Higher spin fields in twistor formulation

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This talk provides a brief description of $D\!=\!3$ and $D\!=\!4$ world-line models for massive particles moving in a new type of enlarged spacetime, with $D\!-\!1$ additional vector coordinates, which after quantization lead to towers of massive higher spin (HS) fields.

Plan of my talk:

- It will be presented the main features of the twistorial formulation of the massless higher-spin fields theory and the required changes for the massive case.
- Two classically equivalent formulations will be presented: one with a hybrid spacetime/bispinor variables and a second described by a free two-twistor dynamics with constraints.
- After first quantization of the pure twistor models in the $D\!=\!3$ and $D\!=\!4$ cases, we will obtain that the massive wave functions are given as functions on the $SL(2,\mathbb{R})$ and $SL(2,\mathbb{C})$ group manifolds respectively, which describe arbitrary on-shell momenta and spin degrees of freedom.
- Combining the Fourier and twistor transformations we will obtain integral twistor transform which compares conventional massive multi-spinor space-time fields to the components in the expansion of the massive twistor field.
- Finally, we will present some comments related to the further studies of twistor description of massive HS fields.

HS particle in tensorial space-time

I.Bandos, J.Lukierski, D.Sorokin, 1999, 2000
M.A.Vasiliev 2001

- The $D{=}4$ HS model in tensorial space $(x^{\alpha\dot\beta}, y^{\alpha\beta}, \bar y^{\dot\alpha\dot\beta})$ is provided by the following action
$$S = \int d\tau \Big(\pi_\alpha \bar\pi_{\dot\beta}\dot x^{\alpha\dot\beta} + \pi_\alpha\pi_\beta \dot y^{\alpha\beta} + \bar\pi_{\dot\alpha}\bar\pi_{\dot\beta}\dot{\bar y}^{\dot\alpha\dot\beta} + \pi_\alpha \dot y^\alpha + \bar\pi_{\dot\alpha}\dot{\bar y}^{\dot\alpha}\Big)$$
$y^{\alpha\beta}$ are three additional complex coordinates; π_α and y^α are *commuting* Weyl spinors.
The constraints lead to unfolded equations on HS field $\Psi_0(x^{\alpha\dot\beta}, y^{\alpha\beta}, \bar y^{\dot\alpha\dot\beta}; y^\alpha, \bar y^{\dot\alpha})$:
$$\Big(p_{\alpha\dot\beta} - \pi_\alpha\bar\pi_{\dot\beta}\Big)\Psi_0 = 0\,, \qquad \Big(p_{\alpha\beta} - \pi_\alpha\bar\pi_\beta\Big)\Psi_0 = 0\,, \qquad \Big(\bar p_{\dot\alpha\dot\beta} - \bar\pi_{\dot\alpha}\bar\pi_{\dot\beta}\Big)\Psi_0 = 0\,.$$
- Transition to the twistor description is achieved by Penrose incidence relation:
$$\omega^\alpha = x^{\alpha\dot\beta}\bar\pi_{\dot\beta} + 2\,y^{\alpha\beta}\pi_\beta + y^\alpha\,, \qquad \bar\omega^{\dot\alpha} = \pi_\beta x^{\beta\dot\alpha} + 2\,\bar y^{\dot\alpha\dot\beta}\bar\pi_{\dot\beta} + \bar y^{\dot\alpha}\,.$$
Then the action can be rewritten (modulo boundary terms) as the one-twistor particle model
$$S = -\tfrac{1}{2}\int d\tau \Big(\bar Z_A \dot Z^A + \mathrm{h.c.}\Big) = -\tfrac{1}{2}\int d\tau \Big(\omega^\alpha\dot\pi_\alpha - \bar\pi_{\dot\alpha}\dot{\bar\omega}^{\dot\alpha} + \mathrm{h.c.}\Big)\,,$$
where the $D{=}4$ twistor Z^A, $A = 1,\ldots,4$ is described by a pair of Weyl spinors
$$Z^A = \Big(\pi_\alpha\,, \bar\omega^{\dot\alpha}\Big)\,.$$
- At quantization in holomorphic representation twistor field $\Phi_0(Z^A)$ combines infinite tower of arbitrary helicity massless fields, which are determined by the integral transformation
$$\psi_{\alpha_1\ldots\alpha_n}(x) = \int d\pi\, e^{\,i\,\pi_\gamma\bar\pi_{\dot\gamma}x^{\gamma\dot\gamma}}\,\pi_{\alpha_1}\ldots\pi_{\alpha_n}\,\Phi_0(\pi,\bar\omega)\big|_{\bar\omega^{\dot\beta} = \pi_\beta x^{\beta\dot\beta}}\,.$$
Massless states with fixed helicity are eigenvectors of the helicity operator
$$h = \tfrac{i}{2}\,\bar Z_A Z^A\,.$$

