Derivation and solution of functional equations for Feynman integrals

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Methods for deriving FE

Three methods for deriving functional equations (FE) were proposed

- FE from recurrence relations
- FE from algebraic relations for propagators
- FE from algebraic relations for deformed propagators

FE from recurrence relations

Functional equations (FE) for Feynman integrals were proposed in O.V.T. Phys.Lett. B670 (2008) 67.

Feynman integrals satisfy recurrence relations which can be written as

$$\sum_j Q_j I_{j,n} = \sum_{k,r<n} R_{k,r} I_{k,r}$$

where Q_j, R_k are polynomials in masses, scalar products of external momenta, space-time dimension d, and powers of propagators. $I_{k,r}$ - are integrals with r external lines. In recurrence relations some integrals are more complicated than the others: $I_{j,n}$ on the l.h.s have more arguments than $I_{k,r}$ on the r.h.s.

General method for deriving functional equations:

By choosing kinematic variables, masses, indices of propagators remove most complicated integrals, i.e. impose conditions :

$$Q_j = 0$$

keeping at least some other coefficients $R_k \neq 0$.

FE from recurrence relations

Example: one-loop n-point integrals

$$I_n^{(d)} = \frac{1}{i\pi^{d/2}} \int \frac{d^d k_1}{[(k_1-p_1)^2 - m_1^2 + i\eta]\ldots[(k_1-p_n)^2 - m_n^2 + i\eta]}$$

$I_n^{(d)}$ satisfy generalized recurrence relations O.T. in Phys.Rev.D54 (1996) p.6479

$$G_{n-1}\mathbf{j}^+ I_n^{(d+2)} - (\partial_j \Delta_n) I_n^{(d)} = \sum_{k=1}^{n} (\partial_j \partial_k \Delta_n)\mathbf{k}^- I_n^{(d)},$$

where $\mathbf{j}^{\pm}$ shifts the indices $\nu_j \to \nu_j \pm 1$, $\partial_j \equiv \frac{\partial}{\partial m_j^2}$,

$$g_{1,2,\ldots n} = G_{n-1} = -2^n \begin{vmatrix} p_1 p_1 & \ldots & p_1 p_{n-1} \\ \vdots & \ddots & \vdots \\ p_1 p_{n-1} & \ldots & p_{n-1} p_{n-1} \end{vmatrix}, \quad \Delta_n = \begin{vmatrix} Y_{11} & \ldots & Y_{1n} \\ \vdots & \ddots & \vdots \\ Y_{1n} & \ldots & Y_{nn} \end{vmatrix} = \lambda_{1,2,\ldots,n},$$

$$Y_{ij} = m_i^2 + m_j^2 - s_{ij}, \qquad s_{ij} = (p_i - p_j)^2,$$

FE from recurrence relations

For fixed j, to obtain functional equation two conditions must be fulfilled:

$$\begin{aligned} G_{n-1} &= 0, \\ \partial_j \Delta_n &= 0. \end{aligned} \tag{1}$$

Integral $I_n^{(d)}$ depends on $n(n+1)/2$ kinematic variables and masses. I.e. $I_n^{(d)}$ depends on n more variables than the integral $I_{n-1}^{(d)}$. The system (1) can be solve by choosing s_{ij}, m_k .

This method is difficult to use for multiloop integrals. Recurrence relations for integrals depending on many scalar invariants and masses are to be derived.

FE from algebraic relations

The simplest is the method based on algebraic relations for propagators.
The following algebraic relation between the products of n propagators was discovered:

$$\prod_{r=1}^{n} \frac{1}{D_r} = \frac{1}{D_0} \sum_{r=1}^{n} x_r \prod_{\substack{j=1 \\ j \neq r}}^{n} \frac{1}{D_j},$$

where

$$D_j = (k_1 - p_j)^2 - m_j^2 + i\eta.$$

This equation can be fulfilled for arbitrary k_1 by imposing conditions on x_j, m_0, p_0.

Algebraic relation for propagators

Example:

$$\frac{1}{D_1 D_2} = \frac{x_1}{D_0 D_2} + \frac{x_2}{D_1 D_0}.$$

Multiplying both sides by $D_0 D_1 D_2$ we get:

$$(k_1 - p_0)^2 - m_0^2 = x_1[(k_1 - p_1)^2 - m_1^2] + x_2[(k_1 - p_2)^2 - m_2^2]$$

or

$$(1 - x_1 - x_2)k_1^2 + 2k_1(x_1 p_1 + x_2 p_2 - p_0) + p_0^2 - m_0^2 + x_1(m_1^2 - p_1^2) + x_2(m_2^2 - p_2^2) = 0.$$

Assuming that k_1 does not depend on p_j we obtain system of equations:

$$1 - x_1 - x_2 = 0, \qquad x_1 p_1 + x_2 p_2 - p_0 = 0, \qquad p_0^2 - m_0^2 + x_1(m_1^2 - p_1^2) + x_2(m_2^2 - p_2^2) = 0.$$

This system can be solved for $x_j, m_0,\ p_0$.

Algebraic relation for propagators

Considering p_j as external momenta and integrating w.r.t. k_1 we get functional equation.

$$\int \frac{d^d k_1}{D_1 D_2} \rightarrow I_2^{(d)}(m_1^2, m_2^2; s_{12}) = x_1 I_2^{(d)}(m_0^2, m_2^2; s_{02}) + x_2 I_2^{(d)}(m_1^2, m_0^2; s_{10}),$$

$$\int \frac{d^d k_1}{D_1 D_2 D_3} \rightarrow I_3^{(d)}(m_1^2, m_2^2, m_3^2; s_{23}, s_{13}, s_{12}) = x_1 I_3^{(d)}(m_0^2, m_2^2, m_3^2; s_{23}, s_{03}, s_{02})$$
$$+ x_2 I_3^{(d)}(m_1^2, m_0^2, m_3^2; s_{03}, s_{13}, s_{10}) + x_3 I_3^{(d)}(m_1^2, m_2^2, m_0^2; s_{20}, s_{10}, s_{12}).$$

Algebraic relation for propagators

For the product of n propagators the following system of equations holds

$$p_0 = \sum_{j=1}^{n} x_j p_j \qquad \sum_{r=1}^{n} x_r \; = 1,$$

$$m_0^2 - \sum_{k=1}^{n} x_k m_k^2 + \sum_{j=1}^{n} \sum_{k=1}^{j-1} x_j \, x_k s_{kj} = 0,$$

where

$$s_{ij} = (p_i - p_j)^2.$$

Solutions of this system of equations will depend on $n-2$ *arbitrary parameters* x_i *and one arbitrary mass* m_0.

Algebraic relations for deformed propagators

Similar algebraic relations can be obtained for products of deformed propagators:

$$\widetilde{D}_j = a_j k_1^2 + \sum_r b_{jr} k_1 p_r + c_j,$$

where a_j, b_{jr}, c_j are arbitrary constants.
Algebraic relations with deformed propagators were used to derive FE for the two-loop vacuum integral with arbitrary masses.

Some comments on the imposed conditions:

1. The condition $p_0 = \sum_k^n x_k p_k$ means that the Gram determinant made of the set of vectors $p_0,\ p_1,\ \dots$ is equal to zero. But this is not enough for constructing functional equation!

