\nu-1)\left(\pi^2+L^2\right)^{(\nu-1)/2}}$$

Here we need only elementary functions.

Properties:

- $\bar{\mathfrak{A}}_0[L] = 1$;
- $\bar{\mathfrak{A}}_{-1}[L] = L$;
- $\bar{\mathfrak{A}}_{-2}[L] = L^2 - \frac{\pi^2}{3}, \quad \bar{\mathfrak{A}}_{-3}[L] = L\left(L^2 - \pi^2\right), \ \ldots$;
- $\bar{\mathfrak{A}}_m[L] = (-1)^m \bar{\mathfrak{A}}_m[-L]$ for $m \geq 2\,,\ m \in \mathbb{N}$;
- $\bar{\mathfrak{A}}_m[\pm\infty] = 0$ for $m \geq 2\,,\ m \in \mathbb{N}$

(F)APT: one-loop Minkowskian $\bar{\mathfrak{A}}_\nu[L]$



Non-power APT expansions

Instead of universal power-in-α_s expansion in APT one should use non-power functional expansions.

In Euclidian space Adler D-function

$$D_{\text{PT}}(Q^2) = d_0 + d_1\,\alpha_s(Q^2) + d_2\,\alpha_s^2(Q^2) + d_3\,\alpha_s^3(Q^2) + d_4\,\alpha_s^4(Q^2)$$

$$\mathcal{D}_{\text{APT}}(Q^2) = d_0 + d_1\,\bar{\mathcal{A}}_1(Q^2) + d_2\,\bar{\mathcal{A}}_2(Q^2) + d_3\,\bar{\mathcal{A}}_3(Q^2) + d_4\,\bar{\mathcal{A}}_4(Q^2)$$

In Minkowskian space R-ratio

$$R_{\text{PT}}(s) = r_0 + r_1\,\alpha_s(s) + r_2\,\alpha_s^2(s) + r_3\,\alpha_s^3(s) + r_4\,\alpha_s^4(s)$$

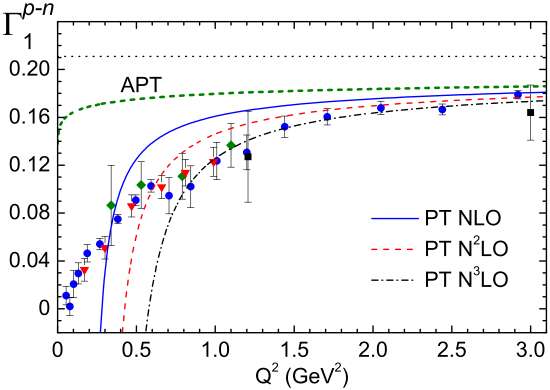
$$\mathcal{R}_{\text{APT}}(s) = d_0 + d_1\,\bar{\mathfrak{A}}_1(s) + d_2\,\bar{\mathfrak{A}}_2(s) + d_3\,\bar{\mathfrak{A}}_3(s) + d_4\,\bar{\mathfrak{A}}_4(s)$$

APT/(F)APT Applications

Loop stabilization

Perturbative power-correction of the polarized Bjorken Sum Rule (see [Khandramai *et. al* (PLB, 2012)])

$$\Gamma_1^{p-n}(Q^2) = \frac{|g_A|}{6} C_{\rm Bj}\,,\; C_{\rm Bj}(Q^2) \equiv 1 - \Delta_{\rm Bj}^{\rm PT}(Q^2)\,,\;\; |g_A| = 1.2701 \pm 0.0025$$



Loop stabilization of IR behavior at two-loop level

Scale-dependence

[Baikov, Chetyrkin, Kühn (2010)]

$$\begin{aligned} C_{\rm Bj}(Q^2, x_\mu &= \mu^2/Q^2) = 1 - 0.318\alpha_{\rm s} - (0.363 + 0.228\ln x_\mu)\alpha_{\rm s}^2 \\ &- (0.652 + 0.649\ln x_\mu + 0.163\ln^2 x_\mu)\alpha_{\rm s}^3 \\ &- (1.804 + 1.798\ln x_\mu + 0.790\ln^2 x_\mu + 0.117\ln^3 x_\mu)\alpha_{\rm s}^4 \end{aligned}$$

