BFKL equation: status and problems

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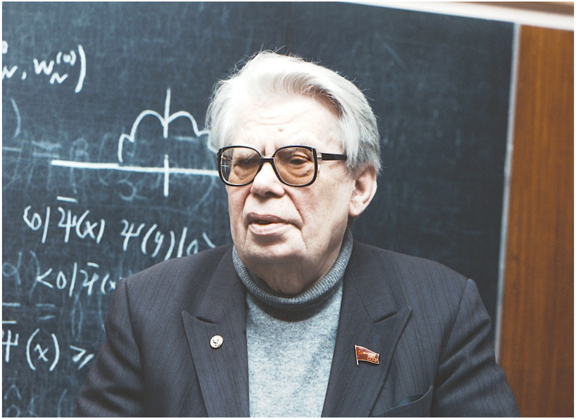
Introduction



Introduction

Among the many outstanding achievements and discoveries of N. N. Bogolyubov, there is the discovery of a new quantum number - colour, the carrier of which are quarks - elementary particles with a strong interaction. Subsequently, it turned out that this quantum number is not just some kind of label that serves to distinguish between quarks, but a source of strong interaction, similar to how an electric charge is a source of electromagnetic interaction. The concept of colour underlies the modern theory of strong interaction, which, by analogy with quantum electrodynamics (QED), is called quantum chromodynamics (QCD). What unites these theories is that both QED and QCD are gauge theories, that is, based on the requirement of invariance of the action with respect to local gauge transformations. But there is a significant difference: the electric charge is a scalar (one-component) quantity, so that the corresponding group of transformations is Abelian, while the colour has three components, the corresponding group of transformations is non-Abelian, so that QCD is non-Abelian gauge theory.

Non-Abelian gauge theories were introduced in physics in 1954

C. N. Yang and R. L. Mills, 1954

but they got real application only after the creation of the theory of electroweak interaction

S. Weinberg, 1967

A. Salam, 1967

and quantum chromodynamics (QCD)

H. Fritzsch, M. Gell-Mann, 1972 H. Fritzsch, M. Gell-Mann and H. Leutwyler, 1973.

The Standard Model of elementary particle contains QCD based on the $SU(3)$ and the theory of electroweak interactions based on the $SU(2) \times U(1)$.

Derivation

Originally the equation which is called BFKL was derived for summation of radiative corrections to elastic scattering amplitudes with the leading logarithmic accuracy (LLA), when in each order of perturbation theory only terms with the highest powers of $\ln s$ are kept, in non-Abelian theories with spontaneously broken symmetry
V.S. F., E.A. Kuraev, L.N. Lipatov, 1975.
Here again, N.N. Bogolyubov should be mentioned: he was first developed a mathematically correct method of describing spontaneous symmetry breaking which was method of quasiaverages.
The derivation was performed in the theories with the Higgs mechanism of mass generation which preserves renormalizability and permits to escape infrared singularities.

Derivation

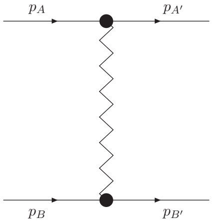
Later, the applicability of the equation in QCD was shown I.I. Balitsky, L.N. Lipatov, 1978.
Currently, the main field of application of BFKL lies in QCD, and hereinafter I will talk specifically about QCD.
The corrections were calculated using the dispersive approach based on the general properties of analyticity, unitarity and renormalizability.
The unitarity was used for calculation of discontinuities of elastic amplitudes, and analyticity for their full restoration.
The dispersion approach requires knowledge of all amplitudes, which contributes to unitarity relations. Of course, their direct calculation is impossible.
Clearly, direct calculation of an infinite number of amplitudes, and even in all orders of the perturbation theory, is impossible.
Something should help.
This "something" was the gluon Reggeization.

Derivation

One of remarkable properties of QCD is the Reggeization of all elementary particles in perturbation theory.
The gluon Reggeization is especially important because it determines the high energy behaviour of non-decreasing with energy cross sections.
Amplitudes with gluon quantum numbers in cross-channels are dominant (have the largest $\ln s$ degrees) in each order of perturbation theory.
They determine the s-channel discontinuites of amplitudes with the same and all other possible quantum numbers.
It is extremely important that both in the leading logarithmic approximation (LLA) and in the next-to-leading one (NLLA) the amplitudes used in the unitarity relations are determined by the Regge pole contributions and have a simple factorized form (pole Regge form).