2. Due to the Lorentz invariance the integral on the left-hand side of the functional equation depends only on the differences of the momenta $s_{ij} = (p_i - p_j)^2$, $i, j = 1 \dots n$. The integrals on the right-hand side also must depend on the same differences, i.e. there should be no $p_1^2,\ p_2^2,\ \dots$. This requirement is fulfilled if

$$\sum_k^n x_k = 1.$$

Examples:

$$I_2^{(d)}((p_1 - p_2)^2) = x_1 \ I_2^{(d)}((p_0 - p_2)^2) + x_2 \ I_2^{(d)}((p_0 - p_1)^2)$$

$$(p_0 - p_1)^2 = (1 - x_1 - x_2)[(1 - x_1)p_1^2 - x_2 p_2^2] + x_2(1 - x_1)s_{12},$$
$$(p_0 - p_2)^2 = (1 - x_1 - x_2)[x_1 p_1^2 + (1 - x_1)p_2^2] + x_1(1 - x_2)s_{12},$$

The meaning of algebraic relation for propagators

$$I_3^{(d)}(s_{23}, s_{13}, s_{12}) = x_1 I_3^{(d)}(s_{23}, s_{03}, s_{02}) + x_2 I_3^{(d)}(s_{03}, s_{13}, s_{01}) + x_3 I_3^{(d)}(s_{02}, s_{01}, s_{12})$$

$$s_{01} = (p_0 - p_1)^2 = (x_1 + x_2 + x_3 - 1)[(x_1 - 1)p_1^2 + x_2\ p_2^2 + x_3 p_3^2]$$
$$- x_2(x_1 - 1)s_{12} - x_3(x_1 - 1)s_{13} - x_2 x_3 s_{23},$$
$$s_{02} = (p_0 - p_2)^2 = (x_1 + x_2 + x_3 - 1)[x_1 p_1^2 + (x_2 - 1)p_2^2 + x_3 p_3^2]$$
$$- x_3(x_2 - 1)s_{23} - x_1 x_2 s_{13} + x_1(1 - x_2)s_{12},$$
$$s_{03} = (p_0 - p_3)^2 = (x_1 + x_2 + x_3 - 1)[x_1 p_1^2 + x_2 p_2^2 + (1 - x_3)p_3^2]$$
$$+ x_2(1 - x_3)s_{23} + x_1(1 - x_3)s_{13} - x_1 x_2 s_{12}.$$

It turns out that if the Lorentz condition $1 - \sum_{j=0}^{n} x_j = 0$ is fulfilled then the Gram determinants for each integral on the r.h.s. are proportional to the Gram determinant on the l.h.s.

$$I_2^{(d)}((p_1 - p_2)^2) = x_1 \; I_2^{(d)}((p_0 - p_2)^2) + x_2 \; I_2^{(d)}((p_0 - p_1)^2)$$

$$g_{12} = -4s_{12}, \quad g_{02} = x_1^2 g_{12}, \quad g_{01} = (1 - x_1)^2 g_{12},$$

$$I_3^{(d)}(s_{23}, s_{13}, s_{12}) = x_1 I_3^{(d)}(s_{23}, s_{03}, s_{02}) + x_2 I_3^{(d)}(s_{03}, s_{13}, s_{01}) + x_3 I_3^{(d)}(s_{02}, s_{01}, s_{12})$$

$$g_{123} = 2s_{12}^2 + 2s_{13}^2 + 2s_{23}^2 - 4s_{12}s_{13} - 4s_{12}s_{23} - 4s_{13}s_{23}.$$

$$g_{023} = x_1^2 g_{123}, \quad g_{103} = x_2^2 g_{123}, \quad g_{120} = (1 - x_1 - x_2)^2 g_{123},$$

This means that the Landau singularities at $g_{ijk\ldots} = 0$ are located at the same position for all integrals in the functional equation.

3. The fulfillment of the third condition

$$m_0^2 - \sum_k^n x_k m_k^2 + \sum_j^n \sum_{k<j} x_k x_j s_{kj} = 0,$$

leads to the proportionality of the modified Cayley determinants $\Delta_n(p_i, p_j, \ldots) = \lambda_{ij\ldots}$ for all integrals in the functional equation.

Examples:

$$I_2^{(d)}: \qquad \lambda_{02} = x_1^2 \lambda_{12}, \qquad \lambda_{02} = (1-x_1)^2 \lambda_{12},$$

$$\lambda_{12} = -m_1^4 - m_2^4 - s_{12}^2 + 2m_1^2 m_2^2 + 2s_{12} m_1^2 + 2s_{12} m_2^2,$$

$$I_3^{(d)}: \qquad \lambda_{023} = x_1^2 \lambda_{123}, \qquad \lambda_{103} = x_2^2 \lambda_{123}, \qquad \lambda_{120} = (1-x_1-x_2)^2 \lambda_{123},$$

$$\lambda_{123} = Y_{11} Y_{22} Y_{33} + 2Y_{22} Y_{23} Y_{13} - Y_{13}^2 Y_{22} - Y_{12}^2 Y_{33} - Y_{23}^2 Y_{11},$$
$$Y_{ij} = m_i^2 + m_j^2 - s_{ij}.$$

Landau singularities for vanishing Cayley determinants are located on the same surfaces for all the integrals in the FE.

Derivation of FE for Feynman integrals to some extent reminds derivation of functional equations for the polylogarithms $\mathrm{Li}_n(x)$.

Derivation of Kummer's type functional equations from the ansatz (G. Wechsung, Wiss.Z. Friedrich-Schiller Univ. Jena Natur. Reihe, **14** (1965) 401-408):

$$\sum_{i=1}^{N} a_i \mathrm{Li}_n(f_i(z)g_i(w)) = \mathcal{E}(z,w), \qquad (2)$$

where a_i are constants, f_i, g_i – rational functions, $\mathcal{E}$ may consist of functions with arguments depending either on z or on w.

Equations (2) share a common structural property:
$f(z)$, $g(z)$ are automorphic functions with respect to subgroups of a finite group of linear automorphisms.
– This invariance restrict the form of arguments and the number of terms in the ansatz.

– The coefficients a_i in (2) were determined requiring cancellation of singularities on the l.h.s. coming from $\mathrm{Li}_n(x)$ at $x = 1$

Comparison of the FE for polylogarithms and Feynman integrals

Similar properties reveal in derivation of FE for Feynman integrals:

– Lorentz invariance of FE for Feynman integrals restricts the form of arguments and location of Landau singularities.
– Cancellation of all types of Landau singularities gives additional condition on parameters x_j, m_0^2.

Observation : for the one-loop integrals the ratio of the Gram and Cayley modified determinants for all integrals in the FE remains the same:

$$\frac{g_{12}}{\lambda_{12}} = \frac{g_{02}}{\lambda_{02}} = \frac{g_{10}}{\lambda_{10}},$$
$$\frac{g_{123}}{\lambda_{123}} = \frac{g_{023}}{\lambda_{023}} = \frac{g_{103}}{\lambda_{103}} = \frac{g_{120}}{\lambda_{120}}.$$

The same is true for integrals with 4-,5- and 6- external legs.

