CGAs and Invariant PDEs

Francesco Toppan

TEO, CBPF (MCTI)
Rio de Janeiro, Brazil

Forum Brazil/JINR, Dubna, June 2015

Based on:

N. Aizawa, Z. Kuznetsova and F. T., **JMP** 56, 031701 (2015).

F.T., **JoP: CS** 597, (2015) 012071.

Extra material (N. Aizawa, Z. Kuznetsova, F.T., in preparation).

Symmetries of PDEs:

$$\Omega = \partial_t + a\partial_x^2 - aV(x),$$
$$\Omega\Psi(x,t) = 0.$$

The invariant condition

$$\Omega\delta\Psi(x,t) = 0,$$
$$\delta\Psi(x,t) = f(x,t)\Psi_t(x,t) + g(x,t)\Psi_x(x,t) + h(x,t)\Psi(x,t)$$

implies

$$f_x = 0,$$
$$f_t - 2g_x + af_{xx} = 0,$$
$$g_t + a(2aVf_x + 2h_x + g_{xx}) = 0,$$
$$agV_x + h_t + 2a^2V_xf_x + aV(f_t + af_{xx}) + ah_{xx} = 0.$$

Maximal number of symmetry generators (non semi-simple) Schrödinger algebra:

i) $V(x) = x^0$ (**free** particle case),
ii) $V(x) = x^1$ (solved by **Airy** functions),
iii) $V(x) = x^2$ (**harmonic oscillator** case).

6-generator algebras $z_{\pm 1}, z_0, w_\pm, c$, with dimensions

$$[z_{\pm 1}] = \pm 1, \quad [w_\pm] = \pm\frac{1}{2}, \quad [z_0] = [c] = 0.$$
$$([t] = -1, \quad [x] = -\frac{1}{2})$$

$$\{z_0, z_{\pm 1}\} \in \mathfrak{sl}(2), \qquad \{w_\pm, c\} \in \mathfrak{h}(1),$$

$$\mathfrak{sch}(1) \;=\; \mathfrak{sl}(2) \ni\!\!\!\!+ \mathfrak{h}(1).$$

Linear Schrödinger Equation:

$$
\begin{aligned}
z_+ &= t^2\partial_t + (tx - \frac{1}{2}t^3)\partial_x + (\frac{1}{2}t - \frac{i}{8}t^4 + \frac{3i}{2}t^2x - \frac{i}{2}x^2),\\
z_0 &= t\partial_t + (\frac{1}{2}x - \frac{3}{2}t^2)\partial_x + \frac{3i}{2}tx - \frac{i}{4}t^3 + \frac{1}{4},\\
z_- &= \partial_t - t\partial_x + ix - \frac{i}{2}t^2,\\
w_+ &= t\partial_x - ix + \frac{i}{2}t^2,\\
w_- &= \partial_x + it,\\
c &= 1.
\end{aligned}
$$

$$
\begin{aligned}
[z_0, z_\pm] &= \pm z_\pm,\\
[z_+, z_-] &= -2z_0,\\
[z_0, w_\pm] &= \pm\tfrac{1}{2}w_\pm,\\
[z_\pm, w_\mp] &= \mp w_\pm,\\
[w_+, w_-] &= ic.
\end{aligned}
$$

Non-vanishing on-shell relations

$$\begin{array}{rcl} [z_+, \Omega] &=& -2t\Omega, \\ [z_0, \Omega] &=& -\Omega. \end{array}$$

$\mathfrak{sl}(2)$ algebra produced by the $\Omega_0, \Omega_{\pm 1}$ invariant PDEs:

$$\Omega_{-1} = \Omega, \quad [z_+, \Omega_{-1}] = \Omega_0, \quad [z_+, \Omega_0] = \Omega_{+1}, \quad ([z_+, \Omega_{+1}] = 0)$$

$$\Omega_{-1} = \Omega = iz_- + \frac{1}{2}w_-^2, \quad \Omega_0 = -2t\Omega_{-1}, \quad \Omega_{+1} = -2t^2\Omega_{-1}.$$

$$\begin{array}{rcl} [\Omega_0, \Omega_{\pm 1}] &=& \mp 2i\Omega_{\pm 1}, \\ [\Omega_+, \Omega_-] &=& 2i\Omega_0. \end{array}$$

