Cosmology in Horndeski Gravity

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Lovelock Gravity

GR equations of motion in vacua should satisfy:

(a) $E^{ij} = E^{ji} \quad (b) \quad E^{ij} = E^{ij}(g_{lk}; \partial_m g_{lk}; \partial_m \partial_n g_{lk})$, \quad (c) $\nabla_i E^{ij} = 0$.

If also (d): equations linear in second derivatives of $g_{ab}$ then $E^{ij} = G^{ij} + \Lambda g^{ij}$ regardless of spacetime dimension, $D$.
Without (d) depends on $D$, (Lovelock, 1971):

$$E^i_j = \sum_{p=1}^{[(D+1)/2]-1} \ a_p \delta_{j k_1 \ldots k_{2p}}^{i h_1 \ldots h_{2p}} R_{h_1 h_2}^{j_1 j_2} \cdots \cdot R_{h_{2p-1} h_{2p}}^{j_{2p-1} j_{2p}} + a \delta^i_j.$$

The Lagrangian read as

$$L = \sum_{p=1}^{[(D+1)/2]-1} \ a_p \delta_{k_1 \ldots k_{2p}}^{h_1 \ldots h_{2p}} R_{h_1 h_2}^{j_1 j_2} \cdots \cdot R_{h_{2p-1} h_{2p}}^{j_{2p-1} j_{2p}} + a.$$
Horndeski theory (in $D = 4$) 

(Horndeski, 1974) The most general scalar-tensor interaction such that equations of motion depend only on

$$
(g_{lk}; \partial_m g_{lk}; \partial_m \partial_n g_{lk}; \phi, \partial_m \phi, \partial_m \partial_n \phi).
$$

$$
\mathcal{L} = \sqrt{(g)} \mathcal{H}_1 \delta^{cde}_{hjk} \phi^j c \, \hat{R}^{j k} - \frac{4}{3} \sqrt{(g)} \mathcal{H}_1 \delta^{cde}_{hjk} \phi^j c \, \phi^d j \, \phi^e k \\
+ \sqrt{(g)} \mathcal{H}_3 \delta^{cde}_{hjk} \phi^j c \, R^{j k} - 4 \sqrt{(g)} \mathcal{H}_3 \delta^{cde}_{hjk} \phi^j c \, \phi^d j \, \phi^e k \\
+ \sqrt{(g)} (\mathcal{F} + 2 \mathcal{W}) \delta^{cd}_{f h} R^{f h} + 2 \sqrt{(g)} (2 \mathcal{H}_3 - 2 \mathcal{H}_1 + 4 \rho \mathcal{H}_3) \delta^{cd}_{f h} \phi^j c \, \phi^d j \\
- 3 \sqrt{(g)} (2 \mathcal{F}' + 4 \mathcal{W}' + \rho \mathcal{H}_8) \phi^j c + 2 \sqrt{(g)} \mathcal{H}_8 \delta^{cd}_{f h} \phi^j c \, \phi^d j \\
+ \sqrt{(g)} \{4 \mathcal{H}_9 - \rho (2 \mathcal{F}'' + 4 \mathcal{W}'' + \rho \mathcal{H}_8 + 2 \mathcal{H}_9)\}
$$

(Horndeski, 1976) The gauge field-tensor interaction such that (a) equations of motion depend only on

$$
(g_{ab}; \partial_m g_{ab}; \partial_m \partial_n g_{ab}; A_i, \partial_m A_i, \partial_m \partial_n A_i); \ b \ 'charge' \ conserves; \ (c) \ flat \ space \ limit \ is \ Maxwell \ theory:
$$

$$
L = L_{EH} + L_{Maxwell} + \tilde{R} FF
$$
Revival of Interest

1990s: late-time cosmic accelerated expansion. Motivation for modifications of gravity: large at long range, negligible at solar system range.

2000s: DGP model, massive gravity e.t.c. Galilean symmetry, $\pi(x) \rightarrow \pi(x) + b_\mu x^\mu + c$, is essential. Galileon models in flat space: both Lagrangian and E.o.M. depend only on second derivatives of scalar field(s), $p$-forms. In $D = 4$ ‘covariantized’ galileons, dimensional reductions of Lovelock gravity and Horndeski theory are almost the same.