Comparison of the FE for polylogarithms and Feynman integrals

Based on these observations one can propose most general functional equation for Feynman integrals

$$I_n^{(d)}(\{m_i\},\{s_{jk}\}) = \sum_r x(d,\{m_i\},\{s_{jk}\}) I_n^{(d)}(\{\overline{m_i}\},\{\bar{s}_{jk}\})$$

This functional equation should respect:

- Lorentz invariance
- invariance with respect to change of variables
  $s_{ij} \to \bar{s}_{ij} = R(\{s_{kr}\},\{m_n\})$
- cancellation of different types of Landau singularities
- invariance of the ratios $g_{ijk\dots}/\lambda_{ijk\dots}$
- invariance with respect to shift of dimension $d \to d+2$.

Work on this ansatz is in progress.

Solution of functional equations

Integrating algebraic relations directly or multiplying it with some weight and then integrating we can get in general relationships for multi-loop, multi-leg integrals depending on different kinematic variables.

What are these relationships?
Can we treat them really as functional equations and solve them?
The answer is yes!

Functional reduction of integrals

A functional equation is an equation which involves independent variables, known functions, unknown functions and constants. Very frequently the equation relates the value of a function (or functions) at some point with its values at other points.

In general solving functional equations can be very difficult, but there are some common methods of solving them. Systematic enumeration of such methods with examples is given in

> Castillo, E., Iglesias, A., Ruiz-Cobo, R.,
> *Functional Equations in Applied Sciences*,
> Elsevier Science, Mathematics in Science and Engineering, 2004.

For solving FE for Feynman integrals the following methods, described in this book, can be used :

1. Replacement of variables by given values
2. Transforming one or several variables
3. Using a more general equation
4. Iterative methods
5. Mixed methods

To some extent all these methods can be used in finding solutions of FE for Feynman integrals.

Let's consider one-loop propagator type integral. Integrating algebraic relation

$$\frac{1}{D_1 D_2} = \frac{x_1}{D_0 D_2} + \frac{x_2}{D_1 D_0}$$

w.r.t. k_1 yields:

$$I_2^{(d)}(m_1^2, m_2^2, s_{12}) = x_1 I_2^{(d)}(m_2^2, m_0^2, s_{20}) + x_2 I_2^{(d)}(m_1^2, m_0^2, s_{10}),$$

where

$$I_2^{(d)}(m_1^2, m_2^2;\ s_{12}) = \int \frac{d^d k_1}{i\pi^{d/2}} \frac{1}{[(k_1-p_1)^2 - m_1^2 + i\eta][(k_1-p_2)^2 - m_2^2 + i\eta]}.$$

$$x_1 = \frac{m_2^2 - m_1^2 + s_{12}}{2s_{12}} \pm \frac{\sqrt{4s_{12}m_0^2 - \lambda_{12}}}{2s_{12}}, \qquad x_2 = 1 - x_1,$$

$$s_{10} = (p_1 - p_0)^2 = \frac{2s_{12}(m_1^2 + m_0^2) - \lambda_{12}}{2s_{12}} \pm \frac{m_2^2 - m_1^2 - s_{12}}{2s_{12}} \sqrt{4s_{12}m_0^2 - \lambda_{12}},$$

$$s_{20} = (p_2 - p_0)^2 = \frac{2s_{12}(m_2^2 + m_0^2) - \lambda_{12}}{2s_{12}} \pm \frac{m_2^2 - m_1^2 + s_{12}}{2s_{12}} \sqrt{4s_{12}m_0^2 - \lambda_{12}},$$

$$\lambda_{ij} = -s_{ij}^2 - m_i^4 - m_j^4 + 2s_{ij}m_i^2 + 2s_{ij}m_j^2 + 2m_i^2 m_j^2.$$

We will consider this equation as a FE for the function $I_2^{(d)}(m_1^2, m_2^2, s_{12})$ and will solve it by the method similar to that was used for solving Sincov's functional equation

$$f(x,y) = f(x,z) - f(y,z).$$

Setting $z = 0$ in this equation, we get the general solution

$$f(x,y) = g(y) - g(x),$$

where

$$g(x) = f(x,0).$$

I.e. we expressed the function $f(x,y)$ in terms of its ' *boundary values*' .

In the same way, setting $m_0 = 0$ in the FE for $I_2^{(d)}$ we get its solution

$$I_2^{(d)}(m_1^2, m_2^2, s_{12}) = \overline{x}_1 I_2^{(d)}(m_2^2, 0, \overline{s}_{20}) + \overline{x}_2 I_2^{(d)}(m_1^2, 0, \overline{s}_{10}),$$

where

$$\overline{x}_j = x_j\big|_{m_0=0}, \qquad \overline{s}_{ij} = s_{ij}\big|_{m_0=0}.$$

Integral with three variables was expressed in terms of integrals with two variables.

One can check that the obtained solution satisfies initial FE for $I_2^{(d)}$ with arbitrary m_0. Substituting solution into both sides of the equation

$$I_2^{(d)}(m_1^2, m_2^2, s_{12}) = x_1 I_2^{(d)}(m_2^2, m_0^2, s_{20}) + x_2 I_2^{(d)}(m_1^2, m_0^2, s_{10}),$$

we obtain two terms on the left - hand side and four terms on the right - hand side of the equation. After simplifying arguments, we find that on the right- hand side two terms with arguments depending on m_0 cancel and the remaining two terms cancel two terms on the left-hand side. With arbitrary m_0^2 this check is not quite trivial because we must substitute into

$$\overline{x}_1 = \frac{m_2^2 - m_1^2 + s_{12}}{2s_{12}} \pm \frac{\sqrt{-\lambda_{12}}}{2s_{12}}, \qquad \overline{x}_2 = 1 - \overline{x}_1,$$

$\overline{s}_{10}$,$\overline{s}_{20}$ instead of s_{12}.

Solution of FE for Feynman integrals

Solution of FE will be expressed in terms of ratios of modified Cayley and Gram determinants:

$$\Delta_n \equiv \Delta_n(\{p_1, m_1\}, \ldots \{p_n, m_n\}) = \begin{vmatrix} Y_{11} & Y_{12} & \ldots & Y_{1n} \\ Y_{12} & Y_{22} & \ldots & Y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \ldots & Y_{nn} \end{vmatrix}, \qquad Y_{ij} = m_i^2 + m_j^2 - s_{ij}$$

$$G_{n-1} \equiv G_{n-1}(p_1, \ldots, p_n) = -2 \begin{vmatrix} S_{11} & S_{12} & \ldots & S_{1\ n-1} \\ S_{21} & S_{22} & \ldots & S_{2\ n-1} \\ \vdots & \vdots & \ddots & \vdots \\ S_{n-1\ 1} & S_{n-1\ 2} & \ldots & S_{n-1\ n-1} \end{vmatrix}, \qquad S_{ij} = s_{in} + s_{jn} - s_{ij},$$

We will use also an indexed notation for Δ_n and G_{n-1}

$$\lambda_{i_1 i_2 \ldots i_n} = \Delta_n(\{p_{i_1}, m_{i_1}\}, \{p_{i_2}, m_{i_2}\}, \ldots, \{p_{i_n}, m_{i_n}\}),$$
$$g_{i_1 i_2 \ldots i_n} = G_{n-1}(p_{i_1}, p_{i_2}, \ldots, p_{i_n}).$$

In what follows solutions of FE will be expressed in terms of

$$r_{ij\ldots k} = -\frac{\lambda_{ij\ldots k}}{g_{ij\ldots k}}, \qquad r_{ij\ldots k}^{(n)} = \frac{\partial\ r_{ij\ldots k}}{\partial m_n^2}$$