- Basic twistor relation $p_{\alpha\dot\beta} \sim \pi_\alpha \bar\pi_{\dot\beta}$ requires to use more than one twistor in the massive case $p^{\alpha\dot\beta} p_{\alpha\dot\beta} \neq 0$ since $\pi^\alpha \pi_\alpha \equiv 0$.
  So, we have to use at least two twistors, including spinors π^i_α, $i = 1, 2$.
- In massless case, used variables reflect a generalized conformal symmetry $Sp(8)$:
  - tensorial space-time $(x^{\alpha\dot\beta}, y^{\alpha\beta}, \bar y^{\dot\alpha\dot\beta})$ is the coset space $\dfrac{Sp(8)}{GL(4) \otimes K}$;
  - $Sp(8)$ transformations are realized as linear transformation of the twistor $Z^A = (\pi_\alpha\,, \bar\omega^{\dot\alpha})$.
  - tensorial coordinates correspond to the generalized momenta generators in M-theory extension of the $N{=}1$ Poincare superalgebra
  $$\{Q_\alpha, \bar Q_{\dot\beta}\} = P_{\alpha\dot\beta}\,, \quad \{Q_\alpha, Q_\beta\} = Z_{\alpha\beta}\,, \quad \{\bar Q_{\dot\alpha}, \bar Q_{\dot\beta}\} = \bar Z_{\dot\alpha\dot\beta}\,.$$
- In massive case we do not know what supergroup is
  HS generalization of the Poincare group.
  The use of two (super) twistors involves consideration
  of $N = 2$ Poincare supersymmetry with generators Q^i_α, $\bar Q^i_{\dot\alpha}$.
  One of the possible generalizations is superalgebras with the most common right-hand side in the following anticommutators

$$\{Q^i_\alpha, \bar Q^j_{\dot\beta}\} = (\sigma_a)^{ij} P^a_{\alpha\dot\beta} = \epsilon^{ij} P_{\alpha\dot\beta} + (\sigma_r)^{ij} P^r_{\alpha\dot\beta}\,, \qquad a = 0, 1, 2, 3\,, \qquad r = 1, 2, 3\,.$$

This involves the use of three additional vector variables $y_r^{\alpha\dot\beta}$ in addition to the position vector $x^{\alpha\dot\beta}$.

- HS massless generalization of Shirafuji model is defined by the action (I.Bandos, J.Lukierski, D.Sorokin, 2000; M.A.Vasiliev 2001)

$$S_{mix}^{(m=0)} = \int d\tau \Big(\pi_\alpha \bar\pi_{\dot\beta}\, \dot x^{\alpha\dot\beta} + \pi_\alpha \pi_\beta\, \dot y^{\alpha\beta} + \bar\pi_{\dot\alpha}\bar\pi_{\dot\beta}\, \dot y^{\dot\alpha\dot\beta} + \pi_\alpha \dot y^\alpha + \bar\pi_{\dot\alpha}\dot{\bar y}^\alpha \Big) .$$

This system possesses the constraints

$$p_{\alpha\dot\beta} - \pi_\alpha\bar\pi_{\dot\beta} \approx 0\,, \qquad p_{\alpha\beta} - \pi_\alpha\pi_\beta \approx 0\,, \qquad p_{\dot\alpha\dot\beta} - \bar\pi_{\dot\alpha}\bar\pi_{\dot\beta} \approx 0\,,$$

which produce unfolded equations for HS wave function.