2010: Horndeski-coupled Higgs field, $(g^{\mu\nu} + \xi G^{\mu\nu}) \partial_\mu \varphi \partial_\nu \varphi$, provides nice inflation.

...Let us further investigate inflationary cosmology in Horndeski gravity!
The key object is the dual of a Riemann tensor,
\[ \tilde{R}^{\alpha \beta \gamma \delta} = \frac{1}{4} \epsilon^{\alpha \beta \mu \nu} \epsilon^{\gamma \delta \rho \sigma} R_{\mu \nu \rho \sigma}, \]
where \( \epsilon^{[\alpha \beta \mu \nu]} \) is the Levi-Civita tensor. It is divergent-free and thus can be used to provide safe coupling of the field strength to gravity:
\[ L_{\text{coupl}} \sim \tilde{R}^{\alpha \beta \mu \nu} F_a^{\alpha \beta} F_a^{\mu \nu} \sim R^{\alpha \beta \mu \nu} \tilde{F}^{a \alpha \beta} \tilde{F}^{a \mu \nu}. \]

Both Riemann and field tensors satisfy the Bianchi identities which annihilate higher order derivatives in the equations of motion.

\[ S = \int \left( \frac{M_{Pl}^2}{2} R - \frac{1}{4} F^{a \mu \nu} F_{a \mu \nu} - \frac{\mu^2 l^2}{8} R^{\alpha \beta \mu \nu} \tilde{F}^{a \alpha \beta} \tilde{F}^{a \mu \nu} \right) \sqrt{-g} d^4 x, \]

where the modified Planck mass is \( M_{Pl} = 1/\sqrt{8\pi G} \), and the appropriate length scale of the theory is \( l \equiv 1/(eM_{Pl}) \).
Equations of Motion

Stress-energy tensor:

\[ T^{\rho\sigma} \equiv \frac{2}{\sqrt{-g}} \frac{\partial (L \sqrt{-g})}{\partial g_{\rho\sigma}} = F^{a\rho\alpha} F^{a\sigma}_{\alpha} - \frac{1}{4} g^{\rho\sigma} F^a_{\mu\nu} F^{a\mu\nu} \]

\[ - \frac{\mu^2}{8} \left[ -g^{\rho\sigma} R_{\alpha\beta\mu\nu} \tilde{F}^{a\alpha\beta} \tilde{F}^{a\mu\nu} + 2 R^\rho_{\beta\mu\nu} \tilde{F}^{a\sigma\beta} \tilde{F}^{a\mu\nu} + 4 \nabla_\beta \nabla_{\mu} \left( \tilde{F}^{a\rho\beta} \tilde{F}^{a\mu\sigma} \right) \right] \]

where

\[ \nabla_\beta \nabla_{\mu} \left( \tilde{F}^{a\rho\beta} \tilde{F}^{a\mu\sigma} \right) = (D_\mu \tilde{F}^{\rho\beta})^a (D_\beta \tilde{F}^{\mu\sigma})^a + [F_{\beta\mu}, \tilde{F}^{\rho\beta}]^a \tilde{F}^{a\mu\sigma} \]

\[ + R^\rho_{\alpha\beta\mu} \tilde{F}^{a\alpha\beta} \tilde{F}^{a\mu\sigma} + R_{\alpha\mu} \tilde{F}^{a\rho\alpha} \tilde{F}^{a\mu\sigma} . \]

Notice that by virtue of Bianchi identities the term which is third-order in field tensor, \( F \tilde{F} \tilde{F} \), arises instead of third-order derivatives. The field equation:

\[ D_\rho \left( F^{\rho\sigma} + \frac{\mu^2}{2} \tilde{R}^{\rho\sigma\mu\nu} F_{\mu\nu} \right) = 0 . \]
Unlike Maxwell field, the Yang–Mills $SU(2)$ configuration is compatible with FLRW metrics,

$$ds^2 = -N^2 dt^2 + a^2 \left[ d\chi^2 + \Sigma_k^2(\chi)(d\theta^2 + \sin^2 \theta d\varphi^2) \right],$$

where $\Sigma_k(\chi) = \{ \sin \chi, \chi, \sinh \chi \}$.