Derivation

Due to this, the Reggeization provides a simple derivation of the BFKL equation in the LLA and in the NLLA.
For elastic scattering processes $A + B \to A' + B'$ in the Regge kinematical region: $s \simeq -u \to \infty$, t fixed (i.e. not growing with s) the Reggeization means that scattering amplitudes with the gluon quantum numbers in the t-channel and negative signature (symmetry with respect to $s \leftrightarrow u$) is written as

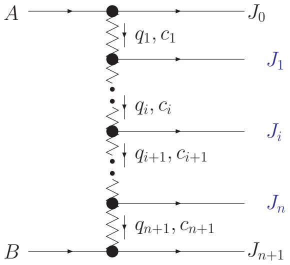


$$\mathcal{A}_{AB}^{A'B'} = \Gamma^{c}_{A'A} \left[\left(\frac{-s}{-t} \right)^{j(t)} - \left(\frac{s}{-t} \right)^{j(t)} \right] \Gamma^{c}_{B'B} \; ;$$

Derivation

$\Gamma^{c}_{P'P}$–particle-particle-Reggeon (PPR) vertices or scattering vertices ("c" are colour indices); $j(t) = 1 + \omega(t)$ – Reggeon trajectory.

The Reggeization means definite form not only of elastic amplitudes, but of inelastic amplitudes in the multi-Regge kinematics (MRK) as well. It can be presented by the picture



and written as

$$\Re\mathcal{A}_{AB}^{\tilde{A}\tilde{B}+n} = 2s\,\Gamma_{\tilde{A}A}^{c_1}\left(\prod_{i=1}^{n}\gamma_{c_i c_{i+1}}^{J_i}(q_i,q_{i+1})\left(\frac{s_i}{s_0}\right)^{\omega(t_i)}\frac{1}{t_i}\right)$$

$$\frac{1}{t_{n+1}}\left(\frac{s_{n+1}}{s_0}\right)^{\omega(t_{n+1})}\Gamma_{\tilde{B}B}^{c_{n+1}}$$

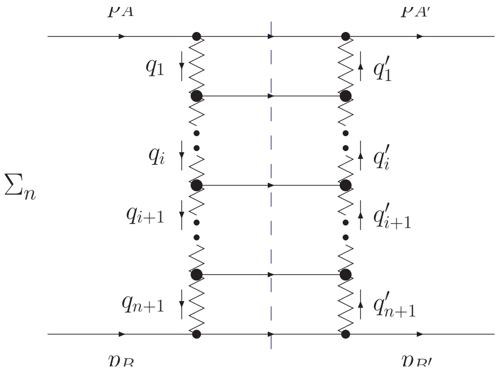
Here $\gamma_{c_i c_{i+1}}^{J_i}(q_i,q_{i+1})$ – the Reggeon-Reggeon-particle (RRP) or production vertices.

MRK is the kinematics where all particles have limited (not growing with s) transverse momenta and are combined into jets with limited invariant mass of each jet and large (growing with s) invariant masses of any pair of the jets.

The MRK gives dominant contributions to cross sections of QCD processes at high energy $\sqrt{s}$. In the LLA only a gluon can be produced. In the NLA one has to account production of $Q\bar{Q}$ and GG jets.

Amplitudes of processes with all possible quantum numbers in the *t*–channel are calculated using *s*-channel unitarity and analyticity .

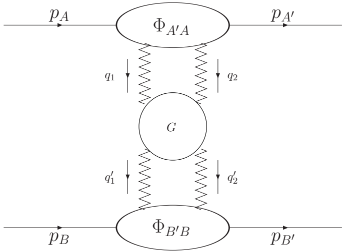
The *s*-channel discontinuities



Derivation

They are presented in the form :

$$\Phi_{A'A} \otimes G \otimes \Phi_{B'B}.$$



Impact factors $\Phi_{A'A}$ and $\Phi_{B'B}$ describe transitions $A \to A'$ $B \to B'$,
G – Green's function for two interacting Reggeized gluons,

Derivation

$$\frac{d}{dY}\hat{\mathcal{G}} = \hat{\mathcal{K}}\hat{\mathcal{G}}\ , \quad \hat{\mathcal{G}} = e^{Y\hat{\mathcal{K}}},$$

$\hat{\mathcal{K}}$ – BFKL kernel, $Y = \ln(s/s_0)$,

$$\hat{\mathcal{K}} = \hat{\omega}_1 + \hat{\omega}_2 + \hat{\mathcal{K}}_r$$

$$\hat{\mathcal{K}}_r = \hat{\mathcal{K}}_G + \hat{\mathcal{K}}_{Q\bar{Q}} + \hat{\mathcal{K}}_{GG}$$

Energy dependence of scattering amplitudes is determined by the BFKL kernel.

The BFKL kernel and the impact factors are expressed in terms of the Reggeon vertices and trajectory.

The kernel is universal (process independent).