Weak scale dependence of observables

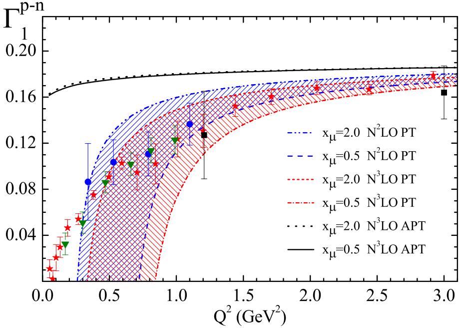


Figure: The μ-scale ambiguities for the perturbative part of the BSR versus Q^2 from [Khandramai *et al.* (2012)]

Convergence

Better loop convergence: the 3^{rd} and 4^{th} terms contribute less than 5% and 1% respectively. Again the 2-loop (N^2LO) level is sufficient.

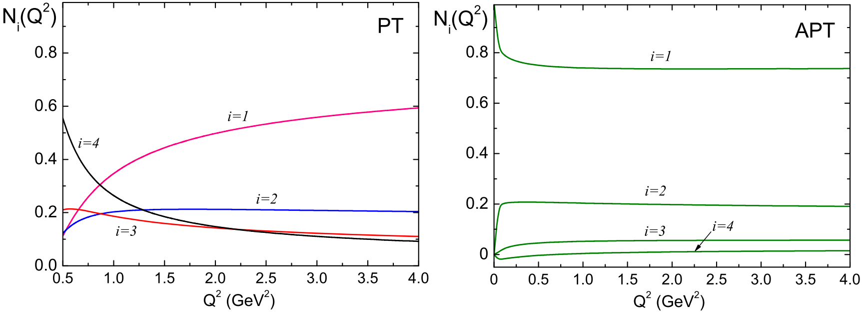


Figure: The relative contributions of separate terms in PT expansion for $\Delta_{\mathrm{Bj}}(Q^2)$, $\mathrm{N}_i(Q^2) = \delta_i(Q^2)/\Delta_{\mathrm{Bj}}(Q^2)$, as a function of Q^2 from [Khandramai *et. al* (2012)]

DIS Sum Rules

See [Pasechnik *et al.* (PRD,2010)]

The total expression for the perturbative part of $\Gamma_1^{p,n}(Q^2)$ including the higher twist contributions reads

$$\Gamma_1^{p,n}(Q^2) = \frac{1}{12}\left[\left(\pm a_3 + \frac{1}{3}a_8\right)E_{NS}(Q^2) + \frac{4}{3}a_0\,E_S(Q^2)\right] + \sum_{i=2}^{\infty}\frac{\mu_{2i}^{p,n}}{Q^{2i-2}},$$

where E_S and E_{NS} are the singlet and nonsinglet Wilson coefficients (for $n_f = 3$):

$$E_{NS}(Q^2) = 1 - \frac{\alpha_s}{\pi} - 3.558\left(\frac{\alpha_\mathrm{s}}{\pi}\right)^2 - 20.215\left(\frac{\alpha_\mathrm{s}}{\pi}\right)^3 - O(\alpha_\mathrm{s}^4),$$
$$E_S(Q^2) = 1 - \frac{\alpha_\mathrm{s}}{\pi} - 1.096\left(\frac{\alpha_\mathrm{s}}{\pi}\right)^2 - O(\alpha_\mathrm{s}^3).$$

In Γ_1^{p-n} the singlet and octet contributions are canceled out, giving rise to more fundamental Bjorken SR:

$$\Gamma_1^{p-n}(Q^2) = \frac{g_A}{6}E_{NS}(Q^2) + \sum_{i=2}^{\infty}\frac{\mu_{2i}^{p-n}(Q^2)}{Q^{2i-2}}.$$

The triplet and octet axial charges $a_3 \equiv g_A = 1.267 \pm 0.004$ and $a_8 = 0.585 \pm 0.025$.