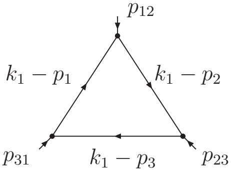
Vertex type integral

Let's consider integral with 3 massless propagators:

$$I_3^{(d)}(s_{23}, s_{13}, s_{12}) = \int \frac{d^d k_1}{i\pi^{d/2}} \frac{1}{P_1 P_2 P_3},$$

where

$$P_i = (k_1 - p_i)^2 + i\eta, \qquad s_{ij} = (p_i - p_j)^2,$$



Vertex type integral

In order to obtain FE for this integral we will use relationship for 3 propagators

$$\frac{1}{D_1 D_2 D_3} = \frac{x_1}{D_0 D_2 D_3} + \frac{x_2}{D_1 D_0 D_3} + \frac{x_3}{D_1 D_2 D_0},$$

where

$$D_j = (k_1 - p_j)^2 - m_j^2 + i\eta, \qquad p_0 = x_1 p_1 + x_2 p_2 + x_3 p_3,$$

Integration of this relationship with respect to k_1 yields FE:

$$\begin{aligned} I_3^{(d)}(m_1^2, m_2^2, m_3^2;\; s_{23}, s_{13}, s_{12}) = {} & x_1\; I_3^{(d)}(m_0^2, m_2^2, m_3^2;\; s_{23}, s_{30}, s_{20}) \\ & + x_2\; I_3^{(d)}(m_1^2, m_0^2, m_3^2;\; s_{30}, s_{13}, s_{10}) \\ & + x_3\; I_3^{(d)}(m_1^2, m_2^2, m_0^2;\; s_{20}, s_{10}, s_{12}). \end{aligned}$$

Vertex type integral

At $m_1 = m_2 = m_3 = m_0 = 0$ from this equation we get FE for the integral with massless propagators

$$
\begin{aligned}
& I_3^{(d)}(0,0,0;\ q_{23}, q_{13}, q_{12}) \\
& \quad = z_1\ I_3^{(d)}(0,0,0;\ q_{23}, q_{30}, q_{20}) \\
& \quad + z_2\ I_3^{(d)}(0,0,0;\ q_{30}, q_{13}, q_{10}) \\
& \quad + z_3\ I_3^{(d)}(0,0,0;\ q_{20}, q_{10}, q_{12}).
\end{aligned}
$$

where

$$
\begin{aligned}
q_{10} &= q_{13} - q_{13} z_1 + (q_{12} - q_{13}) z_2, \\
q_{20} &= q_{23} + (q_{12} - q_{23}) z_1 - z_2 q_{23}, \\
q_{30} &= q_{13} z_1 + q_{23} z_2.
\end{aligned}
$$

and the following equations to be hold:

$$
\begin{aligned}
& z_1 + z_2 + z_3 = 1, \\
& z_1 z_2 q_{12} + z_1 z_3 q_{13} + z_2 z_3 q_{23} = 0.
\end{aligned}
$$

Vertex type integral

In order to obtain solution of this equation we use more general equation. Such an equation we get by setting $m_1 = m_2 = m_3 = 0$ and keeping arbitrary m_0^2 in the equation for the general mass case

$$\begin{aligned} I_3^{(d)}&(0,0,0;\ s_{23}, s_{13}, s_{12}) \\ &= x_1\ I_3^{(d)}(m_0^2, 0, 0;\ s_{23}, s_{30}, s_{20}) \\ &+ x_2\ I_3^{(d)}(0, m_0^2, 0;\ s_{30}, s_{13}, s_{10}) \\ &+ x_3\ I_3^{(d)}(0, 0, m_0^2;\ s_{20}, s_{10}, s_{12}). \end{aligned}$$

where

$$\begin{aligned} s_{10} &= m_0^2 + s_{13} - s_{13}x_1 + (s_{12} - s_{13})x_2, \\ s_{20} &= m_0^2 + s_{23} + (s_{12} - s_{23})x_1 - x_2 s_{23}, \\ s_{30} &= m_0^2 + s_{13}x_1 + s_{23}x_2. \end{aligned}$$

and the following conditions must be satisfied:

$$\begin{aligned} &x_1 + x_2 + x_3 = 1, \\ &x_1 x_2\ s_{12} + x_1 x_3\ s_{13} + x_2 x_3\ s_{23} + m_0^2 = 0. \end{aligned}$$

Vertex type integral

To solve this FE means to express the integral $I_3^{(d)}(0,0,0;\ s_{23}, s_{13}, s_{12})$ in terms of functions with fewer variables.
By choosing arbitrary parameters we will try to express integrals in the right–hand side in terms of integrals with lesser number of variables. This means that we can try to impose some conditions on new variables $s_{j0},\ m_0^2$ like

$$
\begin{gathered}
s_{10} = 0, \quad s_{10} - s_{20} = 0, \quad s_{10} - s_{30} = 0, \quad s_{20} = 0, \quad s_{20} - s_{30} = 0, \quad s_{30} = 0, \\
s_{10} - s_{12} = 0, \quad s_{20} - s_{12} = 0, \quad s_{30} - s_{12} = 0, \quad s_{10} - s_{23} = 0, \quad s_{20} - s_{23} = 0, \\
s_{30} - s_{23} = 0, \quad s_{10} - s_{13} = 0, \quad s_{20} - s_{13} = 0, \quad s_{30} - s_{13} = 0, \quad s_{10} - m_0^2 = 0, \\
s_{20} - m_0^2 = 0, \quad s_{30} - m_0^2 = 0, \quad s_{10} + m_0^2 = 0, \quad s_{20} + m_0^2 = 0, \quad s_{30} + m_0^2 = 0.
\end{gathered}
$$

We considered 1330 systems of equations, each consisting of 3 equations composed out of the above 21 equations
In 35 sec of CPU time, 7 solutions without square roots of Gram determinants were discovered.

Vertex type integral

Substituting these solutions into the FE we found that one of them gives the most compact and simple relationship:

$$I_3^{(d)}(0,0,0;\ s_{23},s_{13},s_{12}) = r_{123}^{(1)}\ \xi_3^{(d)}(\mu_{123},s_{23}) + r_{123}^{(2)}\ \xi_3^{(d)}(\mu_{123},s_{13}) + r_{123}^{(3)}\ \xi_3^{(d)}(\mu_{123},s_{12}),$$

where

$$\xi_3^{(d)}(\mu_{123},s_{ij}) = I_3^{(d)}(0,0,\mu_{123};\ -\mu_{123},-\mu_{123},s_{ij}),$$

$$r_{ijk}^{(n)} = -\frac{\partial\ r_{ijk}}{\partial m_n^2}\Bigg|_{m_i=m_j=m_k=0} = \frac{s_{ij}-s_{ik}-s_{jk}}{s_{ik}s_{jk}}\ \mu_{ijk},$$

$$\mu_{ijk} = r_{ijk}\big|_{m_i=m_j=m_k=0} = \frac{s_{ij}s_{ik}s_{jk}}{s_{ij}^2+s_{ik}^2+s_{jk}^2-2s_{ij}s_{ik}-2s_{ij}s_{jk}-2s_{ik}s_{jk}},$$

Thus, we expressed integral depending on 3 variables in terms integrals depending on 2 variables.