- We consider the following HS massive generalization of Shirafuji model:

$$\begin{aligned} S_{mix}^{(m\neq 0)} \quad = \quad & \int d\tau \Big[\pi^i_\alpha \bar\pi_{\dot\beta i}\, \dot x^{\alpha\dot\beta} + \pi^i_\alpha (\sigma^r)_i{}^j \bar\pi_{\dot\beta j}\, \dot y_r^{\alpha\dot\beta} + \pi^i_\alpha \dot y_i^\alpha + \bar\pi_{\dot\alpha i}\dot{\bar y}^{\alpha i} \\ & + \rho\left(\pi^i_\alpha \pi_i^\alpha + 2M\right) + \bar\rho\left(\bar\pi^i_{\dot\alpha}\bar\pi_i^{\dot\alpha} + 2\bar M\right) \Big] , \end{aligned}$$

where $y_r^{\alpha\dot\beta}$ are three additional vector coordinates. As we will see below, these variables play the role of conjugated coordinates to the phase coordinates, which define Pauli-Lubanski vector.

- Proposed HS massive Shirafuji action leads to the constraints

$$p^a_{\alpha\dot\beta} - u^a_{\alpha\dot\beta} \approx 0\,, \qquad u^a_{\alpha\dot\beta} = \pi^i_\alpha (\sigma^a)_i{}^j \bar\pi_{\dot\beta j}\,, \qquad a = 0,1,2,3\,,$$

which give massive generalization of HS unfolded equations after quatization and imply at $a=0$ massive particle spectrum:

$$p_{\alpha\dot\beta}p^{\alpha\dot\beta} = p^\mu p_\mu = 2|M|^2 = m^2\,.$$

Group-theoretic analysis of the model has a more effective after pass to pure twistor formulation.

- One half of the twistor variables are the spinors π^i_α, $\bar\pi_{\dot\alpha i}$.
  The second half of the twistors is defined by the following HS generalization of the incidence relations

$$\bar\omega^{\dot\alpha i} = \pi^i_\beta x^{\beta\dot\alpha} + \pi^j_\beta(\sigma^r)_j{}^i y_r^{\beta\dot\alpha} + \bar y^{\dot\alpha i}\,, \qquad \omega_i^\alpha = x^{\alpha\dot\beta}\bar\pi_{\dot\beta i} + y_r^{\alpha\dot\beta}(\sigma^r)_i{}^j\bar\pi_{\dot\beta j} + y_i^\alpha\,.$$

- The $D{=}4$ twistors ($Sp(8)$ spinors in HS massless limit and $SU(2,2)$ spinors in fixed helicity limit) can be expressed by two pairs of two-component Weyl spinors

$$Z^{Ai} = \begin{pmatrix} \pi^i_\alpha \\ \bar\omega^{\dot\alpha i} \end{pmatrix}, \qquad \bar Z_{Ai} = (\omega_i^\alpha, -\bar\pi_{\dot\alpha i})\,, \qquad A = 1,2,3,4\,.$$

- Two-twistorial realization of the $D{=}4$ Poincaré algebra is the following

$$P_{\alpha\dot\beta} = \pi^i_\alpha\bar\pi_{\dot\beta i}\,, \qquad M_{\alpha\beta} = \pi^i_{(\alpha}\omega_{\beta)i}\,, \qquad M_{\dot\alpha\dot\beta} = \bar\omega^i_{(\dot\alpha}\bar\pi_{\dot\beta)i}\,.$$

  The Pauli-Lubański four-vector $W_\mu = \frac{1}{2}\,\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}$ can be written as follows

$$W^{\alpha\dot\beta} = S_r\, u_r^{\alpha\dot\beta}\,, \qquad r = 1,2,3\,,$$

  where $\qquad S_r = -\frac{i}{2}\left(\pi^i_\alpha\omega_j^\alpha - \bar\pi_{\dot\alpha j}\bar\omega^{\dot\alpha i}\right)(\sigma_r)_i{}^j\,.$

  It follows that

$$W^\mu W_\mu = -m^2\vec S^{\,2}\,.$$

  After quantization, we obtain the well known relativistic spin square spectrum with $\vec S^{\,2}$ replaced by $s(s+1)$ ($s = 0, \frac{1}{2}, 1, \ldots$).