Main results

Separate contributions to the kernel are infrared singular. They were calculated in the dimensional regularization, at the space-time dimension $D = 4 - 2\epsilon$.
But the in the colour singlet (Pomeron) channel the kernel is infrared safe. For the forward scattering

$$\mathcal{K}(\vec{q}, \vec{l}) = \frac{\alpha_s(\mu^2) N_c}{2\pi^2} \left[\frac{2}{(\vec{q}-\vec{l})^2} - \delta(\vec{q}-\vec{l}) \int \frac{d\vec{l}\, \vec{q}^{\,2}}{(\vec{q}-\vec{l})^2 \vec{l}^{\,2}} \right]$$

$$\times \left[1 + \frac{\alpha_s(\mu^2) N_c}{4\pi} \left(\frac{67}{9} - 2\zeta(2) - \frac{10}{9} \frac{n_f}{N_c} \right) \right]$$

$$+ \frac{\alpha_s^2(\mu^2) N_c^2}{4\pi^3} \left[\frac{1}{(\vec{q}-\vec{l})^2} \left(\frac{\beta_0}{N_c} \ln \left(\frac{\mu^2}{(\vec{q}-\vec{l})^{\,2}} \right) - \ln^2 \left(\frac{\vec{q}^{\,2}}{\vec{l}^{\,2}} \right) \right) + f_1(\vec{q}, \vec{l}) \right.$$

$$\left. + f_2(\vec{q}, \vec{l}) + \delta(\vec{q}-\vec{l}) \left(\frac{\beta_0}{2N_c} \int \frac{d\vec{l}\, \vec{q}^{\,2}}{(\vec{q}-\vec{l})^2 \vec{l}^{\,2}} \ln \left(\frac{(\vec{q}-\vec{l})^2 \vec{l}^{\,2}}{\mu^2 \vec{q}^{\,2}} \right) + 6\pi\zeta(3) \right) \right] ,$$

where $\beta_0 = 11/3 - 2n_f/(3N_c)$ is the first coefficient of the β-function,

$$f_1(\vec{q}_1, \vec{q}_2) = -\frac{2(\vec{q}_1^{\,2} - \vec{q}_2^{\,2})}{\vec{k}^{\,2}(\vec{q}_1 + \vec{q}_2)^2}\left(\frac{1}{2}\ln\left(\frac{\vec{q}_1^{\,2}}{\vec{q}_2^{\,2}}\right)\ln\left(\frac{\vec{q}_1^{\,2}\vec{q}_2^{\,2}\vec{k}^{\,4}}{\left(\vec{q}_1^{\,2}+\vec{q}_2^{\,2}\right)^4}\right)\right.$$

$$\left.+Li_2\left(-\frac{\vec{q}_1^{\,2}}{\vec{q}_2^{\,2}}\right) - Li_2\left(-\frac{\vec{q}_2^{\,2}}{\vec{q}_1^{\,2}}\right)\right) - \left(1 - (\vec{q}_1^{\,2} - \vec{q}_2^{\,2})^2/\vec{k}^{\,2}(\vec{q}_1 + \vec{q}_2)^2\right)$$

$$\times\left(\int_0^1 - \int_1^\infty\right)\frac{dz\,\ln\frac{(z\vec{q}_1)^2}{(\vec{q}_2)^2}}{(\vec{q}_2 - z\vec{q}_1)^2},$$

Main results

$$f_2(\vec{q}_1, \vec{q}_2) = -\left(1 + \frac{n_f}{N_c^3}\right) \frac{2\vec{q}_1^{\,2}\vec{q}_2^{\,2} - 3(\vec{q}_1\vec{q}_2)^2}{16\vec{q}_1^{\,2}\vec{q}_2^{\,2}} \left(\frac{2}{\vec{q}_2^{\,2}} + \frac{2}{\vec{q}_1^{\,2}} + \left(\frac{1}{\vec{q}_2^{\,2}} - \frac{1}{\vec{q}_1^{\,2}}\right) \ln\frac{\vec{q}_1^{\,2}}{\vec{q}_2^{\,2}}\right) - \left(3 + \left(1 + \frac{n_f}{N_c^3}\right)\left(1 - \frac{(\vec{q}_1^{\,2} + \vec{q}_2^{\,2})^2}{8\vec{q}_1^{\,2}\vec{q}_2^{\,2}} - \frac{2\vec{q}_1^{\,2}\vec{q}_2^{\,2} - 3\vec{q}_1^{\,4} - 3\vec{q}_2^{\,4}}{16\vec{q}_1^{\,4}\vec{q}_2^{\,4}}(\vec{q}_1\vec{q}_2)^2\right)\right) \times \int_0^\infty \frac{dx\, \ln\left|\frac{1+x}{1-x}\right|}{\vec{q}_1^{\,2} + x^2\vec{q}_2^{\,2}} \,.$$

This representation greatly simplifies calculation of the eigenvalues of the kernel.