Vertex type integral

We can check that the obtained solution is a solution of the initial FE:

$$
\begin{aligned}
& I_3^{(d)}(0,0,0;\ q_{23},q_{13},q_{12}) \\
& \quad = z_1\ I_3^{(d)}(0,0,0;\ q_{23},q_{30},q_{20}) \\
& \quad + z_2\ I_3^{(d)}(0,0,0;\ q_{30},q_{13},q_{10}) \\
& \quad + z_3\ I_3^{(d)}(0,0,0;\ q_{20},q_{10},q_{12}).
\end{aligned}
$$

Substituting the solution into the left –hand side and the right –hand side of this equation we obtain 12 terms. After algebraic simplification, taking into account algebraic conditions on z- parameters, we found that 6 terms with one arbitrary parameters z on the right –hand side cancel. The remaining 3 terms cancel 3 terms on the left –hand side.

Vertex type integral

Integral $\xi_3^{(d)}(m^2, q^2)$ can be evaluated as a solution of simple dimensional recurrence relation:

$$(d-2)\ \xi_3^{(d+2)}(m^2, q^2) = -2\widetilde{m}^2\ \xi_3^{(d)}(m^2, q^2) - \xi_2^{(d)}(q^2),$$

where

$$\xi_2^{(d)}(q^2) = I_2^{(d)}(0, 0, q^2) = -\frac{\pi^{3/2}(-\widetilde{q}^2)^{\frac{d}{2}-2}}{2^{d-3}\Gamma\left(\frac{d-1}{2}\right)\sin\frac{\pi d}{2}},$$

and

$$\widetilde{q}^2 = q^2 + 4i\eta, \qquad \widetilde{m}^2 = m^2 - i\eta.$$

Solution of dimensional recurrence relation can be written as

$$\xi_3^{(d)}(m^2, q^2) = -\frac{1}{2m^2}\xi_2^{(d)}(q^2)\ {}_2F_1\left[\begin{array}{c} 1, \frac{d-2}{2};\\ \frac{d-1}{2}; \end{array} \frac{-\widetilde{q}^{\,2}}{4\widetilde{m}^2}\right] + \frac{(d-2)\xi_1^{(d)}(m^2)}{2m^2\sqrt{q^2(q^2+4m^2)}}\ \ln\left(1 + \frac{q^2 + \sqrt{q^2(q^2+4m^2)}}{2m^2}\right).$$

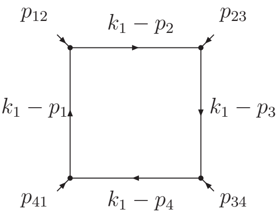
Box type integral

Let's consider integral with 4 propagators:

$$I_4^{(d)}(m_1^2, m_2^2, m_3^2, m_4^2;\ s_{12}, s_{23}, s_{34}, s_{14}, s_{24}, s_{13}) = \int \frac{d^d k_1}{i\pi^{d/2}} \frac{1}{P_1 P_2 P_3 P_4}.$$

where

$$P_i = (k_1 - p_i)^2 - m_i^2 + i\eta, \qquad s_{ij} = p_{ij}^2, \qquad p_{ij} = p_i - p_j.$$



Box type integral

To obtain FE we use relation for product of 4 propagators:

$$\frac{1}{D_1 D_2 D_3 D_4} = \frac{x_1}{D_0 D_2 D_3 D_4} + \frac{x_2}{D_1 D_0 D_3 D_4} + \frac{x_3}{D_1 D_2 D_0 D_4} + \frac{x_4}{D_1 D_2 D_3 D_0},$$

where

$$p_0 = x_1 p_1 + x_2 p_2 + x_3 p_3 + x_4 p_4,$$

and parameters x_j, m_0 obey the following conditions:

$$\begin{aligned} & x_1 + x_2 + x_3 + x_4 = 1, \\ & x_1 x_2 s_{12} + x_1 x_3 s_{13} + x_1 x_4 s_{14} + x_2 x_3 s_{23} + x_2 x_4 s_{24} + x_3 x_4 s_{34} \\ & -x_1 m_1^2 - x_2 m_2^2 - x_3 m_3^2 - x_4 m_4^2 + m_0^2 = 0. \end{aligned}$$

Integration of this relationship with respect to k_1 yields FE:

$$\begin{aligned} & I_4^{(d)}(m_1^2, m_2^2, m_3^2, m_4^2;\; s_{12}, s_{23}, s_{34}, s_{14}, s_{24}, s_{13}) \\ & \quad = x_1\; I_4^{(d)}(m_0^2, m_2^2, m_3^2, m_4^2;\; s_{20}, s_{23}, s_{34}, s_{40}, s_{24}, s_{30}) \\ & \quad + x_2\; I_4^{(d)}(m_1^2, m_0^2, m_3^2, m_4^2;\; s_{10}, s_{30}, s_{34}, s_{14}, s_{40}, s_{13}) \\ & \quad + x_3\; I_4^{(d)}(m_1^2, m_2^2, m_0^2, m_4^2;\; s_{12}, s_{20}, s_{40}, s_{14}, s_{24}, s_{10}) \\ & \quad + x_4\; I_4^{(d)}(m_1^2, m_2^2, m_3^2, m_0^2;\; s_{12}, s_{23}, s_{30}, s_{10}, s_{20}, s_{13}). \end{aligned}$$

Box type integral

At $m_1 = m_2 = m_3 = m_4 = m_0 = 0$ we have FE for massless integral $I_4^{(d)}$

$$
\begin{aligned}
& I_4^{(d)}(0,0,0,0; q_{12}, q_{23}, q_{34}, q_{14}, q_{24}, q_{13}) \\
= & \; z_1 \; I_4^{(d)}(0,0,0,0; q_{20}, q_{23}, q_{34}, q_{40}, q_{24}, q_{30}) \\
& + z_2 \; I_4^{(d)}(0,0,0,0; q_{10}, q_{30}, q_{34}, q_{14}, q_{40}, q_{13}) \\
& + z_3 \; I_4^{(d)}(0,0,0,0; q_{12}, q_{20}, q_{40}, q_{14}, q_{24}, q_{10}) \\
& + z_4 \; I_4^{(d)}(0,0,0,0; q_{12}, q_{23}, q_{30}, q_{10}, q_{20}, q_{13}).
\end{aligned}
$$

In this case

$$
\begin{aligned}
q_{10} &= q_{14} - q_{14} z_1 + (q_{12} - z_{14}) z_2 + (q_{13} - q_{14}) z_3, \\
q_{20} &= q_{24} + (q_{12} - q_{24}) z_1 - z_2 s_{24} + (q_{23} - q_{24}) z_3, \\
q_{30} &= q_{34} + (q_{13} - q_{34}) z_1 + (q_{23} - q_{34}) z_2 - q_{34} z_3, \\
q_{40} &= q_{14} z_1 + q_{24} z_2 + q_{34} z_3,
\end{aligned}
$$

and the following conditions to be hold:

$$
\begin{aligned}
& z_1 + z_2 + z_3 + z_4 = 1, \\
& z_1 z_2 \; q_{12} + z_1 z_3 \; q_{13} + z_1 z_4 \; q_{14} + z_2 z_3 \; q_{23} + z_2 z_4 \; q_{24} + z_3 z_4 \; q_{34} = 0.
\end{aligned}
$$

Box-type integral

Further reduction is possible. Again we used more general equation and computer search of solutions of systems of equations. Second step of functional reduction gives:

$$B_4^{(d)}(\mu_4; s_{ij}, s_{jk}, s_{ik}) = r_{ijk}^{(i)}\ \xi_4^{(d)}(\mu_{ijk}, \mu_4; s_{jk}) + r_{ijk}^{(j)}\ \xi_4^{(d)}(\mu_{ijk}, \mu_4; s_{ik}) + r_{ijk}^{(k)}\ \xi_4^{(d)}(\mu_{ijk}, \mu_4; s_{ij}),$$

where

$$\xi_4^{(d)}(\mu_{ijk}, \mu_4; s_{ij}) = I_4^{(d)}(0, 0, \mu_{ijk}, \mu_4; s_{ij}, -\mu_{ijk}, \mu_{ijk} - \mu_4, -\mu_4, -\mu_4, -\mu_{ijk}).$$

$$\mu_4 = r_{1234}\big|_{m_1=m_2=m_3=m_4=0}$$

The function with 4 variables was expressed in terms of function with 3 variables. Two of them are 'effective masses'.