- Applying incidence relations in mixed action
  we get twistor formulation of HS massive particle:

$$S^{(m\neq 0)}_{twistor} = \int d\tau \Big[\pi^{i}_{\alpha}\, \dot{\omega}^{\alpha}_{i} + \bar{\pi}_{\dot{\alpha} i}\, \dot{\bar{\omega}}^{\dot{\alpha} i} + \mu \left(\pi^{i}_{\alpha} \pi^{\alpha}_{i} + 2M \right) + \bar{\mu} \left(\bar{\pi}^{i}_{\dot{\alpha}} \bar{\pi}^{\dot{\alpha}}_{i} + 2\bar{M} \right) \Big] .$$

  In these formulation there are presented basic constraints

$$G \equiv \pi^{i}_{\alpha}\pi^{\alpha}_{i} + 2M \approx 0 \,, \qquad \bar{G} \equiv \bar{\pi}^{i}_{\dot{\alpha}}\bar{\pi}^{\dot{\alpha}}_{i} + 2\bar{M} \approx 0 \,.$$

- These constraints mean that the spinors

$$g^{i}_{\alpha} \equiv M^{-1/2}\, \pi^{i}_{\alpha} \,, \quad \bar{g}_{\dot{\alpha} i} \equiv M^{-1/2}\, \bar{\pi}_{\dot{\alpha} i} \,; \qquad \epsilon^{\alpha\beta} g^{i}_{\alpha} g^{k}_{\beta} = \epsilon^{ik} \ .$$

  constitute a pair of complex-conjugated spinorial $D\!=\!4$ Lorentz harmonics (I.Bandos, 1990; F.Delduc, A.Galperin, E.Sokatchev, 1992; SF, V.Zima, 1995).
  Composite real four-vectors

$$e^{a}_{\mu} = \tfrac{1}{2M}\, (\sigma_{\mu})^{\alpha\dot{\beta}} u^{a}_{\alpha\dot{\beta}} \,, \qquad e_{\mu\, a} e^{\mu}{}_{b} = \eta_{ab} \,, \quad \eta_{ab} = (1, -1, -1, -1)$$

  describe an orthonormal vectorial Lorentz frame defining $D\!=\!4$ vectorial Lorentz harmonics (E.Sokatchev, 1986).

- For the local gauge transformations generated by the constraints $G, \bar{G}$ we introduce the gauge fixing conditions

$$i\left(\pi^i_\alpha \bar{\omega}^\alpha_i - \bar{\pi}_{\dot{\alpha}i}\omega^{\dot{\alpha}i}\right) \approx 0\,, \qquad \pi^i_\alpha \bar{\omega}^\alpha_i + \bar{\pi}_{\dot{\alpha}i}\omega^{\dot{\alpha}i} \approx 0\,.$$

The Dirac brackets for the twistor components are the following

$$\{\pi^k_\alpha, \pi^j_\beta\}_* = \{\pi^k_\alpha, \bar{\pi}_{\dot{\beta}j}\}_* = 0\,,$$

$$\{\omega^\alpha_k, \pi^j_\beta\}_* = \delta^\alpha_\beta \delta^j_k + \tfrac{1}{2M}\pi^\alpha_k \pi^j_\beta\,, \qquad \{\bar{\omega}^{\dot{\alpha}k}, \pi^j_\beta\}_* = 0\,,$$

$$\{\omega^\alpha_k, \omega^\beta_j\}_* = -\tfrac{1}{M}\left(\pi^\alpha_k \bar{\omega}^\beta_j - \pi^\beta_j \bar{\omega}^\alpha_k\right)\,, \qquad \{\omega^\alpha_k, \bar{\omega}^{\dot{\beta}j}\}_* = 0\,.$$

- We will consider the $(\pi, \bar{\pi})$-realization of quantized version of the DB algebra. In such a realization, after using the ordering with π's at the left and ω's at the right, we obtain $\hat{\pi}^k_\alpha = \pi^k_\alpha$, $\hat{\bar{\pi}}_{\dot{\alpha}k} = \bar{\pi}_{\dot{\alpha}k}$ and

$$\hat{\omega}^\alpha_k = i\frac{\partial}{\partial \pi^k_\alpha} + \frac{i}{2M}\,\pi^\alpha_k\,\pi^j_\beta \frac{\partial}{\partial \pi^j_\beta}\,, \qquad \hat{\bar{\omega}}^{\dot{\alpha}k} = i\frac{\partial}{\partial \bar{\pi}_{\dot{\alpha}k}} - \frac{i}{2\bar{M}}\,\bar{\pi}^{\dot{\alpha}k}\,\bar{\pi}_{\dot{\beta}j}\frac{\partial}{\partial \bar{\pi}_{\dot{\beta}j}}\,.$$

The quantum counterparts of the spin operators

$$\hat{S}_r = \frac{1}{2}\left(\pi^i_\alpha \frac{\partial}{\partial \pi^k_\alpha} - \bar{\pi}_{\dot{\alpha}k}\frac{\partial}{\partial \bar{\pi}_{\dot{\alpha}i}}\right)(\sigma_r)_i{}^k\,.$$

The square of the Pauli-Lubański vector becomes $\hat{W}^\mu \hat{W}_\mu = -m^2 \hat{S}^r \hat{S}^r$, which will be used below to define spin states.