With the NLO accuracy

$$\alpha_s(\mu^2) = \alpha_s(\vec{q}^{\,2}) \left(1 + \beta_0 \alpha_s(\vec{q}^{\,2}) \ln\left(\frac{\vec{q}^{\,2}}{\mu^2}\right)\right) ,$$

$$\int d^2l\, \mathcal{K}(\vec{q},\vec{l}) \left(\frac{\vec{l}^{\,2}}{\vec{q}^{\,2}}\right)^{\gamma-1} = \omega(\vec{q}^{\,2},\gamma) = \frac{\alpha_s(\vec{q}^{\,2})\, N_c}{\pi}\chi(\gamma) ,$$

$$\chi(\gamma) = \chi_B(\gamma) + \frac{\alpha_s N_c}{\pi}\chi^{(1)}(\gamma) .$$

Here $\chi_B(\gamma)$ gives the LO eigenvalues,

$$\chi_B(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma)$$

Main results

and the correction $\chi^{(1)}(\gamma)$ is:

$$\chi^{(1)}(\gamma) \;=\; -\frac{1}{4}\left[\frac{\beta_0}{2N_c}\left(\chi_B^2(\gamma)(\gamma)+\chi_B'(\gamma)-\;6\zeta(3)\right)\right.$$

$$+\frac{\pi^2\cos(\pi\gamma)}{\sin^2(\pi\gamma)(1-2\gamma)}\left(3+\left(1+\frac{n_f}{N_c^3}\right)\frac{2+3\gamma(1-\gamma)}{(3-2\gamma)(1+2\gamma)}\right)$$

$$\left.-\left(\frac{67}{9}-\frac{\pi^2}{3}-\frac{10}{9}\frac{n_f}{N_c}\right)\chi_B(\gamma)+\;\chi_B''(\gamma))-\frac{\pi^3}{\sin(\pi\gamma)}+4\phi(\gamma)\right],$$

where

$$\phi(\gamma)=-\int\limits_0^1\frac{dx}{1+x}\left(x^{\gamma-1}+x^{-\gamma}\right)\int_x^1\frac{dt}{t}\ln(1-t)$$

$$=\sum_{n=0}^{\infty}(-1)^n\left[\frac{\psi(n+1+\gamma)-\psi(1)}{(n+\gamma)^2}+\frac{\psi(n+2-\gamma)-\psi(1)}{(n+1-\gamma)^2}\right].$$

Main results

This result was firstly obtained
V.S. F., L.N. Lipatov, 1998.
The relative correction $r(\gamma)$

$$\chi^{(1)}(\gamma) = -r(\gamma)\chi_B(\gamma)$$

in the symmetrical point $\gamma = 1/2$, corresponding to the largest eigenvalue of the LO kernel,

$$\omega_P^{(1)} = -r(\gamma)\frac{\alpha_s N_c}{\pi}\omega_P^B$$

$$\omega_P^B = 4N_c\frac{\alpha_s}{\pi} \ln 2$$

$$r\left(\frac{1}{2}\right) \simeq 6.46 + 0.05\frac{n_f}{N_c} + 0.96\frac{n_f}{N_c^3}.$$

The correction is very large.
Various treatments have been developed for this problem. They include partial summation of higher approximations.

Main results

The representation written above is useful also for finding all eigenvalues of the kernel. Defining them as

$$\int d^2l\, K(\vec{q},\vec{l})\left(\frac{\vec{l}^{\,2}}{\vec{q}^{\,2}}\right)^{\gamma-1} e^{in(\phi_l-\phi_q)}$$

$$=\frac{\alpha_s(\vec{q}^{\;2})\,N_c}{\pi}\left(\chi_B(\gamma,n)+\frac{\alpha_s N_c}{\pi}\chi^{(1)}(\gamma,n)\right)\ ,$$

where

$$\chi_B(\gamma,n)=2\psi(1)-\psi(\gamma+\frac{|n|}{2})-\psi(1-\gamma+\frac{|n|}{2})$$

one comes to the result
A.V. Kotikov,L.N. Lipatov, 2000

Main results

$$4\chi^{(1)}(\gamma,n) = -\frac{\beta_0}{2N_c}\Big(\chi_B^2(\gamma,n)+\chi_B'(\gamma,n)\Big)+6\zeta(3)-\chi_B''(\gamma,n)$$

$$+\left(\frac{67}{9}-\frac{\pi^2}{3}-\frac{10}{9}\frac{n_f}{N_c}\right)\chi_B(\gamma,n)-2\Phi(n,\gamma)-2\Phi(n,1-\gamma)+F(n,\gamma)$$