Box-type integral

Combining formulae obtained on the first and second steps of reductions we find that one-loop box type integral with massless propagators depending on 6 variables is a combination of 12 integrals depending on 3 variables.

$$
\begin{aligned}
& I_4(s_{12}, s_{23}, s_{34}, s_{14}, s_{24}, s_{13}) \\
& = r_{1234}^{(1)} \left[r_{234}^{(2)} \xi_4^{(d)}(\mu_{234}, \mu_4, s_{34}) + r_{234}^{(3)} \xi_4^{(d)}(\mu_{234}, \mu_4, s_{24}) + r_{234}^{(4)} \xi_4^{(d)}(\mu_{234}, \mu_4, s_{23}) \right] \\
& + r_{1234}^{(2)} \left[r_{134}^{(1)} \xi_4^{(d)}(\mu_{134}, \mu_4, s_{34}) + r_{134}^{(3)} \xi_4^{(d)}(\mu_{134}, \mu_4, s_{14}) + r_{134}^{(4)} \xi_4^{(d)}(\mu_{134}, \mu_4, s_{13}) \right] \\
& + r_{1234}^{(3)} \left[r_{124}^{(1)} \xi_4^{(d)}(\mu_{124}, \mu_4, s_{24}) + r_{124}^{(2)} \xi_4^{(d)}(\mu_{124}, \mu_4, s_{14}) + r_{124}^{(4)} \xi_4^{(d)}(\mu_{124}, \mu_4, s_{12}) \right] \\
& + r_{1234}^{(4)} \left[r_{123}^{(1)} \xi_4^{(d)}(\mu_{123}, \mu_4, s_{23}) + r_{123}^{(2)} \xi_4^{(d)}(\mu_{123}, \mu_4, s_{13}) + r_{123}^{(3)} \xi_4^{(d)}(\mu_{123}, \mu_4, s_{12}) \right].
\end{aligned}
$$

We checked that this formula is a solution of the FE containing $I_4^{(d)}$ only with massless propagators. Substituting the above solution into the left –hand side and the right– hand side of the FE with two arbitrary parameters gives 60 terms. After nontrivial algebraic simplification 36 terms on the right –hand side cancel and the remaining 12 terms cancel 12 terms in the left – hand side.

Box-type integral

To evaluate $\xi_4^{(d)}$ we used simple dimensional recurrence relation

$$(d-3)\xi_4^{(d+2)}(\mu_3,\mu_4,s_{ij}) = -2\widetilde{\mu}_4\xi_4^{(d)}(\mu_3,\mu_4,s_{ij}) - \xi_3^{(d)}(\mu_3,s_{ij}).$$

Exploiting the method similar to that used for $\xi_3^{(d)}$ we get

$$\xi_4^{(d)}(\mu_3,\mu_4;s_{ij}) = \frac{\pi^{\frac{1}{2}}\Gamma\left(\frac{d}{2}\right)}{2\mu_4^2\Gamma\left(\frac{d-3}{2}\right)}\frac{\xi_1^{(d)}(\mu_4)}{(s_{ij}+4\mu_3)Z}\ln\left(\frac{1-Z}{1+Z}\right)$$
$$+\frac{(d-2)(d-4)}{4\mu_3\mu_4 R}\xi_1^{(d)}(\mu_3)\,{}_2F_1\left[\begin{matrix}1,\ \frac{d-3}{2};\\ \frac{3}{2};\end{matrix}\ 1-\frac{\widetilde{\mu}_3}{\widetilde{\mu}_4}\right]\ln\left(1+\frac{s_{ij}+R}{2\mu_3}\right)$$
$$+\frac{1}{2\mu_3\mu_4}\left(\frac{\widetilde{\mu}_3}{\widetilde{s}_{ij}+4\widetilde{\mu}_3}\right)^{\frac{1}{2}}\xi_2^{(d)}(s_{ij})F_1\left(\frac{d-3}{2},\frac{1}{2},1,\frac{d-1}{2};\frac{-\widetilde{s}_{ij}}{4\widetilde{\mu}_3},\frac{-\widetilde{s}_{ij}}{4\widetilde{\mu}_4}\right).$$

where

$$R=\sqrt{s_{ij}(s_{ij}+4\mu_3)} \qquad Z=\left(\frac{s_{ij}(\mu_4-\mu_3)}{\mu_4(s_{ij}+4\mu_3)}\right)^{1/2},$$

and F_1 is the Appell function. Numerical comparison of our result for $I_4^{(d)}(s_{12},s_{23},s_{34},s_{14},s_{24},s_{13})$ with the result obtained by using package SecDec reveals perfect agreement.

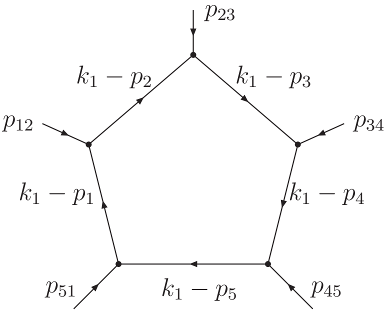
Pentagon type integral

Now let's consider pentagon type integral:

$$I_5^{(d)}(m_1^2, m_2^2, m_3^2, m_4^2, m_5^2; s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, s_{13}, s_{14}, s_{24}, s_{25}, s_{35})$$
$$= \int \frac{d^d k_1}{i\pi^{d/2}} \frac{1}{D_1 D_2 D_3 D_4 D_5}.$$

where

$$D_i = (k_1 - p_i)^2 - m_i^2 + i\eta, \qquad s_{ij} = p_{ij}^2, \qquad p_{ij} = p_i - p_j.$$



Pentagon type integral

To obtain FE we use relation for product of 5 propagators:

$$\frac{1}{D_1 D_2 D_3 D_4 D_5} = \frac{x_1}{D_0 D_2 D_3 D_4 D_5} + \frac{x_2}{D_1 D_0 D_3 D_4 D_5} + \frac{x_3}{D_1 D_2 D_0 D_4 D_5} + \frac{x_4}{D_1 D_2 D_3 D_0 D_5} + \frac{x_5}{D_1 D_2 D_3 D_4 D_0}$$