The Hamiltonian is

$$H = -\frac{1}{2}{\partial_x}^2 + x = -\frac{1}{2}{w_-}^2 + iw_+.$$

Free case:

$$
\begin{aligned}
z_{+1} &= \partial_t, \\
z_0 &= t\partial_t + \frac{1}{2}x\partial_x + \frac{1}{4}, \\
z_{-1} &= t^2\partial_t + tx\partial_x - \frac{ix^2}{4} + \frac{1}{2}t, \\
w_+ &= \partial_x, \\
w_- &= t\partial_x - \frac{ix}{2}, \\
c &= 1.
\end{aligned}
$$

Harmonic oscillator:

$$
\begin{aligned}
z_{+1} &= e^{2it}\left(\partial_t + ix\partial_x + \frac{i}{2} - ix^2\right), \\
z_0 &= \partial_t, \\
z_{-1} &= e^{-2it}\left(\partial_t - ix\partial_x - \frac{i}{2} - ix^2\right), \\
w_+ &= e^{it}\left(\partial_x - x\right), \\
w_- &= e^{-it}\left(\partial_x + x\right), \\
c &= 1.
\end{aligned}
$$

1^{st} and 2^{nd} order differential operators:

$$
\begin{aligned}
w_{+1} &= \{w_+, w_+\}, \\
w_0 &= \{w_+, w_-\}, \\
w_{-1} &= \{w_-, w_-\}.
\end{aligned}
$$

Algebra-superalgebra duality:

• the non-simple Lie algebra $\mathfrak{esch}$ with 9 generators $\{z_{\pm1}, z_0, w_\pm, w_{\pm1}, w_0, c\}$ and

• the Lie superalgebra $\mathfrak{ssch} = S_0 \oplus S_1$, with 7 even ($z_{\pm1}, z_0, w_{\pm1}, w_0, c \in S_0$) and 2 odd generators ($w_\pm \in S_1$).

In $\mathfrak{ssch}$ we have the anti-commutators

$$
\{w_+, w_+\} = w_{+1}, \quad \{w_+, w_-\} = w_0, \quad \{w_-, w_-\} = w_{-1}.
$$

$$
\begin{aligned}
x^0 : \quad \Omega &= z_{+1} + \frac{1}{2} a w_{+1}, \\
x^1 : \quad \Omega &= z_{-1} + \frac{a}{2} w_{-1}, \\
x^2 : \quad \Omega &= z_0 + \frac{a}{2} w_0 .
\end{aligned}
$$

$V(x) = x^0$ example: all commutators with Ω are vanishing, apart

$$
\begin{aligned}
[z_0, \Omega] &= -z_{+1} - \frac{a}{2} w_{+1}, \\
[z_{-1}, \Omega] &= -2z_0 - a w_0 .
\end{aligned}
$$

Representation-dependent formulas for the commutators above:

$$
\begin{aligned}
[z_0, \Omega] &= -\Omega, \\
[z_{-1}, \Omega] &= -2t\Omega .
\end{aligned}
$$

Representation-dependent commutators (on-shell condition):

$$
[g, \Omega] = f_g \cdot \Omega .
$$

$d = 1$ $\ell = \frac{1}{2} + \mathbb{N}_0$ Centrally Extended CGAs

$$\mathfrak{cga}_\ell \quad = \quad \mathfrak{sl}(2) \niplus \mathfrak{h}(\ell + \tfrac{1}{2}).$$

Features:

- Spin ℓ representation of $\mathfrak{sl}(2)$.
- $\ell + \frac{1}{2}$ copies of Heisenberg algebras.
- $\mathbb{Z}_2$-grading.
- Schrödinger algebra recovered at $\ell = \frac{1}{2}$.

Generators:

$\mathfrak{sl}(2)$: $z_\pm, z_0$,
central charge: c ,
creation/annihilation operators: w_j with $j = -\ell, \ell + 1, \ldots, \ell$.