The RG evolution of the axial singlet charge $a_0(Q^2)$

$$a_0^{\mathrm{PT}}(Q^2) = a_0^{\mathrm{PT}}(Q_0^2)\left\{1+\frac{\gamma_2}{(4\pi)^2\beta_0}\left[\alpha_s(Q^2)-\alpha_s(Q_0^2)\right]\right\},$$

$$a_0^{\mathrm{APT}}(Q^2) = a_0^{\mathrm{APT}}(Q_0^2)\left\{1+\frac{\gamma_2}{(4\pi)^2\beta_0}\left[\mathcal{A}_1(Q^2)-\mathcal{A}_1(Q_0^2)\right]\right\}, \quad \gamma_2 = 16 n_f.$$

The evolution from 1 GeV^2 to Λ_{QCD} in the APT increases the absolute value of a_0 by about 10 %.

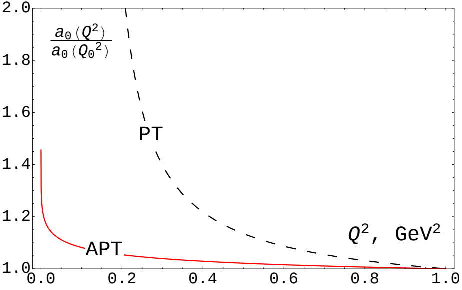


Figure: Evolution of $a_0(Q^2)$ normalized at $Q_0^2 = 1$ GeV^2 in APT and PT.

The RG evolution of the axial singlet charge $a_0(Q^2)$

Note, the Q^2-evolution of $\mu_4^p(Q^2)$ leads to close fit results within error bars. Therefore considered only of $a_0(Q^2)$

Table: Combined fit results of the proton $\Gamma_1^p(Q^2)$ data (elastic contribution excluded). APT fit results a_0 and $\mu_{4,6,8}^{APT}$ (at the scale $Q_0^2 = 1$ GeV^2) are given without and with taking into account the RG Q^2 evolution of $a_0(Q^2)$.

Method	Q^2_{min} GeV^2	a_0	μ_4/M^2	μ_6/M^4	μ_8/M^6
NNLO APT no evolution	0.47	0.35(4)	-0.054(4)	0	0
	0.17	0.39(3)	-0.069(4)	0.0081(8)	0
	0.10	0.43(3)	-0.078(4)	0.0132(9)	-0.0007(5)
NNLO APT with evolution	0.47	0.33(4)	-0.051(4)	0	0
	0.17	0.31(3)	-0.059(4)	0.0098(8)	0
	0.10	0.32(4)	-0.065(4)	0.0146(9)	-0.0006(5)

The fit results become more stable with respect to Q_{min} variations

Obtained values are very close to the corresponding COMPASS [Alexakhin *et al.* (2007)] and HERMES [Airapetian *et al.* (2007)] results 0.35 ± 0.06.

The RG evolution of the higher-twist $\mu_4^{p-n}(Q^2)$

$$\mu_{4,\mathrm{PT}}^{p-n}(Q^2) = \mu_{4,\mathrm{PT}}^{p-n}(Q_0^2)\left[\frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)}\right]^{\nu},$$

$$\mu_{4,\mathrm{APT}}^{p-n}(Q^2) = \mu_{4,\mathrm{APT}}^{p-n}(Q_0^2)\,\frac{\mathcal{A}_{\nu}^{(1)}(Q^2)}{\mathcal{A}_{\nu}^{(1)}(Q_0^2)}, \quad \nu = \frac{\gamma_0}{8\pi\beta_0}, \quad \gamma_0 = \frac{16}{3}C_F\,, \quad C_F = \frac{4}{3}\,.$$

The evolution from 1 GeV^2 to Λ_{QCD} in the APT increases the absolute value of μ_4^{p-n} by about 20 %.