Integration of this relationship with respect to k_1 yields FE:

$$\begin{aligned}
& I_5^{(d)}(m_1^2, m_2^2, m_3^2, m_4^2, m_5^2; s_{12}, s_{23}, s_{34}, s_{45}, s_{15}; s_{13}, s_{14}, s_{24}, s_{25}, s_{35}) \\
&= x_1 I_5^{(d)}(m_0^2, m_2^2, m_3^2, m_4^2, m_5^2; s_{20}, s_{23}, s_{34}, s_{45}, s_{50}, s_{30}, s_{40}, s_{24}, s_{25}, s_{35}) \\
&+ x_2 I_5^{(d)}(m_1^2, m_0^2, m_3^2, m_4^2, m_5^2; s_{10}, s_{30}, s_{34}, s_{45}, s_{15}; s_{13}, s_{14}, s_{40}, s_{50}, s_{35}) \\
&+ x_3 I_5^{(d)}(m_1^2, m_2^2, m_0^2, m_4^2, m_5^2; s_{12}, s_{20}, s_{40}, s_{45}, s_{15}, s_{10}, s_{14}, s_{24}, s_{25}, s_{50}) \\
&+ x_4 I_5^{(d)}(m_1^2, m_2^2, m_3^2, m_0^2, m_5^2; s_{12}, s_{23}, s_{30}, s_{50}, s_{15}; s_{13}, s_{10}, s_{20}, s_{25}, s_{35}) \\
&+ x_5 I_5^{(d)}(m_1^2, m_2^2, m_3^2, m_4^2, m_0^2; s_{12}, s_{23}, s_{34}, s_{40}, s_{10}, s_{13}, s_{14}, s_{24}, s_{20}, s_{30}).
\end{aligned}$$

Pentagon type integral

Setting all masses $m_j^2 = 0$ we get FE for the integral $I_5^{(d)}$ with massless propagators

$$
\begin{aligned}
&I_5^{(d)}(0,0,0,0,0; q_{12}, q_{23}, q_{34}, q_{45}, q_{15}; q_{13}, q_{14}, q_{24}, q_{25}, q_{35}) \\
&= z_1 I_5^{(d)}(0,0,0,0,0; q_{20}, q_{23}, q_{34}, q_{45}, q_{50}, q_{30}, q_{40}, q_{24}, q_{25}, q_{35}) \\
&+ z_2 I_5^{(d)}(0,0,0,0,0; q_{10}, q_{30}, q_{34}, q_{45}, q_{15}; q_{13}, q_{14}, q_{40}, q_{50}, q_{35}) \\
&+ z_3 I_5^{(d)}(0,0,0,0,0; q_{12}, q_{20}, q_{40}, q_{45}, q_{15}, q_{10}, q_{14}, q_{24}, q_{25}, q_{50}) \\
&+ z_4 I_5^{(d)}(0,0,0,0,0; q_{12}, q_{23}, q_{30}, q_{50}, q_{15}; q_{13}, q_{10}, q_{20}, q_{25}, q_{35}) \\
&+ z_5 I_5^{(d)}(0,0,0,0,0; q_{12}, q_{23}, q_{34}, q_{40}, q_{10}, q_{13}, q_{14}, q_{24}, q_{20}, q_{30}).
\end{aligned}
$$

To solve this FE we will use more general FE setting in the initial one $m_1^2 = m_2^2 = m_3^2 = m_4^2 = m_5^2 = 0$, keeping $m_0^2 \neq 0$

Pentagon with off-shell momenta

The one-loop massless pentagon integral depending on 10 variables was reduced to 60 integrals depending on 4 variables:

$$
\begin{aligned}
& I_5^{(d)}(s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, s_{13}, s_{14}, s_{24}, s_{25}, s_{35}) \\
&= [\overline{r}_{2345}^{(5)}\overline{r}_{234}^{(3)}\xi_5^{(d)}(\overline{r}_{234}, \overline{r}_{2345}, \overline{\mu}_5, s_{24}) + \overline{r}_{2345}^{(5)}\overline{r}_{234}^{(2)}\xi_5^{(d)}(\overline{r}_{234}, \overline{r}_{2345}, \overline{\mu}_5, s_{34}) \\
&+\overline{r}_{2345}^{(4)}\overline{r}_{235}^{(5)}\xi_5^{(d)}(\overline{r}_{235}, \overline{r}_{2345}, \overline{\mu}_5, s_{23}) + \overline{r}_{2345}^{(4)}\overline{r}_{235}^{(3)}\xi_5^{(d)}(\overline{r}_{235}, \overline{r}_{2345}, \overline{\mu}_5, s_{25}) \\
&+\overline{r}_{2345}^{(4)}\overline{r}_{235}^{(2)}\xi_5^{(d)}(\overline{r}_{235}, \overline{r}_{2345}, \overline{\mu}_5, s_{35}) + \overline{r}_{2345}^{(3)}\overline{r}_{245}^{(5)}\xi_5^{(d)}(\overline{r}_{245}, \overline{r}_{2345}, \overline{\mu}_5, s_{24}) \\
&+\overline{r}_{2345}^{(3)}\overline{r}_{245}^{(4)}\xi_5^{(d)}(\overline{r}_{245}, \overline{r}_{2345}, \overline{\mu}_5, s_{25}) + \overline{r}_{2345}^{(3)}\overline{r}_{245}^{(2)}\xi_5^{(d)}(\overline{r}_{245}, \overline{r}_{2345}, \overline{\mu}_5, s_{45}) \\
&+\overline{r}_{2345}^{(2)}\overline{r}_{345}^{(5)}\xi_5^{(d)}(\overline{r}_{345}, \overline{r}_{2345}, \overline{\mu}_5, s_{34}) + \overline{r}_{2345}^{(2)}\overline{r}_{345}^{(4)}\xi_5^{(d)}(\overline{r}_{345}, \overline{r}_{2345}, \overline{\mu}_5, s_{35}) \\
&+\overline{r}_{2345}^{(2)}\overline{r}_{345}^{(3)}\xi_5^{(d)}(\overline{r}_{345}, \overline{r}_{2345}, \overline{\mu}_5, s_{45}) + \overline{r}_{2345}^{(5)}\overline{r}_{234}^{(4)}\xi_5^{(d)}(\overline{r}_{234}, \overline{r}_{2345}, \overline{\mu}_5, s_{23})]\overline{r}_{12345}^{(1)}
\end{aligned}
$$

$+$ 4 similar sets of terms where

$$
\begin{aligned}
& \xi_5^{(d)}(\mu_3, \mu_4, \mu_5, u_{ij}) \\
& \quad = I_5^{(d)}(\mu_5, \mu_4, 0, 0, \mu_3; \mu_4 - \mu_5, -\mu_4, u_{ij}, -\mu_3, \mu_3 - \mu_5, -\mu_5, -\mu_5, -\mu_4, \mu_3 - \mu_4, -\mu_3),
\end{aligned}
$$

Pentagon type integral

The on-shell case:

$$s_{12} = 0, \quad s_{23} = 0, \quad s_{34} = 0, \quad s_{45} = 0, \quad s_{15} = 0$$