Inverse problem: diff. realizations are given. Are there invariant PDEs?

Differential realizations from

$$e^{\mathcal{G}_-} e^{\mathcal{G}_0} e^{\mathcal{G}_+} |l.w.r >$$

Invariant PDEs:

- Verma modules (difficult),

- on-shell condition (easy).

$d=1$ $\ell=\frac{3}{2}$ - deformation of the free system:

$$
\begin{aligned}
\overline{z}_{+} &= \partial_\tau,\\
\overline{z}_{0} &= -2i\tau\partial_\tau - ix\partial_x - 3iy\partial_y - 2i,\\
\overline{z}_{-} &= -4\tau^2 - 4(\tau x - \frac{3}{\gamma}y)\partial_x - 12\tau y\partial_y - 8(\tau - ix^2),\\
\overline{w}_{+3} &= \partial_y,\\
\overline{w}_{+1} &= -2i\tau\partial_y + \frac{2i}{\gamma}\partial_x,\\
\overline{w}_{-1} &= -4\tau^2\partial_y + \frac{8}{\gamma}\tau\partial_x - \frac{8i}{\gamma}x,\\
\overline{w}_{-3} &= 8i\tau^3\partial_y - \frac{24i}{\gamma}\tau^2\partial_x - 48(\frac{1}{\gamma}\tau x + \frac{1}{\gamma^2}y),\\
\overline{c} &= 1.
\end{aligned}
$$

Algebra:

$$
\begin{array}{rcl}
[\overline{z}_0, \overline{z}_\pm] &=& \pm 2i\overline{z}_\pm, \\
[\overline{z}_+, \overline{z}_-] &=& -4i\overline{z}_0, \\
[\overline{z}_0, \overline{w}_k] &=& i\frac{k}{2}\overline{w}_k, \\
[\overline{z}_\pm, \overline{w}_k] &=& (k \mp 3)i\overline{w}_{k\pm 2}, \\
[\overline{w}_{|k|}, \overline{w}_{-|k|}] &=& (3-2k)\frac{16}{\gamma^2}c.
\end{array}
$$

$$\begin{aligned}
\overline{\Omega}_{+1} &= i\partial_\tau - i\gamma\partial_y + \frac{1}{2}\partial_x^2 = i\overline{z}_+ - \overline{H}_+ = i\overline{z}_+ + \frac{\gamma^2}{16}\left(\{\overline{w}_{+3},\overline{w}_{-1}\} - \{\overline{w}_{+1},\overline{w}_{+1}\}\right),\\
\overline{\Omega}_0 &= -2i\tau\overline{\Omega}_{+1} = i\overline{z}_0 - \overline{H}_0 = i\overline{z}_0 + \frac{\gamma^2}{32}\left(\{\overline{w}_{+3},\overline{w}_{-3}\} - \{\overline{w}_{+1},\overline{w}_{-1}\}\right),\\
\overline{\Omega}_{-1} &= -4\tau^2\overline{\Omega}_{+1} = i\overline{z}_- - \overline{H}_- = i\overline{z}_- + \frac{\gamma^2}{16}\left(\{\overline{w}_{+1},\overline{w}_{-3}\} - \{\overline{w}_{-1},\overline{w}_{-1}\}\right).
\end{aligned}$$

Non-vanishing (on-shell invariant) commutators involving the $\overline{\Omega}$'s:

$$\begin{aligned}
[\overline{z}_0,\overline{\Omega}_{+1}] &= 2i\overline{\Omega}_{+1},\\
[\overline{z}_-,\overline{\Omega}_{+1}] &= 4i\overline{\Omega}_0 = 8\tau\overline{\Omega}_{+1},\\
[\overline{z}_+,\overline{\Omega}_0] &= -2i\overline{\Omega}_{+1} = \tau^{-1}\overline{\Omega}_0,\\
[\overline{z}_-,\overline{\Omega}_0] &= 2i\overline{\Omega}_{-1} = 4\tau\overline{\Omega}_0,\\
[\overline{z}_+,\overline{\Omega}_{-1}] &= -4i\overline{\Omega}_0 = 2\tau^{-1}\overline{\Omega}_{-1},\\
[\overline{z}_0,\overline{\Omega}_{-1}] &= -2i\overline{\Omega}_{-1}.
\end{aligned}$$