$$F(n,\gamma)=\frac{\pi^2\cos(\pi\gamma)}{\sin^2(\pi\gamma)(1-2\gamma)}\left[\frac{\gamma(1-\gamma)\left(\delta_{n,2}+\delta_{n,-2}\right)}{2(3-2\gamma)(1+2\gamma)}\right.$$

$$\left.\times\left(1+\frac{n_f}{N_c^3}\right)-\left(\frac{3\gamma(1-\gamma)+2}{(3-2\gamma)(1+2\gamma)}\left(1+\frac{n_f}{N_c^3}\right)+3\right)\delta_{n,0}\right],$$

and

Main results

$$\Phi\left(n,\gamma\right)=\int_0^1\frac{dt}{1+t}t^{\gamma-1+n/2}\left\{\frac{\pi^2}{12}-\frac{1}{2}\psi'\left(\frac{n+1}{2}\right)-Li_2\left(t\right)\right.$$

$$-Li_2\left(-t\right)-\left(\psi\left(n+1\right)-\psi\left(1\right)+\ln\left(1+t\right)+\sum_{k=1}^{\infty}\frac{\left(-t\right)^k}{k+n}\right)\ln t$$

$$\left.-\sum_{k=1}^{\infty}\frac{t^k}{\left(k+n\right)^2}\left[1-\left(-1\right)^k\right]\right\}.$$

Main results

The relation between the eigenvalues of the NLO BFKL kernel and the anomalous dimension γ_ω of the twist-2 operators
V.S. F., L.N. Lipatov, 1998.
The NLO kernel in in maximally extended supersymmetric Yang-Mills theory (N=4 SYM)
A.V. Kotikov, L.N. Lipatov, 2000.
The NLO kernel for $t \neq 0$
V. S. F., R. Fiore, 2005.
Representation of the NLO kernel in the form where the conformal invariance is violated only by renormalization
V. S. F., R. Fiore, A. V. Grabovsky, A. Papa, 2009.
Remainder function to the BDS ansatz in N=4 SYM
V.S. F., L.N. Lipatov, 2012.
Moebius invariant BFKL equation for the adjoint representation in N=4 SUSY
V.S. F., R. Fiore, L. N. Lipatov, A. Papa, 2013.

Justification

The BFKL approach is based on the gluon Reggeization hypothesis.

The hypothesis is extremely powerful since an infinite number of amplitudes is expressed in terms of the gluon Regge trajectory and several Reggeon vertices.

Calculation of the vertices and trajectory is greatly simplified assuming the Reggeization.

In the LLA, to find the PPR vertices it is sufficient to calculate the simplest elastic scattering amplitude with the $P \to P'$ transition in the Born approximation. Of course, other processes can be used to test that the Regge form is valid.

To find a trajectory it is sufficient to calculate with logarithmic accuracy one-loop correction to elastic scattering amplitude with corresponding quantum numbers in the t–channel.

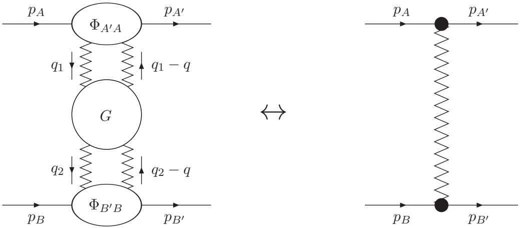
Of course, neither the calculation, nor the results are not so simple in the next-to-leading order (NLO).

Justification

The Reggeization hypothesis appeared on the basis of direct calculations at three-loop level for elastic amplitudes and one-loop level for one-gluon production amplitudes. Evidently, its proof was extremely desirable.

It was performed in the NLLA using bootstrap relations following from the bootstrap requirement (compatibility of the Reggeized form with the *s*–channel unitarity)

V.S.F., R. Fiore, M.G. Kozlov, A. V. Reznichenko, 2006



Justification

With the NLO accuracy

$$\frac{1}{-2\pi i}\mathrm{disc}_s\left(\ln^n(-s)+\ln^n s\right)=\frac{1}{2}\frac{\partial}{\partial \ln s}\mathrm{Re}\left[\ln^n(-s)+\ln^n s\right]\ ,$$

that gives the differential relation:

$$\frac{1}{-2\pi i s}\mathrm{disc}_s\,\mathcal{A}_{2\to 2}=\frac{1}{2}\frac{\partial}{\partial \ln s}\mathrm{Re}\,\mathcal{A}_{2\to 2}\ .$$

Using in the right hand side the Regge form gives the bootstrap relation:

$$\frac{1}{-2\pi i s}\mathrm{disc}_s\mathcal{A}_{2\to 2}\ =\ \frac{1}{2}\omega(t)\mathcal{A}^R_{2\to 2}$$

Evidently, there is an infinite number of bootstrap relations, because there is an infinite number of amplitudes $\mathcal{A}_{2\to n+2}$. At first sight, it seems a miracle to satisfy all of them, since all these amplitudes are expressed through several reggeon vertices and the gluon Regge trajectory.