The most general cosmological ansatz preserving the isotropy and homogeneity of the metrics can be written in terms of a single function $f(t)$:

$$A = f(t) T_\chi d\chi + \left[ f(t) \Sigma_k T_\theta + (\Sigma_k' - 1) T_\varphi \right] d\theta + \left[ f(t) \Sigma_k T_\varphi - (\Sigma_k' - 1) T_\theta \right] \sin \theta d\varphi.$$

The group generators, $T_a$, are the Pauli matrices, $\tau_k/(2i)$, contracted with spherical unit vectors, $n^k_{(\chi, \theta, \varphi)}$. 
Effective Lagrangian

Let us introduce ‘electric’, \( E = f/Na \), and ‘magnetic’, \( H = (k - f^2)/a^2 \), components of the YM field tensor. The pure YM Lagrangian then read as:

\[
L_{YM} = -\frac{1}{4} F^a_{\mu \nu} F^{a \mu \nu} = \frac{3}{2} (E^2 - H^2).
\]

The standard Einstein-Hilbert term (in the gauge \( N = 1 \)) is:

\[
L_{EH} = 3 \left[ \frac{\dot{a}^2 + k}{a^2} + \frac{\ddot{a}}{a} \right],
\]

The coupling term looks like a ‘safe’ combination of above:

\[
L_{coupl} \sqrt{-g} = -\frac{3\mu^2}{2} \left[ \frac{\dot{a}^2 + k}{a^2} E^2 - \frac{\ddot{a}}{a} H^2 \right].
\]
Equations of Motion

In the inflationary cosmology, the usual notations are: $\psi \equiv f/a$, $H \equiv \dot{a}/a$, and assume $k = 0$. The energy density and pressure read as:

$$\rho_{ym} = \frac{3}{2} \left( \dot{\psi}^2 + 2H\psi\dot{\psi} + H^2\psi^2 + \psi^4 \right), \quad p_{ym} = \frac{\rho_{ym}}{3},$$

$$\rho_c = -\frac{3\mu^2}{2} \left[ H^2(3\dot{\psi}^2 + 3H^2\psi^2 + 2\psi^4) + 2H\psi\dot{\psi}(3H^2 + 2\psi^2) \right],$$

$$p_c = \frac{\mu^2}{2} \left[ 3\dot{\psi}^2(3H^2 + 4\psi^2) + 2H\psi\dot{\psi}(7H^2 + 8\psi^2) + H^2\psi^2(5H^2 + 2\psi^2) + 4\dot{\psi}(H^2\psi + H\dot{\psi} + \psi^3) + 2\dot{H}(\dot{\psi}^2 + 4H\psi\dot{\psi} + 3H^2\psi^2) \right].$$

Mention that the energy density corresponding to coupling term, $\rho_c$, is not positive-defined. The gauge field equation takes the following form:

$$(1 - \mu^2H^2) \left( \dot{\psi} + H\psi \right) + 2 \left[ 1 - \mu^2(\dot{H} + H^2) \right] \left( H\dot{\psi} + H^2\psi + \psi^3 \right) = 0.$$
De Sitter Space

Mention that the Riemann tensor in de Sitter space read as

\[ R_{\alpha\beta\mu\nu} = -\tilde{R}_{\alpha\beta\mu\nu} = H^2 (g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}). \]

Since \( \tilde{F}^2 = -F^2 \), the full field Lagrangian is proportional to the conventional \( F^2 \) term:

\[ L_{\text{coupl}} + L_{YM} = -\frac{1}{4} (1 - \mu^2 H^2) F^a_{\mu\nu} F^{a\mu\nu}. \]

And the equation of motion for the gauge field also acquires the same factor:

\[ (1 - \mu^2 H^2) D_\rho F^{\rho\sigma} = 0. \]

De Sitter space with \( H = H_c \equiv \mu^{-1} \) is a special case for Horndeski model. However, the energy density and pressure are not vanishing!
Exact Solutions