$$\begin{aligned}
[\overline{\Omega}_0,\overline{\Omega}_{\pm 1}] &= \mp 2\overline{\Omega}_{\pm 1},\\
[\overline{\Omega}_{+1},\overline{\Omega}_{-1}] &= 4\overline{\Omega}_0.
\end{aligned}$$

$d = 1$ $\ell = \frac{3}{2}$ - deformation of the "oscillator" system:

$$
\begin{array}{rcl}
z_0 &=& \partial_t, \\
z_+ &=& e^{2it}(\partial_t + ix\partial_x + 3iy\partial_y + ix^2 + 2i), \\
z_- &=& e^{-2it}(\partial_t - ix\partial_x - 3iy\partial_y + \frac{12}{\gamma}y\partial_x + 7ix^2 + \frac{12}{\gamma}xy - 2i), \\
w_{+3} &=& e^{3it}\partial_y, \\
w_{+1} &=& e^{it}(\partial_y + \frac{2i}{\gamma}\partial_x + \frac{2i}{\gamma}x), \\
w_{-1} &=& e^{-it}(\partial_y + \frac{4i}{\gamma}\partial_x - \frac{4i}{\gamma}x), \\
w_{-3} &=& e^{-3it}(\partial_y + \frac{6i}{\gamma}\partial_x - \frac{18i}{\gamma}x - \frac{48i}{\gamma^2}y), \\
c &=& 1.
\end{array}
$$

$$
\begin{aligned}
\Omega_{+1} &= e^{2it}\Omega_0 = iz_+ - H_+ = iz_+ + \frac{\gamma^2}{16}\left(\{w_{+3}, w_{-1}\} - \{w_{+1}, w_{+1}\}\right),\\
\Omega_0 &= i\partial_t + \frac{1}{2}{\partial_x}^2 - \frac{1}{2}x^2 - 3y\partial_y - i\gamma x\partial_y - \frac{3}{2} =\\
&= iz_0 - H_0 = iz_0 + \frac{\gamma^2}{32}\left(\{w_{+3}, w_{-3}\} - \{w_{+1}, w_{-1}\}\right),\\
\Omega_{-1} &= e^{-2it}\Omega_0 = iz_- - H_- = iz_- + \frac{\gamma^2}{16}\left(\{w_{+1}, w_{-3}\} - \{w_{-1}, w_{-1}\}\right).
\end{aligned}
$$

$$
\begin{aligned}
[z_0, \Omega_{+1}] &= 2i\Omega_{+1},\\
[z_-, \Omega_{+1}] &= 4i\Omega_0 = 4ie^{-2it}\Omega_{+1},\\
[z_+, \Omega_0] &= -2i\Omega_{+1} = -2ie^{2it}\Omega_0,\\
[z_-, \Omega_0] &= 2i\Omega_{-1} = 2ie^{-2it}\Omega_0,\\
[z_+, \Omega_{-1}] &= -4i\Omega_0 = -4ie^{2it}\overline{\Omega}_{-1},\\
[z_0, \Omega_{-1}] &= -2i\Omega_{-1};\\
[\Omega_0, \Omega_{\pm1}] &= \mp 2\Omega_{\pm1},\\
[\Omega_{+1}, \Omega_{-1}] &= 4\Omega_0.
\end{aligned}
$$

Connection of the two systems:

$g \mapsto \overline{g}$ via similarity transform. and $t \mapsto \tau = \frac{i}{2}e^{-2it}$ redefinition of time.