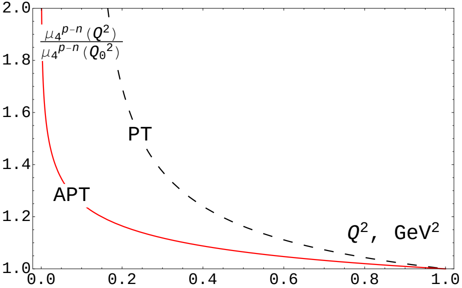


Figure: Evolution of $\mu_4^{p-n}(Q^2)$ normalized at $Q_0^2 = 1$ GeV^2 in APT and PT.

The RG evolution of the higher-twist $\mu_4^{p-n}(Q^2)$

Table: Combined fit results of the Γ_1^{p-n} data. APT fit results $\mu_{4,6,8}^{APT}$ (at the scale $Q_0^2 = 1$ GeV^2) are given without and with taking into account the RG Q^2-evolution of μ_4^{p-n}.

Method	Q^2_{min} GeV^2	μ_4/M^2	μ_6/M^4	μ_8/M^6
NNLO APT no evolution	0.47	-0.055(3)	0	0
	0.17	-0.062(4)	0.008(2)	0
	0.10	-0.068(4)	0.010(3)	-0.0007(3)
NNLO APT with evolution	0.47	-0.051(3)	0	0
	0.17	-0.056(4)	0.0087(4)	0
	0.10	-0.058(4)	0.0114(6)	-0.0005(8)

Account of this evolution, which is most important at low Q^2, improves the stability of the extracted parameters whose Q^2 dependence diminishes

The M_4 and M_8 moments evolution

[Bakulev, Ayala (In preparation)]

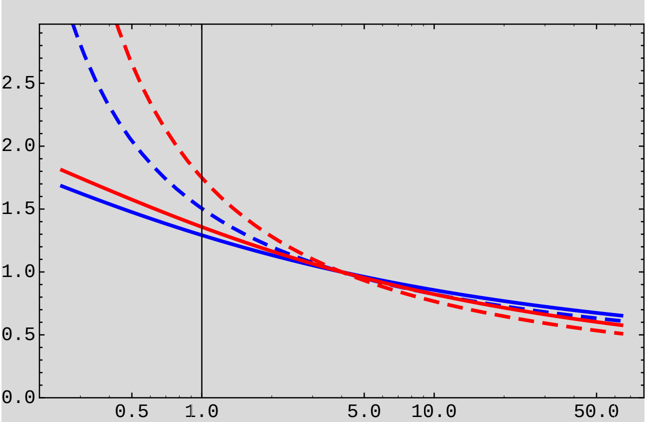


Figure: The M_4 (solid curves) and M_8 (dashed curves) moments evolution normilized at the scale $Q_0^2 = 4$ GeV^2 in the APT (blue curves) and PT (red curves).

Adler D-function analysis

$$\Pi_{\mu\nu}(q^2) = i \int e^{iqx} \langle | T\{J_\mu(x) J_\nu(0)\} | 0 \rangle \, d^4x, \quad \Pi_{\mu\nu}(q^2) = \left(q_\mu q_\nu - g_{\mu\nu} q^2\right) \Pi(Q^2).$$

$$D(Q^2) = -Q^2 \frac{d\Pi(-Q^2)}{dQ^2} = Q^2 \int_0^\infty ds \frac{R(s)}{(s+Q^2)^2}, \quad R(s) = \mathbf{Im}\, \Pi(s)/\pi.$$

The OPE-representation for the D-function

$$\begin{aligned} D_{\rm OPE}(Q^2) &= D_{\rm PT}(Q^2) + D_{\rm NP}(Q^2) \\ &\to 1 + 0.318\alpha_{\rm s} + 0.166\alpha_{\rm s}^2 + 0.205\alpha_{\rm s}^3 + 0.504\alpha_{\rm s}^4 + \frac{\mathbf{A}}{Q^4} + \cdots \end{aligned}$$

