Knots: from Hadron Physics to Biology

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Based on the collaboration of groups at

- International Institute of Physics – UFRN (Natal, Brazil)
- Institute for Theoretical and Experimental Physics (Moscow, Russia)
- other institutions worldwide

Series of international events

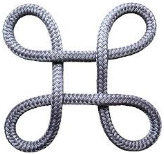
- workshop “Strings, knots and related aspects", Nov’13 – Natal
- conference “Group theory and knots", Nov’14 – Natal
- workshop “Group theory and knots III", 2015 – ???
- conference “Group theory and knots IV", 2016 – Natal
- conference “Group theory and knots IV", 2017 – Russia???

Sponsors

- UFRN (FUNPEC)
- CNPq – MCTI
- CAPES – MEC
- RFBR
- binational cooperation ???

Outline

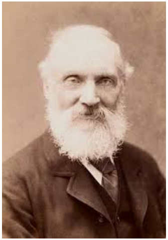
1. Historical overview
2. Invariants
3. Beyond torus knots
4. Other directions



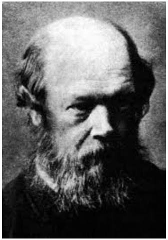
Historical Overview

First Mention

Atoms as knots



In 1867 lord Kelvin put forward the idea that atoms may represent knotted vortices of aether



- Systematic study and first classification of knots by Tait

Knots from the second part (1884) of Tait's trailblazing paper.

Many Years Later

Hadrons as knots

Faddeev and Niemi considered the idea that glueballs of QCD may correspond to knotted configurations of the QCD flux tubes

- Predictions of the glueball spectrum (have not been verified)

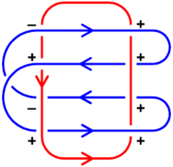
Other applications

- Vortices in liquid helium III
- Soft matter (DNA/RNA/proteins)

Mathematics

Classification of knots

The first knot (link) invariant was introduced by Gauss (linking number)



$$\mathcal{L}(\gamma_1, \gamma_2) = \frac{1}{4\pi} \oint_{\gamma_1} \oint_{\gamma_2} \frac{\mathbf{r}_1 - \mathbf{r}_1}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \, d\mathbf{r}_1 \times d\mathbf{r}_2$$

- 1923 – Alexander introduces the first knot polynomial (invariant with integer coefficients)
- 1984 – Jones introduces the second polynomial

Jones polynomials is not enough to distinguish an arbitrary pair of knots

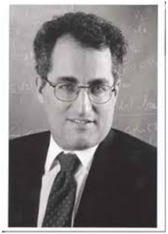
Breakthrough

Topological quantum field theories

Chern-Simons theory $$\frac{k}{4\pi}\int d^3x\, \epsilon^{ijk}\mathrm{Tr}\left(A_i\partial_j A_k + \frac{2}{3}A_iA_jA_k\right)$$

$$\left\langle \mathrm{Tr}_R\, \mathrm{P}\exp\left(\oint_{C(\mathcal{K})} A_\mu\, dx^\mu\right)\right\rangle_k = \mathcal{J}(\mathcal{K}|k,R)$$

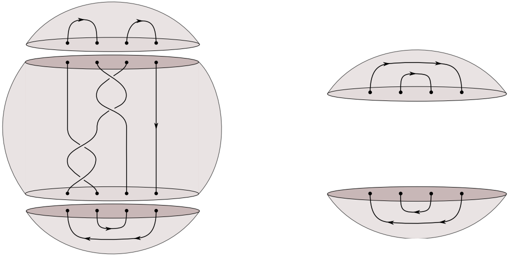
Expectation value of the Wilson along the knot $\mathcal{K}$ in $SU(2)$ level k Chern-Simons theory on S^3 is equal to the Jones polynomial of $\mathcal{K}$



$$\mathcal{J}(\;|k,\Box) = -q^4 + q^3 + q\,, \qquad q = \exp\left(\frac{2\pi i}{k+2}\right)$$

Breakthrough

Conformal field theories



$$= \langle \Psi |$$

$$= | \Psi \rangle$$

Quantum states in the TQFT = conformal blocks of CFT

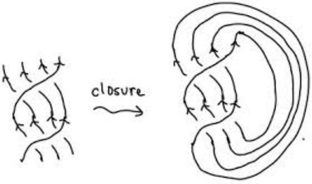
$$\langle \prod_{i=1}^{n} \phi_{R_i}(z_i) \rangle = \sum_{\lambda} |\mathcal{F}_{\lambda;R_1,\ldots,R_n}(z_1,\ldots,z_n)|^2$$

Breakthrough

Integrable models

- Knot invariants can be constructed as traces of certain representation of braid groups

$$\mathrm{Inv}\mathcal{K} = \mathrm{Tr}_{\mathcal{R}}\beta \qquad \beta \in B_n$$



- Yang-Baxter equation

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} \qquad b \in B_n$$

Invariants

New Invariants

HOMFLY

$$\mathcal{H}_R(\mathcal{K}, G) = \left\langle \mathrm{Tr}_R \, \mathrm{P} \exp\left(\int_{C(\mathcal{K})} A_\mu \, dx^\mu \right) \right\rangle_k$$

- $N = 2$, $R = \Box$ – Jones polynomials
- $N = 2$, general R – colored Jones
- general N, $R = \Box$ – HOMFLY
- general N, general R – colored HOMFLY

$$\mathcal{J}(q) = \mathcal{H}(q, a = q^2), \qquad a = q^N$$

It is believed that HOMFLY are general enough to distinguish any two knots if R is *large* enough

New Invariants

Superpolynomials

All the above polynomials can be obtained as Euler characteristics of certain complexes of vector spaces

There is a yet more general type of invariants which arise from the homologies of those complexes (Khovanov-Rozansky homologies), associated to knots – superpolynomials $P(q, a, t)$.

$$H(q, a) = P(q, a, t = -1)$$

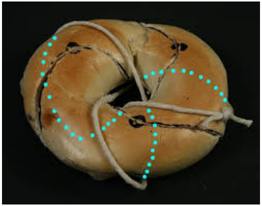
Which topological theory has $P(q, a, t)$ as its amplitudes?