Moreover, it is quite nontrivial to satisfy even some definite bootstrap relation for a definite amplitude, because it connects two infinite series in powers of rapidities, and therefore it leads to an infinite number of equalities between coefficients of these series.

In fact, two miracles must occur in order to satisfy all the bootstrap relations: first, for each particular amplitude $\mathcal{A}_{2\to n+2}$ it must be possible to reduce the bootstrap relation to a limited number of restrictions (bootstrap conditions) on the gluon trajectory and the reggeon vertices, and secondly, starting from some $n = n_0$ these bootstrap conditions must be the same as those obtained for amplitudes with $n < n_0$. Finally, all bootstrap conditions must be satisfied by the known expressions for the trajectory and the vertices.

It was shown that it is indeed the case

V.S.F., R. Fiore, M.G. Kozlov, A. V. Reznichenko, 2006–2015 .

Problems

It is extremely important that both in the LLA and in the NLLA the amplitudes used in the unitarity relations are determined by the Regge pole contributions and have a simple factorized form (pole Regge form).
Due to this, the Reggeization provides a simple derivation of the BFKL equation in the LLA and in the NLLA.
Validity of the pole Regge form is proved now in all orders of perturbation theory in the coupling constant g both in the LLA and in the NLLA.
The pole Regge form is violated in the NNLLA.
The first observation of the violation was done
V. Del Duca, E.W.N. Glover, 2001
at consideration of the high-energy limit of the two-loop amplitudes for gg, gq and qq scattering. The discrepancy appears in non-logarithmic terms.

Problems

If the pole Regge form would be correct in the NNLLA, they should satisfy a definite condition (factorization condition), because three amplitudes should be expressed in terms of two Reggeon-Particle-Particle vertices.

Detailed consideration of the terms responsible for breaking of the pole Regge form in two-loop and three-loop amplitudes of elastic scattering in QCD was performed by

V. Del Duca, G. Falcioni, L. Magnea, L. Vernazza, 2013-2015.

In particular, the non-logarithmic terms violating the pole Regge form at two-loops were recovered and not satisfying the factorization condition single-logarithmic terms at three loops were found using the techniques of infrared factorization.

It is necessary to say that, in general, breaking the pole Regge form is not a surprise.

Problems

It is well known that Regge poles in the complex angular momenta plane generate Regge cuts. Moreover, in amplitudes with positive signature the Regge cuts appear already in the LLA. In particular, the BFKL Pomeron is the two-Reggeon cut in the complex angular momenta plane. But in amplitudes with negative signature Regge cuts must be at least three-Reggeon ones and can appear only in the NNLLA.

It was natural to expect that the observed violation of the pole Regge form can be explained by existence of the cut.

Indeed, all known cases of breaking of the pole Regge form are now explained by the three-Reggeon cuts

V.S. F., L.N. Lipatov, 2016

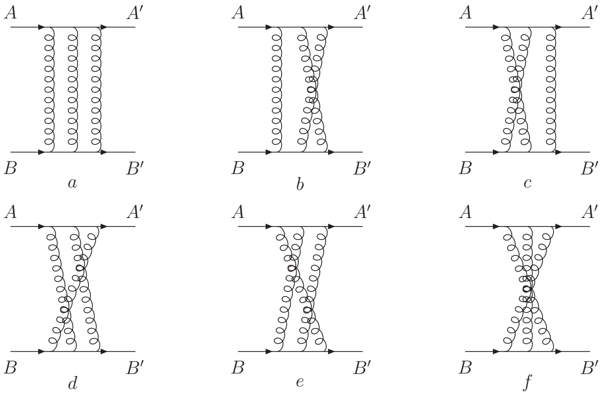
S. Caron-Huot, E. Gardi, L. Vernazza, 2017

Unfortunately, the approaches used and the explanations given in these papers are different.

Their results coincide in three loops but may diverge in more loops. It requires further investigation.

Problems

Due to the signature conservation the cut with negative signature must be the three-Reggeon one.
Since our Reggeon is the Reggeized gluon, the three-Reggeon cut first contribute to amplitudes corresponding to the diagrams



Problems

The amplitude of the process $\mathcal{A}_{AB}^{A'B'}$ can be written as the sum over permutations σ of products of colour factors and colour-independent matrix elements:

$$\mathcal{A}_{AB}^{A'B'} = \sum_{\sigma} \left(C_{AB}^{(0)\sigma} \right)^{\alpha'\beta'}_{\alpha\beta} M_{AB}^{(0)\sigma}(s,t) \ .$$