Thus, the twistorial wave function is defined on the space parametrized by π^i_α, $\bar\pi_{\dot\alpha i}$ which satisfy the constraints G, $\bar G$ and the matrix $g_\alpha{}^i = M^{-1/2}\pi^i_\alpha$ defines the $SL(2,\mathbb{C})$ group manifold.
Thus, the twistorial wave function $\Psi = \Psi(\pi^i_\alpha, \bar\pi_{\dot\alpha i})$ is defined on $SL(2,\mathbb{C})$ parametrized by π^i_α.

Let us analyze the twistorial wave function.

- One can use the well known decomposition of $SL(2,\mathbb{C})$ elements
$$g = h\,v\,, \qquad g_\alpha{}^i = h_\alpha{}^{\mathrm{k}} v_{\mathrm{k}}{}^i\,,$$
in terms of the product of an hermitian matrix $h = h^\dagger$ with unit determinant and an $SU(2)$ matrix v, $v^\dagger v = 1$.
- The three parameters of the matrix h parametrize the coset $SL(2,\mathbb{C})/SU(2)$ which defines the three-dimensional mass hyperboloid for timelike four-momenta:
$$p_{\alpha\dot\beta} = h_\alpha{}^{\mathrm{i}}\bar h_{\dot\beta\mathrm{i}}\,.$$
- The unitary matrix v paramerizes $\mathbb{S}^3 \sim SU(2)$ and is linked with the spin degrees of a massive particle. In particular, the spin operators take the form
$$\hat S_r = \tfrac12\,(\sigma_r)_j{}^k\, v_{\mathrm{i}}{}^j \frac{\partial}{\partial v_{\mathrm{i}}{}^k}\,.$$
- We can consider the variables $v_{\mathrm{i}}{}^k$ as the harmonic variables that were introduced early to describe $N{=}2$ superfield formulations (GIKOS). In the notation
$$v_{\mathrm{i}}{}^k = (v_{\mathrm{i}}{}^1, v_{\mathrm{i}}{}^2) = (v^+_{\mathrm{i}}, v^-_{\mathrm{i}})\,, \qquad v^{+\mathrm{i}}v^-_{\mathrm{i}} = 1\,, \qquad (v^\pm_{\mathrm{i}})^* = \mp v^{\mp\mathrm{i}}\,,$$
the spin operators take the form
$$D^0 \equiv 2\hat S_3 = v^+_{\mathrm{i}}\frac{\partial}{\partial v^+_{\mathrm{i}}} - v^-_{\mathrm{i}}\frac{\partial}{\partial v^-_{\mathrm{i}}}\,, \qquad D^{\pm\pm} \equiv \hat S_1 \pm \mathrm{i}\hat S_2 = v^\pm_{\mathrm{i}}\frac{\partial}{\partial v^\mp_{\mathrm{i}}}\,,$$
and the square of the Pauli-Lubański vector is given by the formula
$$\hat W^\mu \hat W_\mu = -\tfrac{m^2}{4}\left[\left(D^0\right)^2 + 2\left\{D^{++}, D^{--}\right\}\right].$$

- Since the variables $v_{\rm i}^{\pm}$ parametrize a compact space, the general wave function on $SL(2,\mathbb{C})$ has the following harmonic expansion (we use the $SU(2)$-covariant expansion)

$$\Psi(h_\alpha{}^{\rm i}, v_{\rm i}{}^{k}) = \sum_{K,N=0}^{\infty} v^{+}_{({\rm i}_1} \ldots v^{+}_{{\rm i}_N} v^{-}_{{\rm j}_1} \ldots v^{-}_{{\rm j}_K)} f^{{\rm i}_1 \ldots {\rm i}_N {\rm j}_1 \ldots {\rm j}_K}(h)\,,$$

where the coefficient fields $f^{({\rm i}_1 \ldots {\rm i}_N {\rm j}_1 \ldots {\rm j}_K)}(h)$ are symmetric with respect to all indices and depend on the on-shell four-momenta p_μ.