To the "oscillatorial" D-module rep $z_\pm = e^{\pm 2it}(\partial_t + X_\pm), \quad z_0 = \partial_t$
we apply the similarity transformation

$$\begin{aligned} g \mapsto \overline{g} &= e^S g e^{-S}, \qquad (e^S = e^{S_2}e^{S_1}), \\ S_1 &= tX_+, \\ S_2 &= \frac{1}{2}x^2 \end{aligned}$$

Explanation: $z_+ \mapsto \hat{z}_+ = e^{S_1} z_+ e^{-S_1} = e^{2it}\partial_t = \partial_\tau,$

$$\begin{aligned} \Omega_{+1} \mapsto \widehat{\Omega}_{+1} &= e^{S_1}\Omega_{+1}e^{-S_1} = ie^{2it}\partial_t - \widehat{H}_{+1}, \\ \widehat{H}_{+1} &= e^{2it}\left(iX_+ + e^{tX_+}H_0 e^{-tX_+}\right). \end{aligned}$$

Magic identity: $[X_+, H_0] = 2iK_+$, $[X_+, K_+] = -2iK_+ \Rightarrow$
$\Rightarrow iX_+ + H_0 + K_+ = 0 \Rightarrow \widehat{H}_{+1}$ **does not depend on time.**

The commutative diagram

$$\begin{array}{lccc} \textit{coupled}\ (\gamma\neq 0): & \mathbf{Free}_{\gamma}^{0,\pm 1}(\tau) & \overset{\mathbf{S}}{\longleftrightarrow} & \mathbf{Osc}_{\gamma}^{0,\pm 1}(t) \\ & {}^{\mathbf{r}}\downarrow & & \downarrow^{\mathbf{r}} \\ \textit{decoupled}\ (\gamma=0): & \mathbf{Free}^{0,\pm 1}(\tau) & \overset{\mathbf{S}}{\longleftrightarrow} & \mathbf{Osc}^{0,\pm 1}(t) \end{array}$$

• left: equations from the "free" *D*-module rep.

• right: equations obtained from the "oscillator" *D*-module rep.

Each three invariant PDEs (deg $0, \pm 1$) are mapped into each other.

Left: Schrödinger-type invariant PDE corresponds to *deg* 1 and possesses a continuous spectrum.

Right: Schrödinger-type invariant PDE corresponds to *deg* 0 and possesses a discrete spectrum.

Vertical arrows: $g \mapsto RgR^{-1}$, $R = e^{\alpha y \partial_y}$.

Therefore $\gamma \to e^{-\alpha}\gamma$. The $\alpha \to \infty$ limit is singular.

Four Schrödinger-type equations:

- deformed oscillator:

$$\Omega_0(\gamma)\Psi(t,x,y)=0 \quad \equiv \quad (i\partial_t+\frac{1}{2}\partial_x^2-\frac{1}{2}x^2-3y\partial_y-i\gamma x\partial_y-\frac{3}{2})\Psi(t,x,y);$$

- decoupled oscillator:

$$\Omega_0\Psi(t,x,y)=0 \quad \equiv \quad (i\partial_t+\frac{1}{2}\partial_x^2-\frac{1}{2}x^2-3y\partial_y-\frac{3}{2})\Psi(t,x,y);$$

- free equation:

$$\overline{\Omega}_1\Psi(\tau,x,y)=0 \quad \equiv \quad (i\partial_\tau+\frac{1}{2}\partial_x^2)\Psi(\tau,x,y)=0;$$

- deformed free equation:

$$\overline{\Omega}_1(\gamma)\Psi(\tau,x,y)=0 \quad \equiv \quad (i\partial_\tau+\frac{1}{2}\partial_x^2+i\gamma x\partial_y)\Psi(\tau,x,y)=0.$$

Fundamental domain: $\gamma \in]0, +\infty[$

- Left PDEs: hermitian.
- Right PDEs: non-hermitian.