A simple model for the function $R_V(s)$ (see [Peris, Perrottet, de Rafael (1998), Dorokhov (2004)])

$$R_V^{\rm had}(s) = \frac{2\pi}{g_V^2} m_V^2 \, \delta(s - m_V^2) + \left(1 + \frac{\alpha_s^{(0)}}{\pi}\right) \theta(s - s_0),$$

$$D_V^{\rm had}(Q^2) = \frac{2\pi}{g_V^2} \frac{Q^2 \, m_V^2}{(Q^2 + m_V^2)^2} + \left(1 + \frac{\alpha_s^{(0)}}{\pi}\right) \frac{Q^2}{Q^2 + s_0},$$

which reproduces well the "experimental" curve $D_V^{\rm exp}(Q^2)$ with the parameters: $m_V = 770$ MeV, $g_V^{-2} \simeq 2.1$, $\alpha_s^{(0)} \simeq 0.4$, and $s_0 \simeq 1.77$ GeV^2.

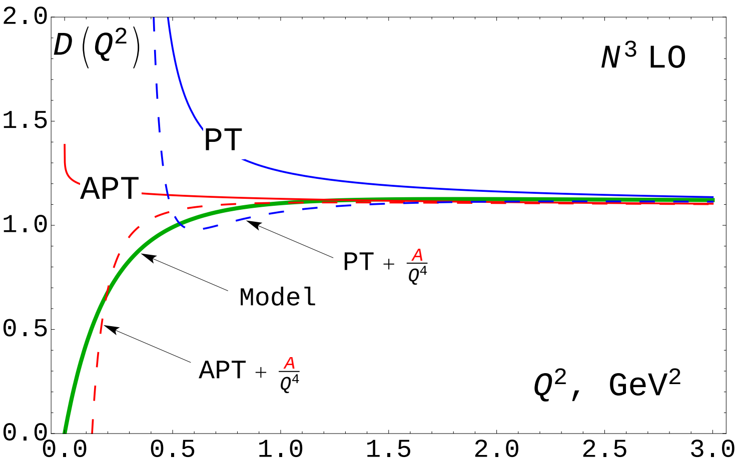
Adler D-function analysis

Table: Fit results of the Adler D-function data based on hadron model.

Method	Order	Q^2_{min}, GeV^2	**A**, GeV^4	$\chi^2_{d.o.f}$
PT	LO	0.2	−0.020	0.711
	NLO	0.3	−0.061	0.626
	N^{2}LO	0.4	−0.114	0.343
	N^{3}LO	0.5	−0.196	0.538
APT	LO	0.2	−0.018	0.508
	NLO	0.2	−0.019	0.896
	N^{2}LO	0.2	−0.019	0.912
	N^{3}LO	0.2	−0.019	0.905

- Standard PT provides: the results strongly changes from order to order;
- APT gives stable values of non-perturbative $\mathcal{O}\left(1/Q^4\right)$-correction and allow to describe data up to $Q_{min} = 0.2$ GeV^2.

Adler D-function analysis



Package ‘‘FAPT’’

‘‘FAPT’’ package review

Title of program: FAPT

Available from:

http://theor.jinr.ru/~bakulev/fapt.mat/FAPT.m

http://theor.jinr.ru/~bakulev/fapt.mat/FAPT_Interp.m

Computer for which the program is designed and others on which it is operable: Any work-station or PC where Mathematica is running.

Operating system or monitor under which the program has been tested: Windows XP, Mathematica (versions 5,7,8).