Each monomial in the expansion is an eigenvector of the Casimir operator:

$$\hat{W}^{\mu}\hat{W}_{\mu}\,(v^{+}_{\rm i})^{N}(v^{-}_{\rm j})^{K} f^{({\rm i})_N({\rm j})_K} = -m^2\, s(s+1)\,(v^{+}_{\rm i})^{N}(v^{-}_{\rm j})^{K} f^{({\rm i})_N({\rm j})_K}\,, \quad s = \frac{N+K}{2}\,.$$

So, the wave function expression is the general expansion into arbitrary spin states.

- By means of the nonsingular transformation $v_{\rm i}^{\pm} \to \pi^{\pm}_{\alpha}$ or $v_{\rm i}^{\mp} \to \bar\pi^{\pm}_{\dot\alpha}$ where

$$(\pi^{+}_{\alpha}, \pi^{-}_{\alpha}) = (\pi^{1}_{\alpha}, \pi^{2}_{\alpha})\,, \qquad (\bar\pi^{+}_{\dot\alpha}, \bar\pi^{-}_{\dot\alpha}) = (\bar\pi_{\dot\alpha 2}, -\bar\pi_{\dot\alpha 1})\,,$$

and by redefining component fields the expansion can be rewritten in $SL(2,\mathbb{C})$-covar. form.

- But we would like to stress that the spin content in the expansion is degenerate.
This degeneracy can be however removed by the harmonic condition $D^{++}\,\tilde\Psi^{(+)} = 0\,.$
As a solution of this condition, we obtain the following wave function

$$\tilde\Psi^{(+)}(h_\alpha{}^{\rm i}, v^{\pm}_{\rm i}) = \sum_{N=0}^{\infty} v^{+}_{{\rm i}_1} \ldots v^{+}_{{\rm i}_N} f^{{\rm i}_1 \ldots {\rm i}_N}(h)\,.$$

This twistor wave function rewritten in Lorentz covariant way takes the form

$$\tilde\Psi^{(+)}(\pi^{\pm}_{\alpha}, \bar\pi^{\pm}_{\dot\alpha}) = \sum_{N=0}^{\infty} \pi^{+}_{\alpha_1} \ldots \pi^{+}_{\alpha_N}\, \psi^{\alpha_1 \ldots \alpha_N}(p_\mu)\,.$$

Note that these twistor wave function also depends on π^{-}_{α} and $\bar\pi^{\pm}_{\dot\alpha}$ through p_μ.

- Spin $s{=}L/2$ massive particles are described by the fields $\psi^{\alpha_1\cdots\alpha_L}(p_\mu)$.
  The corresponding multispinor spacetime fields are obtained by an integral Fourier-twistor transform which combines the Fourier and twistor transformations:

$$
\begin{aligned}
\phi_{\alpha_1\ldots\alpha_L}(x) &= \int d^6\pi\, e^{-ix^\mu p_\mu}\,\pi^-_{\alpha_1}\ldots\pi^-_{\alpha_L}\tilde{\Psi}^{(+)}(\pi^\pm,\bar\pi^\pm)\,,\\
\phi_{\alpha_1\ldots\alpha_{L-1}}{}^{\dot\beta_1}(x) &= \int d^6\pi\, e^{-ix^\mu p_\mu}\,\pi^-_{\alpha_1}\ldots\pi^-_{\alpha_{L-1}}\bar\pi^{-\dot\beta_1}\tilde{\Psi}^{(+)}(\pi^\pm,\bar\pi^\pm)\,,\\
&\qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\phi^{\dot\beta_1\ldots\dot\beta_L}(x) &= \int d^6\pi\, e^{-ix^\mu p_\mu}\,\bar\pi^{-\dot\beta_1}\ldots\bar\pi^{-\dot\beta_L}\tilde{\Psi}^{(+)}(\pi^\pm,\bar\pi^\pm)
\end{aligned}
$$

  where p_μ is defined as a bilinear product of twistors. In the integrals for a given L, only the term $\pi^+_{\alpha_1}\ldots\pi^+_{\alpha_L}\,\psi^{\alpha_1\cdots\alpha_L}(p_\mu)$ in the twistorial wave function gives non-zero contribution.
- We can show that the multispinors $\phi_{\alpha_1\ldots\alpha_N}{}^{\dot\beta_1\ldots\dot\beta_M}$ satisfy automatically the following sequence of Dirac-Fierz-Pauli field equations