All operators $K = -\frac{1}{2}\partial_x^2 + \frac{1}{2}x^2 + \omega y\partial_y - i\gamma x\partial_y + C$, $\forall C$ and $\forall \gamma \neq 0$ correspond to $\ell = \frac{3}{2}$ CGA invariant PDE if

$$\omega = \pm\frac{1}{3}, \pm 3$$

The $\omega \leftrightarrow -\omega$ change of sign explained by the hermitian conjugation.
The $\omega \leftrightarrow \frac{1}{\omega}$ transformation explained by $x \leftrightarrow y$ exchange.
Explicit check: to get $z_+ = e^{i\lambda t}\partial_t + \ldots$, the following necessary (and sufficient) condition has to be satisfied:

$$\begin{aligned} \lambda(\omega^2 + 1 - \frac{5}{2}\lambda^2) &= 0, \\ -3\lambda^2 + 3\lambda^4 + 2\lambda\omega + 4\lambda^3\omega - \lambda^2\omega^2 - 2\lambda\omega^3 &= 0. \end{aligned}$$

All three equations in $1+1$ invariant under the Schrödinger algebra are recovered as contractions of the $\ell = \frac{3}{2}$ invariant PDEs.

Symmetries of two decoupled oscillators

Without loss of generality, $\omega \geq 1$, for

$$\Omega = i\partial_t + \frac{1}{2}\partial_x^2 - \frac{1}{2}x^2 + \omega y \partial_y .$$

At generic ω, nine invariant operators can be encountered at degree $0, \pm\frac{1}{2}, \pm\frac{\omega}{2}, \pm 1$ ($\overline{d}$ is the degree operator):

$$\begin{aligned}
\overline{z}_\pm &= e^{\pm 2it}(\partial_t \pm ix\partial_x + i\omega y\partial_y + ix^2 \pm \tfrac{i}{2}),\\
\overline{z}_0 &= \partial_t + i\omega y\partial_y,\\
\overline{d} &= -\frac{i}{2}\partial_t,\\
\overline{c} &= 1,\\
\overline{w}_\omega &= e^{i\omega t}\partial_y,\\
\overline{w}_1 &= e^{it}(\partial_x + x),\\
\overline{w}_{-1} &= e^{-it}(\partial_x - x),\\
\overline{w}_{-\omega} &= e^{-i\omega t}y.
\end{aligned}$$

The 9-generator algebra closes the $\mathfrak{u}(1)\niplus(\mathfrak{sch}(1)\oplus\mathfrak{h}(1))$ algebra.

Enhanced symmetry at the critical values $\omega = 1$ and $\omega = 3$.

• $\omega = 3$: three extra generators $\overline{r}_{-j}$, $j = 1, 2, 3$, of degree $-j$,

$$\begin{aligned} \overline{r}_{-1} &= e^{-2it} y(\partial_x + x), \\ \overline{r}_{-2} &= e^{-4it} y(\partial_x - x), \\ \overline{r}_{-3} &= e^{-6it} y^2. \end{aligned}$$

• $\omega = 1$: three extra generators at degree 0 and -1:

$$\begin{aligned} q_1 &= y(\partial_x + x), \\ q_2 &= e^{-2it} y^2, \\ q_3 &= e^{-2it} y(\partial_x - x). \end{aligned}$$

In both cases, we obtain a (different) 12-generator closed symmetry algebra.

The contraction algebra

In the $\gamma \to 0$ limit, a contraction algebra is recovered by rescaling the generators ($g \mapsto \widetilde{g} = \gamma^s g$), where

$$
\begin{array}{rcl}
s=0 & : & z_0, z_+, w_3, c, \\
s=1 & : & z_-, w_1, w_{-1}, \\
s=2 & : & w_{-3}.
\end{array}
$$

$$
\begin{array}{rcl}
\widetilde{z}_+ & = & e^S \overline{z}_+ e^{-S} = e^{2it}(\partial_t + ix\partial_x + 3iy\partial_y + ix^2 + 2i), \\
\widetilde{z}_0 & = & e^S (az_0 + bd) e^{-S} = \partial_t, \\
\widetilde{z}_- & = & e^S (12i\overline{r}_{-1}) e^{-S} = 12ie^{-2it} y(\partial_x + x), \\
\widetilde{w}_{+3} & = & e^S \overline{w}_{+3} e^{-S} = e^{3it}\partial_y, \\
\widetilde{w}_{+1} & = & e^S (-2\overline{w}_{+1}) e^{-S} = -2e^{it} y(\partial_x + x), \\
\widetilde{w}_{-1} & = & e^S (-4\overline{w}_{-1}) e^{-S} = -4e^{-it} y(\partial_x - x), \\
\widetilde{w}_{-3} & = & e^S (48\overline{w}_{-3}) e^{-S} = 48e^{-3it} y, \\
\widetilde{c} & = & e^S \overline{c} e^{-S} = 1.
\end{array}
$$

The contraction algebra is $\mathfrak{e}(2) \Supset \mathfrak{h}(2)$.