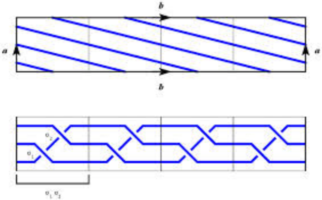
Beyond Torus Knots

Torus Knots

Torus knots

Torus knots can be drawn on the surface of a torus without self-intersection.
Parameterized by mutually prime numbers (m, n)





Torus Knots

General Results

- For torus knots one can derive general formulae for the invariants

$$H_R(\mathcal{K}_{m,n}) = \sum_{R'} c^R_{R',m} q^{n/m C_2(R')} \dim_q R'$$

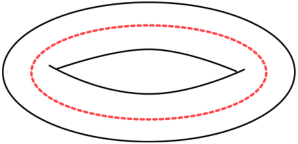
- Matrix model

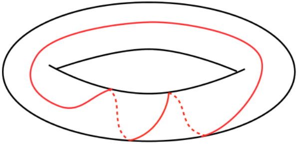
$$H_R(\mathcal{K}_{m,n}) = \frac{1}{Z_{m,n}} \int du\, e^{-u^2/2gmn} \prod_{\alpha>0} 4 \sinh\left(\frac{u\cdot\alpha}{2m}\right) \sinh\left(\frac{u\cdot\alpha}{2n}\right) \mathrm{ch}_R(e^u)$$

- DAHA (Cherednik) construction

Torus Knots

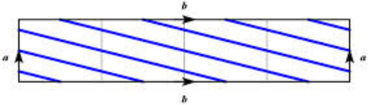
Modular transformations





Torus knots transform one into another via $PSL(2,Z)$ transformations

$$\begin{pmatrix} m & \star \\ n & \star \end{pmatrix} = \cdots \tau_+^{b_2}\tau_-^{a_2}\tau_+^{b_1}\tau_-^{a_1}$$



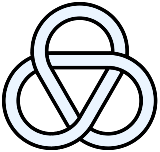
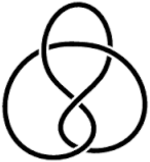
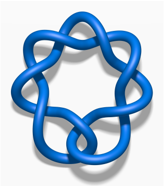
where $a_i, b_i \in \mathbb{Z}_+$

$$\tau_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \tau_- = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

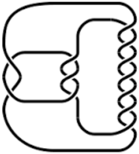
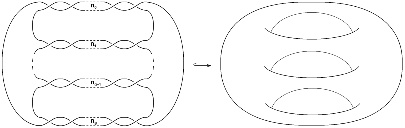
Beyond Torus

Other families

- Twist knots

- Pretzel knots

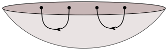
- ???

Can we extend the general results for torus knots to other families?

Other Directions

Quantum Hall Effect

Wavefunctions

Conformal blocks $\Leftrightarrow$ Chern-Simons  $\Leftrightarrow$ QHE wave-functions

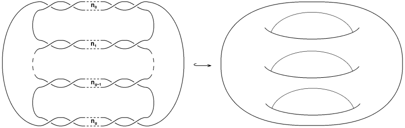
- $\nu = 5/2$ state is believed to be a Moore-Read state
- Jack polynomials are extensively studied as candidates of excited state wavefunctions

Quantum Information

- It is natural to invoke topology in the discussion of quantum computers as topology may protect from quantum decoherence

- Majorana Fermions can realize representations of the braid group. *(Special thanks to F.Toppan)*

- Can knot invariants compute entanglement entropy?



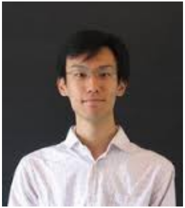
Higher Dimensions

AGT correpondense

$$\text{CFT} \quad \longrightarrow \quad \text{Chern-Simons} \quad \longrightarrow \quad \mathcal{N} = 2 \text{ SYM} \quad \longrightarrow \quad \cdots$$

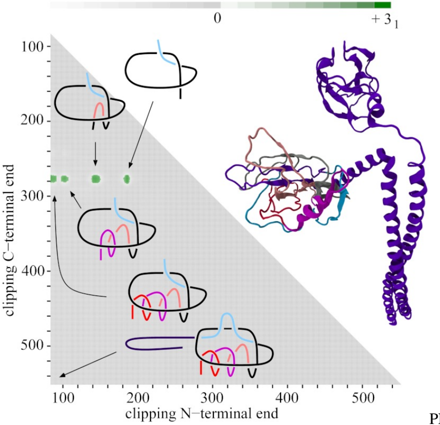
Conformal blocks (of CFT) are fundamental objects in these theories

- Knot invariants can be constructed with the use of Liouville conformal blocks
- Integrability story behind
- New interesting way knot invariants can enter the game

Protein folding

In the process of folding proteins can form topologically non-trivial configurations. Are those topologically protected?



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Conclusions