‘‘FAPT’’ package contains:

1. $\bar{\alpha}_{\rm s}^{(\ell)}[L, n_f]$, $\bar{\alpha}_{\rm s}^{(\ell);\text{glob}}$
2. $\bar{\rho}^{(\ell)}[L_\sigma, n_f, \nu]$, $\rho^{(\ell);\text{glob}}[L_\sigma, \nu, \Lambda_{n_f=3}]$
3. $\bar{\mathcal{A}}_\nu^{(\ell)}[L, n_f]$, $\mathcal{A}_\nu^{(\ell);\text{glob}}[L, \nu, \Lambda_{n_f=3}]$
4. $\bar{\mathfrak{A}}_\nu^{(\ell)}[L, n_f]$, $\mathcal{A}_\nu^{(\ell);\text{glob}}[L, \nu, \Lambda_{n_f=3}]$

Numerical parameters

- The pole masses of heavy quarks and Z-boson, collected in the set NumDefFAPT (all mass variables and parameters are measured in GeVs):

$$\begin{array}{llll} \texttt{MQ4}: & M_c = 1.65~\text{GeV}\,, & \texttt{MQ5}: & M_b = 4.75~\text{GeV}\,; \\ \texttt{MQ6}: & M_t = 172.5~\text{GeV}\,, & \texttt{MZboson}: & M_Z = 91.19~\text{GeV}\,. \end{array}$$

** The package RunDec is using the set NumDef with slightly different values of these parameters ($M_c = 1.6$ GeV, $M_b = 4.7$ GeV, $M_t = 175$ GeV, $M_Z = 91.18$ GeV).*

- Collection in the set setbetaFAPT the following rules of substitutions $b_i \to b_i(n_f)$

$$\texttt{b0}: b_0 \;\to\; 11 - \frac{2}{3}\,n_f\,, \quad \texttt{b1}, \quad \texttt{b2}, \quad \texttt{b3}.$$

** Here we follow the same substitution strategy as in RunDec, but our b_i differ from b_i^{RunDec} by factors 4^{i+1}: $b_i = 4^{i+1}\, b_i^{RunDec}$.*

α_s calculations

The QCD scales $\Lambda\ell[\Lambda, n_f]$:

$$\backslash\texttt{[CapitalLambda]}\,\ell[\Lambda, k] = \Lambda\ell[\Lambda, n_f = k] = \Lambda_k^{(\ell)}(\Lambda)\,,\ (\ell = 1 \div 4, 3\text{P}\,;\ k = 4 \div 6)\,,$$

The threshold logarithms — as $\lambda\ell4[\Lambda]$, $\lambda\ell5[\Lambda]$, and $\lambda\ell6[\Lambda]$:

$$\backslash\texttt{[Lambda]}\,\ell k[\Lambda] = \lambda\ell k[\Lambda] = \ln\left(\Lambda^2/\Lambda\ell[\Lambda, k]^2\right)\,, \quad (\ell = 1 \div 4, 3\text{P}\,;\ k = 4 \div 6)\,,$$

The running QCD couplings with fixed n_f — as $\alpha\text{Bar}\ell[Q^2, n_f, \Lambda]$:

$$\backslash\texttt{[Alpha]}\,\text{Bar}\ell[Q^2, n_f, \Lambda] = \alpha\text{Bar}\ell[Q^2, n_f, \Lambda] = \alpha_s^{(\ell)}[\ln(Q^2/\Lambda^2); n_f],\ (\ell = 1 \div 4, 3\text{P})\,,$$

The global running QCD couplings $\alpha\text{Glob}\ell[Q^2, \Lambda]$, :

$$\backslash\texttt{[Alpha]}\,\text{Glob}\ell[Q^2, \Lambda] = \alpha\text{Glob}\ell[Q^2, \Lambda] = \alpha_s^{\text{glob};(\ell)}(Q^2, \Lambda)\,,\ (\ell = 1 \div 4, 3\text{P})\,,$$

Example 1

We assume that the two-loop QCD scale Λ_3 is fixed at the value $\Lambda_3 = 0.387$ GeV. We want to evaluate the corresponding values of the coupling $\alpha_s^{glob;(\ell)}(Q^2, \Lambda)$ at the scale $Q = M_5$.