Cryptohermiticity

Two types of operators, same CCR, but different conjugation properties

$$K(\overline{\gamma}) = a^\dagger a + 3b^\dagger b + \frac{1}{2} + \overline{\gamma}(a + a^\dagger)b.$$

$$[a, a^\dagger] = [b, b^\dagger] = 1 \quad .$$

The operator $K(\overline{\gamma})$ acts on the Hilbert space $\mathcal{L}^2(\mathbb{R}^2)$, spanned by the (unnormalized) states $|n, m >= (a^\dagger)^n (b^\dagger)^m |vac >$, where $|vac >\equiv |0, 0 >$ is the Fock vacuum ($a|vac >= b|vac >= 0$).

Excitation mode creation $[K(\overline{\gamma}), A_\lambda] = \lambda A_\lambda$.
For any $\overline{\gamma} \neq 0$, $\lambda = \pm 3, \pm\frac{1}{3}$:

$$\begin{aligned} A_{-3} &= b, \\ A_{-1} &= a + \frac{1}{2}\overline{\gamma} b, \\ A_{+1} &= a^\dagger - \frac{1}{4}\overline{\gamma} b, \\ A_{+3} &= b^\dagger - \frac{1}{2}\overline{\gamma} a^\dagger - \frac{1}{4}\overline{\gamma} a + \frac{1}{24}\overline{\gamma}^2 b. \end{aligned}$$

Non-vanishing commutators

$$[A_{-i}, A_j] = \delta_{ij}, \qquad (i, j = 1, 3).$$

The non-hermitian operator $K(\overline{\gamma})$ commutes with the "non-hermitian analog of the Number operator", $N(\overline{\gamma})$.

$$\begin{aligned} K(\overline{\gamma}) &= 3A_3A_{-3} + A_1A_{-1} + \frac{1}{2}, \\ N(\overline{\gamma}) &= A_3A_{-3} + A_1A_{-1}, \\ [K(\overline{\gamma}), N(\overline{\gamma})] &= 0. \end{aligned}$$

The Fock vacuum $|vac>$ satisfies

$$a|vac>= b|vac>= 0 \quad , \quad A_{-1}|vac>= A_{-3}|vac>= 0.$$

The Hilbert space $\mathcal{L}^2(\mathbb{R}^2)$ can be spanned by both sets of (unnormalized) states,

$$\begin{aligned} |n, m> &= (a^\dagger)^n (b^\dagger)^m |vac>, \\ |\overline{n}, \overline{m}> &= A_1^n A_3^m |vac>, \end{aligned}$$

so that $|vac>= |0, 0>= |\overline{0}, \overline{0}>$.

The spectrum of $K(\overline{\gamma})$, $N(\overline{\gamma})$ is real, discrete and bounded. It coincides with the spectrum of the Hamiltonian and Number operator of the decoupled harmonic oscillators.

$|\overline{n},\overline{m}>$ is an eigenvector for $K(\overline{\gamma})$, $N(\overline{\gamma})$ The respective eigenvalues are $n+3m+\frac{1}{2}$ and $n+m$.