The colour factors can be decomposed into irreducible representations $\mathcal{R}$ of the colour group in the t-channel:

$$\left(C_{AB}^{(0)\sigma} \right)^{\alpha'\beta'}_{\alpha\beta} = \sum_{R} [\mathcal{P}_{AB}^{R}]^{\alpha'\beta'}_{\alpha\beta} \sum_{\sigma} \mathcal{G}(R)_{AB}^{(0)\sigma},$$

where

$$[\mathcal{P}_{AB}^{R}]^{\alpha'\beta'}_{\alpha\beta} = \sum_{n} [\mathcal{P}_{A}^{R,n}]^{\alpha'}_{\alpha} [\mathcal{P}_{B}^{R,n}]^{\beta'}_{\beta}$$

$\hat{\mathcal{P}}^{R,n}$ is the projection operator on the state n in the representation $\mathcal{R}$.

Problems

It turns out that for the representations R different from the Reggeized gluon one the colour coefficients $\mathcal{G}(R)_{AB}^{(0)\sigma}$ do not depend on σ, so that momentum dependent factors for them summed up to the eikonal amplitude

$$\sum_{\sigma} M_{AB}^{(0)\sigma}(s,t) = A^{eik} = g^6 \frac{s}{t} \left(\frac{-4\pi^2}{3} \right) \vec{q}^{\,2} A_2(q_\perp) \; ,$$

where $A_2(q_\perp)$ is depicted by the diagram

and is written as

$$A_2(q_\perp) = \int \frac{d^{2+2\epsilon} l_1 \, d^{2+2\epsilon} l_2}{(2\pi)^{2(3+2\epsilon)} \vec{l}_1^{\,2} \vec{l}_2^{\,2} (\vec{q} - \vec{l}_1 - \vec{l}_2)^2} \, .$$

This result is very important, because contribution of the cut must be gauge invariant, whereas $M_{AB}^{(0)\sigma}$ taken separately are gauge dependent.

In the Reggeized gluon channel the colour coefficients $\mathcal{G}(R)_{AB}^{(0)\sigma}$ depend on σ. However, this dependence has a specific form, such that

the terms violating the pole factorization have σ-independent colour coefficients, so that momentum factors for them summed up to the eikonal amplitude.

Problems

Separation of the pole and cut contributions is impossible in the two-loop approximation because of the ambiguity of the allocation of the part of the amplitudes violating the factorization. The separation becomes possible in higher loops, due to different energy dependence of the pole and cut contributions. Energy dependence of the pole contribution is determined by the Regge factor of the Reggeized gluon $\exp(\omega(t)\ln s)$, where $\omega(t)$ is the gluon trajectory, whereas for the three-Reggeon cut it is

$$e^{[(\hat{\omega}_1+\hat{\omega}_2+\hat{\omega}_3+\hat{\mathcal{K}}_r(1,2)+\hat{\mathcal{K}}_r(1,3)+\hat{\mathcal{K}}_r(2,3))\ln s]}\ ,$$

where $\hat{\mathcal{K}}_r(m,n)$ is the real part of the BFKL kernel describing interaction between Reggeons m and n.

Problems

With the help of the integral representation of the trajectory

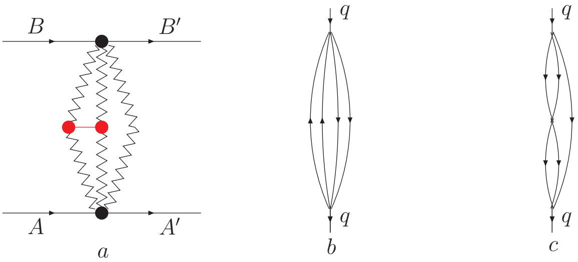
$$\omega(t) = -g^2 N_c \vec{q}^{\,2} \int \frac{d^{2+2\epsilon} l}{2(2\pi)^{(3+2\epsilon)} \vec{l}^{\,2} (\vec{q} - \vec{l})^2}$$

and the explicit form of the real part of the kernel describing interaction between two Reggeons with transverse momenta $\vec{l}_1$ and $\vec{l}_2$ and colour indices c_1 and c_2

$$\left[\mathcal{K}_r(\vec{q}_1, \vec{q}_2; \vec{k})\right]_{c_1 c_2}^{c_1' c_2'} = -T^a_{c_1 c_1'} T^a_{c_2 c_2'} \frac{g^2}{(2\pi)^{D-1}} \left[\frac{\vec{q}_1^{\,2} \vec{q}_2^{\,\prime\,2} + \vec{q}_2^{\,2} \vec{q}_1^{\,\prime\,2}}{\vec{k}^{\,2}} - \vec{q}^{\,2}\right],$$

Problems

the first order corrections are expressed through the diagrams b and c.