Solution of the FE reads:

$$\begin{aligned} I_5^{(d)}(0,0,0,0,0,s_{13},s_{14},s_{24},s_{25},s_{35}) &= \widetilde{r}_{12345}^{(1)} F^{(d)}(\widetilde{\mu}_5;0,0,0,s_{24},s_{25},s_{35}) \\ &+ \widetilde{r}_{12345}^{(2)} F^{(d)}(\widetilde{\mu}_5;0,0,0,s_{14},s_{13},s_{35}) + \widetilde{r}_{12345}^{(3)} F^{(d)}(\widetilde{\mu}_5;0,0,0,s_{14},s_{24},s_{25}) \\ &+ \widetilde{r}_{12345}^{(4)} F^{(d)}(\widetilde{\mu}_5;0,0,0,s_{13},s_{35},s_{25}) + \widetilde{r}_{12345}^{(5)} F^{(d)}(\widetilde{\mu}_5;0,0,0,s_{13},s_{14},s_{24}) \end{aligned}$$

where

$$\begin{aligned} &F^{(d)}(\widetilde{\mu}_5;0,0,0,s_{ik},s_{in},s_{jn}) \\ &= -s_{in}\rho_{ijkn}\widetilde{\xi}_5^{(d)}(s_{in},\mu_{ijkn},\widetilde{\mu}_5) + s_{jn}\rho_{ijkn}\widetilde{\xi}_5^{(d)}(s_{jn},\mu_{ijkn},\widetilde{\mu}_5) + s_{ik}\rho_{ijkn}\widetilde{\xi}_5^{(d)}(s_{ik},\mu_{ijkn},\widetilde{\mu}_5), \end{aligned}$$

Pentagon-type integral

The function $\widetilde{\xi}_5^{(d)}(s_{in}, \mu_{ijkn}, \widetilde{\mu}_5)$ is $I_5^{(d)}$ integral with one massive line

$$\widetilde{\xi}_5^{(d)}(s_{in}, \mu_{ijkn}, \widetilde{\mu}_5) = I_5^{(d)}(0,0,0,0,\widetilde{\mu}_5;\ 0,0,0,-\widetilde{\mu}_5,-\widetilde{\mu}_5, s_{in}, \mu_{ijkn}, \mu_{ijkn}, -\widetilde{\mu}_5, -\widetilde{\mu}_5),$$

where

$$\mu_{ijkn} = s_{ik}s_{jn}\rho_{ijkn}, \qquad \rho_{ijkn} = \frac{1}{s_{ik} - s_{in} + s_{jn}}, \qquad \widetilde{\mu}_5 = -\frac{\delta_5}{g_4},$$

and δ_5, g_4 are the on-shell values of Δ_5, G_4:

$$\delta_5 = -2s_{13}s_{14}s_{24}s_{25}s_{35},$$
$$g_4 = 2s_{24}s_{35}(2s_{13}s_{24} - 2s_{13}s_{25} - s_{24}s_{35}) - 4(s_{25}+s_{35})s_{13}s_{14}s_{24}$$
$$- 2s_{13}^2(s_{24}-s_{25})^2 + 2(s_{25}-s_{35})s_{14}[2s_{25}s_{13} - 2s_{24}s_{35} - (s_{25}-s_{35})s_{14}].$$

The on-shell massless pentagon type integral depending on 5 variables is a combination of 15 integrals depending on 3 variables.

Pentagon type integral

Analytical result for the most elementary integral

$$\widetilde{\xi}_5^{(d)}(s_{13}, s_{14}, m^2) \equiv I_5^{(d)}(0,0,0,0,m^2;0,0,0,-m^2,-m^2,s_{13},s_{14},s_{14},-m^2,-m^2),$$

can be obtained as a solution of a simple dimensional recurrence relation:

$$(d-4)\widetilde{\xi}_5^{(d+2)}(s_{13}, s_{14}, m^2) = -2\widetilde{m}^2\widetilde{\xi}_5^{(d)}(s_{13}, s_{14}, m^2) - I_4^{(d)}(s_{13}, s_{14}),$$

where

$$I_4^{(d)}(s_{13}, s_{14}) \equiv I_4^{(d)}(0,0,0,0;0,0,0,s_{14},s_{14},s_{13})$$

$$= -\frac{4(d-3)}{s_{13}s_{14}(d-4)} I_2^{(d)}(s_{13})\, {}_2F_1\left[\begin{matrix} 1, \frac{d-4}{2}; \\ \frac{d-2}{2}; \end{matrix} \frac{\widetilde{s}_{13}}{\widetilde{s}_{14}}\right] - \frac{(d-3)}{s_{13}s_{14}} I_2^{(d)}(s_{14}) \ln\left(1 - \frac{\widetilde{s}_{13}}{\widetilde{s}_{14}}\right).$$

Pentagon type integral

For the case of $|\widetilde{s}_{13}/4/\widetilde{m}^2| < 1$, $|\widetilde{s}_{14}/4/\widetilde{m}^2| < 1$, the solution is:

$$I_5^{(d)}(m^2; s_{13}, s_{14}) = \bar{I}_5^{(d)}(m^2; s_{13}, s_{14}) + \frac{(d-3)}{m^2 s_{13} s_{14}} \, {}_2F_1\left[\begin{array}{c} 1, \frac{d}{2}-2; \\ \frac{d-3}{2}; \end{array} \frac{-\widetilde{s}_{14}}{4\widetilde{m}^2} \right]$$

$$\times \left\{ \frac{2}{(d-4)} \xi_2^{(d)}(s_{13}) \, {}_2F_1\left[\begin{array}{c} 1, \frac{d-4}{2}; \\ \frac{d-2}{2}; \end{array} \frac{\widetilde{s}_{13}}{\widetilde{s}_{14}} \right] + \xi_2^{(d)}(s_{14}) \ln\left(1 - \frac{\widetilde{s}_{13}}{\widetilde{s}_{14}}\right) \right\}$$

$$+ \frac{2\xi_2^{(d)}(s_{13})}{\widetilde{s}_{13}\widetilde{m}^2(\widetilde{s}_{14} + 4\widetilde{m}^2)} F_3\left(1, 1, \frac{1}{2}, \frac{d-4}{2}, \frac{d-1}{2}; \frac{\widetilde{s}_{14}}{\widetilde{s}_{14} + 4\widetilde{m}^2}, \frac{-\widetilde{s}_{13}}{4\widetilde{m}^2}\right),$$

where

$$\bar{I}_5^{(d)}(m^2; s_{13}, s_{14}) = \frac{(d-2)(d-4)tw}{4m^8(1-t)^2(w^2-1)} \xi_1^{(d)}(m^2) \left[\phi\left(\frac{w}{t}\right) + \phi(tw) - 2\phi(w) - \frac{1}{2}\ln^2 t \right].$$

Here

$$\phi(x) = \mathrm{Li}_2(x) + \ln x \ln(1-x),$$

$$t = 1 + \frac{\widetilde{s}_{13} + R_{13}}{2\widetilde{m}^2}, \qquad w = 1 + \frac{\widetilde{s}_{14} + R_{14}}{2\widetilde{m}^2}, \qquad \xi_1^{(d)}(m^2) = -\frac{\pi(\widetilde{m}^2)^{\frac{d}{2}-1}}{\Gamma\left(\frac{d}{2}\right)\sin\frac{\pi d}{2}},$$

$$\widetilde{q}^{\,2} = q^2 + 4i\eta, \qquad (3)$$

Conclusions and outlook

- Lorentz invariance and cancellation of Landau singularities between different terms determine form of functional equations for Feynman integrals
- Solution of FE significantly simplify analytic evaluation of Feynman integrals.
- Master integrals are not the simplest integrals. By applying the method of functional reduction master integrals can be reduced to elementary integrals.
- The next direction of investigation will be functional reduction of multiloop integrals.
- New ansatz for FE respecting different invariances is under investigation