$$
\begin{aligned}
i\partial_{\alpha\dot\beta_M}\phi_{\alpha_1\ldots\alpha_N}{}^{\dot\beta_1\ldots\dot\beta_M} &= m\,\phi_{\alpha\alpha_1\ldots\alpha_N}{}^{\dot\beta_1\ldots\dot\beta_{M-1}}\,,\\
i\partial^{\alpha\dot\beta_M}\phi_{\alpha\alpha_1\ldots\alpha_N}{}^{\dot\beta_1\ldots\dot\beta_{M-1}} &= m\,\phi_{\alpha_1\ldots\alpha_N}{}^{\dot\beta_1\ldots\dot\beta_M}
\end{aligned}
$$

  and the generalized Lorenz conditions

$$
\partial^{\dot\beta\alpha}\phi_{\alpha_1\ldots\alpha_{N-1}\alpha\dot\beta}{}^{\dot\beta_1\ldots\dot\beta_{M-1}} = 0\,.
$$

In D=3 case twistorial formulation of massive HS particle is similar to the D=4 case.

- D=3 twistors are real four-dimensional $Sp(4;\mathbb{R}) = \overline{SO(3,2)}$ spinors:
$$t^{Ai} = \begin{pmatrix} \lambda^i_\alpha \\ \mu^{\alpha i} \end{pmatrix}, \qquad \alpha = 1,2\,, \quad i = 1,2\,, \quad A = 1,\ldots,4\,.$$
Pure twistor action of massive HS particle takes the form
$$S^{(D=3)}_{twistor} = \int d\tau \left[\lambda^i_\alpha \dot{\mu}^{\alpha i} + \ell \left(\lambda^i_\alpha \lambda^\alpha_i + \sqrt{2}\, m \right) \right].$$
- Due to the mass constraint $\lambda^i_\alpha \lambda^\alpha_i + \sqrt{2}\, m \approx 0$ the real matrices $h_\alpha{}^i = 2^{1/4} m^{-1/2} \lambda_\alpha{}^i$ have determinant equal to one and characterize the $SL(2;\mathbb{R})$ group manifold.
So, HS D=3 twistor wave function is given as function on the $SL(2;\mathbb{R})$ group manifold.
Spin operator $\frac{1}{2m}\, \epsilon_{\mu\nu\rho} p^\mu M^{\nu\rho} = S$ is realized as follows $\hat{S} = \frac{i}{2}\, \epsilon_{ij}\, \lambda^i_\alpha \dfrac{\partial}{\partial \lambda^j_\alpha}$.
- Let us decompose twistor wave function into a superposition of momentum-dependent eigenfunctions of the spin operator.
For it we pass to corresponding $SU(1,1)$ matrix
$$g = U\, h\, U^{-1}, \qquad U = e^{-i\pi\sigma_1/4}, \qquad g = \begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix}, \qquad |a|^2 - |b|^2 = 1\,.$$
One can introduce the natural parametrization of the $SU(1,1)$ matrices
$$a = \cosh(r/2) e^{i(\psi+\varphi)/2}, \qquad b = \sinh(r/2) e^{i(\psi-\varphi)/2},$$
where $0 \leq \varphi \leq 2\pi\,, \quad 0 < r < \infty\,, \quad -2\pi \leq \psi < 2\pi\,.$
In terms of the angle ψ, the spin operator takes the simple form
$$\hat{S} = i\, \frac{\partial}{\partial \psi}$$
i.e., it describes the D=3 $U(1)$ spin.

- To obtain the Hilbert space of the quantized model we use the $SU(1,1)$ regular representation on the $SU(1,1)$ manifold when the wave function is square-integrable and satisfies the periodicity conditions

$$\Psi(\varphi, r, \psi) = \Psi(\varphi + 4\pi, r, \psi) = \Psi(\varphi, r, \psi + 4\pi) = \Psi(\varphi + 2\pi, r, \psi + 2\pi)\,.$$

One can use the double Fourier expansion

$$\Psi(\varphi, r, \psi) = \sum_{k,n=-\infty}^{\infty} f_{kn}(r)\, e^{-i(k\varphi + n\psi)} = \sum_{n=-\infty}^{\infty} e^{-in\psi} F_n(r, \varphi)\,,$$

where the pairs (k, n) such that the numbers k and n are both integer or half-integer.