Problems

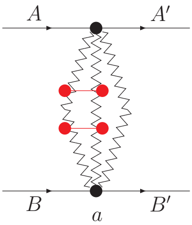
The calculation of the three-loop corrections
V.S. F., L.N. Lipatov, 2016.
shows that
all known results for violation of the pole Regge form can be explained by the pole and cut contributions.
It should be noted that this result is limited to three loops and can not be considered as a proof that in the NNLLA the only singularities in the J plane are the Regge pole and the three-Reggeon cut.
Moreover, the explanation of the violation of the pole Regge form given in
S. Caron-Huot, E. Gardi, L. Vernazza, 2017
differs from
V.S. F., L.N. Lipatov, 2016.
Besides the cut the Reggeon-cut mixing is introduced.

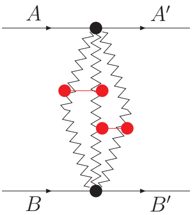
Problems

To my mind, the approach used in
S. Caron-Huot, E. Gardi, L. Vernazza, 2017
has some weak points.
– The parton-parton-cut vertex was in fact postulated, without any derivation and justification.
– No connection with Feynman diagrams was established.
– The cut contribution is not suppressed at large N_c, i.e. it exists in the planar $N = 4$ SYM, in contradiction with the common wisdom.
In our approach, the Reggeon-cut mixing is not necessary in the three-loop approximation.
Whether mixing is necessary can be verified in the four-loop approximation.

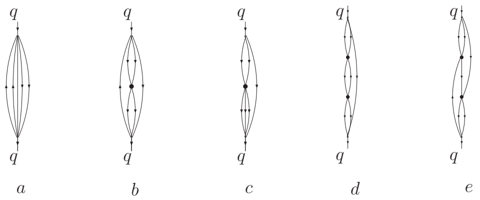
Problems

In the four loops there are three types of corrections. The first (simplest) ones come from account of the Regge factors of each of three Reggeons. The second type of the corrections are given by the products of the trajectories and real parts of the BFKL kernel, and the third come from account of Reggeon-Reggeon interactions.





All the types of the corrections are expressed through the integrals in the transverse momentum space corresponding to the diagrams



Problems

$$I_i = \int \frac{d^{2+2\epsilon} l_1 \, d^{2+2\epsilon} l_2 \, d^{2+2\epsilon} l_3}{(2\pi)^{3(3+2\epsilon)} \vec{l}_1^{\,2} \vec{l}_2^{\,2} \vec{l}_3^{\,2}} F_i \delta^{2+2\epsilon}(\vec{q} - \vec{l}_1 - \vec{l}_2 - \vec{l}_3) \,,$$

$$F_a = f_1(\vec{l}_1) f_1(\vec{l}_2) \,, \quad F_b = f_1(\vec{l}_1) f_1(\vec{l}_1) \,, \quad F_c = f_2(\vec{l}_1 + \vec{l}_2) \,,$$

$$F_d = f_1(\vec{l}_1 + \vec{l}_2) f_1(\vec{l}_1 + \vec{l}_2) \,, \quad F_e = f_1(\vec{q} - \vec{l}_1) f_1(\vec{q} - \vec{l}_3) \,,$$

$$f_1(\vec{k}) = \vec{k}^{\,2} \int \frac{d^{2+2\epsilon} l}{(2\pi)^{(3+2\epsilon)} \vec{l}^{\,2} (\vec{l} - \vec{k})^2} \,, \quad f_2(\vec{k}) = \int \frac{d^{2+2\epsilon} l f_1(\vec{l})}{(2\pi)^{(3+2\epsilon)} \vec{l}^{\,2} (\vec{l} - \vec{k})^2} \,.$$

These integrals enter in the total four-loop correction with different colour factors in the approaches with or without the Reggeon-cut mixing.

Problems

Calculation of the colour factors $\mathcal{G}(\mathbf{8_a})^{(2)\sigma}_{AB}$ shows that as well as in the two and three loops the terms violating the pole factorization have σ-independent colour coefficients.
The question of whether the four-loop amplitudes of elastic scattering in QCD are given by the Regge pole and cut contributions, with or without mixing, can be solved by comparing of these corrections with the result obtained using the infrared factorization.

Summary

- The BFKL equation was derived assuming the pole Regge form of amplitudes with gluon quantum numbers in cross channels and negative signature.
- It is proved now in all orders of perturbation theory that this form is valid both in the leading and in the next-to-leading logarithmic approximations.
- However, this form is violated in the NNLLA.
- It was shown that the observed violation can be explained by the cut contributions.
- But the assertion that the QCD amplitudes with gluon quantum numbers in cross-channels and negative signature are given in the NNLLA by the contributions of the Regge pole and the three-Reggeon cut is only a hypotheses, and as yet there is no general proof of it, it should be checked in each order of the perturbation theory.