Possible Mathematica realization of this task

```
In[1]:= SetDirectory[NotebookDirectory[]];
<< FAPT.m

In[2]:=  L23=0.387;

In[3]:=  Mb=MQ5/.NumDefFAPT
Out[3]=  4.75

In[4]:=  \[Alpha]Glob2[Mb^2,L23]
Out[4]=  0.218894
```

ρ_ν calculations

RhoBar$\ell[L, n_f, \nu]$ returns ℓ-loop spectral density $\bar{\rho}_\nu^{(\ell)}$ ($\ell = 1, 2, 3, 3\text{P}, 4$) of fractional-power ν at $L = \ln(Q^2/\Lambda^2)$ and at fixed number of active quark flavors n_f:

$$\texttt{RhoBar}\ell[L, k, \nu] = \bar{\rho}_\nu^{(\ell)}[L; n_f = k]\,, \quad (\ell = 1 \div 4, 3\text{P}\,;\ k = 3 \div 6)$$

RhoGlob$\ell[L, \nu, \Lambda_3]$ returns the global ℓ-loop spectral density $\bar{\rho}_\nu^{(\ell);\text{glob}}[L; \Lambda_3]$ ($\ell = 1, 2, 3, 3\text{P}, 4$) of fractional-power ν at $L = \ln(Q^2/\Lambda_3^2)$, cf. and with Λ_3 being the QCD $n_f = 3$-scale:

$$\texttt{RhoGlob}\ell[L, \nu, \Lambda_3] = \bar{\rho}_\nu^{(\ell);\text{glob}}[L; \Lambda_3]\,, \quad (\ell = 1 \div 4, 3\text{P})$$

$\bar{\mathcal{A}}_\nu$ and $\bar{\mathfrak{A}}_\nu$ calculations

AcalBar$\ell[L, n_f, \nu]$ returns ℓ-loop ($\ell = 1, 2, 3, 3\text{P}, 4$) analytic image of fractional-power ν coupling $\bar{\mathcal{A}}_\nu^{(\ell)}[L; n_f]$ in Euclidean domain,

$$\texttt{AcalBar}\ell[L, k, \nu] = \bar{\mathcal{A}}_\nu^{(\ell)}[L; n_f = k]\,, \quad (\ell = 1 \div 4, 3\text{P}\,;\ \ k = 3 \div 6)$$

AcalGlob$\ell[L, \nu, \Lambda_3]$ returns ℓ-loop analytic image of fractional-power ν coupling $\mathcal{A}_\nu^{(\ell);glob}[L, \Lambda_3]$ in Euclidean domain

$$\texttt{AcalGlob}\ell[L, \nu, \Lambda_3] = \mathcal{A}_\nu^{(\ell);\text{glob}}[L, \Lambda_3]\,, \quad (\ell = 1 \div 4, 3\text{P})$$

UcalBar$\ell[L, n_f, \nu]$ returns ℓ-loop ($\ell = 1, 2, 3, 3\text{P}, 4$) analytic image of fractional-power ν coupling $\bar{\mathfrak{A}}_\nu^{(\ell)}[L, n_f]$ in Minkowski domain

$$\texttt{UcalBar}\ell[L, k, \nu] = \bar{\mathfrak{A}}_\nu^{(\ell)}[L; n_f = k]\,, \quad (\ell = 1 \div 4, 3\text{P}\,;\ \ k = 3 \div 6)$$

UcalGlob$\ell[L, \nu, \Lambda_3]$ returns ℓ-loop analytic image of fractional-power ν coupling $\mathfrak{A}_\nu^{(\ell);glob}[L, \Lambda_3]$ in Minkowski domain

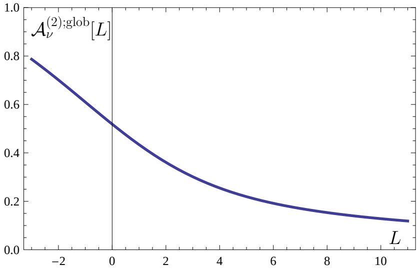
$$\texttt{UcalGlob}\ell[L, \nu, \Lambda_3] = \mathfrak{A}_\nu^{(\ell);glob}[L, \Lambda_3]\,, \quad (\ell = 1 \div 4, 3\text{P})$$