- The eigenvalues of the operator $\hat{S}$ coincide with parameter n in the expansion. As a result, the spin in our model takes *quantized* integer and half-integer values. The functions $F_n(r, \varphi) = \tilde{F}_n(p_\mu; m)$ describe states with definite $D{=}3$ spin equal to n.
- Lorentz covariant expansion is obtained after transition to the $SU(1,1)$ spinor coordinates

$$\xi_\alpha = \sqrt{m}\begin{pmatrix} a \\ b \end{pmatrix}, \qquad \bar{\xi}^\alpha = (\xi_\alpha)^\dagger = \sqrt{m}\,(\bar{a}, \bar{b})\,, \qquad \bar{\xi}^\alpha(\sigma_3)_\alpha{}^\beta \xi_\beta = m\,,$$

when anti-holomorphic wave functions $\left(\frac{\partial}{\partial \xi_\alpha}\Psi(\xi, \bar{\xi}) = 0\right)$ is given by the power series

$$\Psi(\bar{\xi}) = \sum_{N=0}^{\infty} \bar{\xi}^{\alpha_1} \ldots \bar{\xi}^{\alpha_N}\, \psi^{(+)}{}_{\alpha_1 \ldots \alpha_N}(p_\mu)\,.$$

- The corresponding spacetime fields are then given by

$$\phi_{\alpha_1 \ldots \alpha_N}(x) = \int \mu^3(\xi)\, e^{-i(\tilde{\xi}\gamma_\mu \xi)\, x^\mu}\, \xi_{\alpha_1} \ldots \xi_{\alpha_N}\, \Psi(\xi)\,.$$

where $\widetilde{\xi}^\alpha = \bar{\xi}^\beta(\gamma_0)_\beta{}^\alpha$ is the Dirac conjugated spinor. These fields satisfy automatically the $D{=}3$ Bargmann-Wigner equations

$$\partial_\mu(\gamma^\mu)_\beta{}^{\alpha_1}\phi_{\alpha_1\alpha_2\ldots\alpha_N} - m\phi_{\beta\alpha_2\ldots\alpha_N} = 0\,,$$

- We present $D{=}3$ and $D{=}4$ massive HS particle models in a mixed formulation as well as in a pure twistor formulation.
- In pure twistor formulation the $D{=}3$ and $D{=}4$ wave functions are given as functions on the $SL(2,\mathbb{R})$ and $SL(2,\mathbb{C})$ group manifolds respectively, which describe arbitrary on-shell momenta and spin degrees of freedom.
- There were presented massive field twistor transformations, which associate the HS twistor wave functions with tower of conventional space-time fields.
- In mixed formulation the wave functions are defined by a massive version of Vasiliev's free unfolded equations. For example, considering the differential realization of the spinorial variables $\hat{\pi}^i_\alpha = -i\partial/\partial y_i^\alpha$ in $D{=}4$, and leaving only position space-time vector we will obtain the unfolded equation supplemented with the mass quantum constraints:

$$\left(i\,\partial_{\alpha\dot\beta} - \frac{\partial^2}{\partial y_i^\alpha \partial \bar{y}^{\dot\beta i}} \right) \Psi(x,y,\bar{y}) = 0\,,$$

$$\left(\frac{\partial^2}{\partial y_i^\alpha \partial y^i_\alpha} - 2M \right) \Psi(x,y,\bar{y}) = 0\,, \qquad \left(\frac{\partial^2}{\partial \bar{y}_i^{\dot\alpha} \partial \bar{y}^i_{\dot\alpha}} - 2\bar{M} \right) \Psi(x,y,\bar{y}) = 0\,.$$

  It is interesting to make a generalization of these equations to the case of the (A)dS spacetime and HS gravity background.
- The discussed models give the same mass for all HS fields, that is very strict condition. In a physical HS case, when considering *e.g*. spin excitations in string theory, the masses are spin-dependent: $m^2 \quad \rightarrow \quad m^2(\vec{S}^2)\,.$
  In the twistor formulation, the spinorial mass-shell conditions may be considered as 'complex roots' of the standard mass-shell condition. It is an interesting problem to see how to introduce, in the complex mass parameter M, a dependence on the twistor variables that could lead to HS multiplets with masses on a Regge trajectory.

THANK YOU FOR YOUR ATTENTION!