$$
\begin{aligned}
(\frac{1}{2},0): &\qquad |\overline{0},\overline{0}>=|0,0>=|vac>,\\
(\frac{3}{2},1): &\qquad |\overline{1},\overline{0}>=|1,0>,\\
(\frac{5}{2},2): &\qquad |\overline{2},\overline{0}>=|2,0>,\\
(\frac{7}{2},1): &\qquad |\overline{0},\overline{1}>=|0,1>-\frac{1}{2}\overline{\gamma}|1,0>,\\
(\frac{7}{2},3): &\qquad |\overline{3},\overline{0}>=|3,0>,\\
(\frac{9}{2},2): &\qquad |\overline{1},\overline{1}>=|1,1>-\frac{1}{2}\overline{\gamma}|2,0>-\frac{1}{4}\overline{\gamma}|0,0>,\\
(\frac{9}{2},4): &\qquad |\overline{4},\overline{0}>=|4,0>.
\end{aligned}
$$

Since the operators are non-hermitian, their eigenvectors are non-orthogonal.

Physical consequence. Let us suppose we prepare a system in a given common eigenvector of $K(\overline{\gamma})$, $N(\overline{\gamma})$, let's say the state $|\overline{1},\overline{1}>$. We can compute for instance compute the probability that in a measurement the state can be found in the vacuum state. A simple quantum mechanical computation gives for this probability $p=|_N<\overline{1},\overline{1}|\overline{0},\overline{0}>_N|^2$ ($|\overline{0},\overline{0}>_N$, $|\overline{1},\overline{1}>_N$ are the normalized states). We obtain

$$p = \frac{\overline{\gamma}^2}{16+9\overline{\gamma}^2}.$$

This probability is restricted in the range $0\leq p<\frac{1}{9}<1$. The parameter $\overline{\gamma}^2$ has measurable consequence.

Any $\ell = \frac{1}{2} + \mathbb{N}_0$:

$$i\partial_\tau \Psi(\tau, \mathbf{x}_j) = \mathbf{H}^{(\ell)} \Psi(\tau, \mathbf{x}_j),$$

for the ℓ-oscillator

$$\mathbf{H}^{(\ell)} = -\frac{1}{2m}\partial^2_{\mathbf{x}_1} + \frac{m}{2}{\mathbf{x}_1}^2 + \sum_{j=1}^{\ell-\frac{1}{2}} \left((2j+1)\mathbf{x}_{j+1}\partial_{\mathbf{x}_{j+1}} - i\gamma_j \mathbf{x}_j \partial_{\mathbf{x}_{j+1}}\right) + C$$

Spectrum:

$$E_{\vec{n}} = \sum_{j=1}^{\ell+\frac{1}{2}} \omega_j n_j + \omega_0, \quad \text{with} \quad \omega_j = (2j-1).$$

Energy modes: $1, 3, 5, 7, \ldots$.
For $\ell = \frac{5}{2}$: $\pm 1, \pm 3, \pm 5$; sign flipping for generic ℓ? Pais-Uhlenbeck Hamiltonian: $\omega_1, -\omega_2, \omega_3, -\omega_4, \ldots$ ($\omega_j \in \mathbb{R}^+$, $\quad \omega_{j+1} > \omega_j$).

Spectrum generating off-shell invariant algebra:

$B(0,n) = osp(1|2n)$ superalgebra in terms of n bosonic generators,

$$n = \ell + \frac{1}{2}.$$

• First-order differential operators w_j
(n copies of creation/annihilation operators).
• Second-order differential operators $\{w_i, w_j\} = w_{i,j}$
closing the $sp(2\ell+1)$ algebra, while $w_{i,j}, w_j$ close $osp(1|2\ell+1)$.

In the $\ell \to \infty$ limit we obtain $sp(\infty)$.

The spectrum is recovered from $osp(1|2n)$ l.w.r. .

Conclusions

Boring extensions: centrally extended CGAs $\ell \in \frac{1}{2} + \mathbb{N}_0$, $d > 1$: several copies of the $d = 1$ systems at given ℓ.

New Features: the exceptional centrally extended CGAs at $d = 2$ and $\ell \in \mathbb{N}_0$ ($\ell = 1, 2, 3, \ldots$).

Future investigations:

- connection with higher spin theory,
- non-linear Equations invariant under CGAs,
- Virasoro-Galilei algebra (BMS, asymptotic symmetries and soft theorems),
- ...

Thanks for the attention!