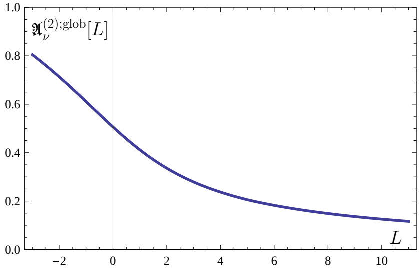
Example 2

Creation of a two-dimensional plot of $\mathcal{A}_\nu^{(2);\text{glob}}[L, \text{L23APT}]$ and $\mathfrak{A}_\nu^{(2);\text{glob}}[L, \text{L23APT}]$ for $L \in [-3, 11]$ with indication of the needed time:

```
In[5]:=  Plot[AcalGlob2[L,1,L23APT],{L,-3,11}]//Timing
Out[5]=  {19.843, Graphics
(see in the left panel of Fig. below)}

In[6]:=  Plot[UcalGlob2[L,1,L23APT],{L,-3,11}]//Timing
Out[6]=  {14.656, Graphics
(see in the right panel of Fig. below)}
```





Interpolation

To obtain the results much faster one can use module "FAPT_Interp" which consists of procedures AcalGlobℓi[L, ν, Λ_3] and UcalGlobℓi[L, ν, Λ_3], which are based on interpolation using the basis of the precalculated data.

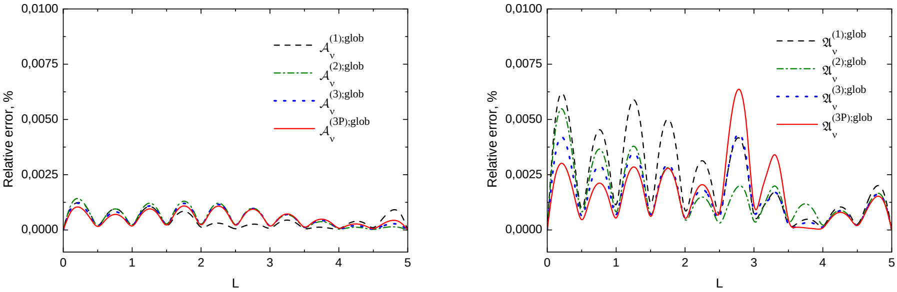


Figure: Relative error of the interpolation procedure for $\mathcal{A}^{\boldsymbol{glob}}_{\nu=\mathbf{1}.\mathbf{1}}$ (left panel) and $\mathfrak{A}^{\boldsymbol{glob}}_{\nu=\mathbf{1}.1}$ (right panel), calculated at various loop orders with $\Lambda_{\mathbf{3}} = 0.36$ GeV for $N = 11$ number of points.

Summary

APT provides natural way for coupling and related quantities with

- Universal (loop & scheme independent) IR limit;
- Weak loop dependence;
- Practical scheme independence.

(F)APT provides effective tool to apply APT approach for renormgroup improved perturbative amplitudes.

This approaches are used in many applications, for example:

- Higgs boson decay [Bakulev, Mikhailov, Stefanis (2007)];
- calculation of binding energies and masses of quarkonia [Ayala, Cvetič (2013)];
- analysis of the structure function $F_2(x)$ behavior at small values of x [Kotikov, Krivokhizhin, Shaikhatdenov (2012)];
- resummation approach [Bakulev, Potapova (2011)].

I collect in "FAPT" package all the procedures in APT and (F)APT which are needed to compute analytic images of the standard QCD coupling powers up to 4-loops of renormalization group running and to use it for both schemes: with fixed number of active flavours n_f, $\mathcal{A}_\nu(Q^2; n_f), \mathfrak{A}_\nu(s; n_f)$, and the global one with taking into account all heavy-quark thresholds, $\mathcal{A}_\nu^{glob}(Q^2), \mathfrak{A}_\nu^{glob}(s)$ based on the system "Mathematica".

Thanks for your attention!