With the ansatz $H = H_c$ for the metrics, the gauge field equation is identically satisfied. One has to solve only the Friedmann equation for $\dot{\psi}$:

$$
\dot{\psi}_\pm = -\frac{1}{2H_c} \left( \psi^3 + H_c^2 \psi \pm \sqrt{\psi^6 + (3/2)\psi^4 H_c^2 - H_c^4} \right).
$$

With any additional matter, $\rho_m$, in a state of a perfect fluid:

$$
\dot{\psi}_\pm = -\frac{1}{2H_c} \left( \psi^3 + H_c^2 \psi \pm \sqrt{\psi^6 + 2\psi^4 H_c^2 / 3 - H_c^4 + H_c^2 \rho_m / 3} \right).
$$

And the second Friedmann equation then holds if

$$
\rho_m + 3H_c(\rho_m + p_m) = 0.
$$
## Properties of Exact Solution

<table>
<thead>
<tr>
<th>dominating gauge field</th>
<th>dominating matter</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \dot{\psi}_+ \simeq -\frac{2\psi^3}{H_c} )</td>
<td>( \dot{\psi}_- \simeq -\frac{H_c\psi}{4} )</td>
</tr>
<tr>
<td>( \psi_+ \simeq \sqrt{\frac{H_c}{4(t-t_0)}} )</td>
<td>( \psi_- \simeq \psi_0 \exp (-H_c t/4) )</td>
</tr>
<tr>
<td>( \dot{\psi}_\pm \simeq \pm \sqrt{\frac{\rho_m}{3}} )</td>
<td>( \psi_\pm \simeq \psi_0 \mp \sqrt{\frac{\rho_m}{3}} t )</td>
</tr>
</tbody>
</table>

Condensate solution: \( \rho_g \simeq \rho_m, \psi_c^4 \simeq \frac{2\rho_m}{3} \).

<table>
<thead>
<tr>
<th>Solution</th>
<th>Eigenvalues</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \psi = \sqrt{\frac{H_c}{4(t-t_0)}} ) ( \psi = \psi_0 \exp (-H_c t/4) ) ( \psi = \psi_c ) ( \psi = \psi_0 \mp \sqrt{\frac{\rho_m}{3}} t )</td>
<td>( \frac{12\psi^2}{H_c}, \frac{2\sqrt{15}\psi^2}{H_c}, -\frac{2\sqrt{15}\psi^2}{H_c} ) ( -2H_c, -\frac{H_c}{4}, -\frac{5H_c}{4} ) ( -H_c, -2H_c, -2H_c ) ( \frac{3\psi^2}{H_c} - 2H_c, -\frac{3}{2H_c}(H_c^2 + \psi^2) \pm \sqrt{\frac{\pm 2\psi}{H_c}(3\rho_m)^{1/4}} )</td>
</tr>
</tbody>
</table>
Gauge inflation

FIG. 1: The solutions for the gauge field, \( \psi \). A non-minimal coupling scale, \( H_c \), divides exponentially decaying solutions and oscillations.

FIG. 2: The solution for the metrics, \( Ht \) represents the inflationary stage with Hubble parameter value \( H = H_c \), and radiation-dominated universe, \( Ht = 1/2 \).

FIG. 3: The solutions for the gauge field, \( \psi \), with initial state \( \psi_i \gg H_c \) demonstrate a continuous exponentially decaying mode.

FIG. 4: The decaying mode of gauge field corresponds to de Sitter metrics with Hubble parameter value \( H = H_c \); the oscillating gauge field gives rise to the radiation-dominated universe, \( Ht = 1/2 \).
Horndeski theory is expected to be valid in the range $\psi_i \sim 1$, $\psi_e \sim H_c$.

For the inflating mode, $\psi \simeq \psi_i e^{-H_c t/4}$, one finds:

$$N_{\text{e-folds}} \simeq H_c t \simeq 4 \ln \frac{\psi_i}{\psi_e}.$$  

With $H \sim 10^{-6}..10^{-5}$ (from Planck) one obtains $N_{\text{e-folds}} = 50..60$.

I.e. gauge inflation could play a significant role during the observed inflation stage.
Gauge Inflation with Matter

The duration of inflation is approximately

$$\Delta t \approx \int_{\rho_m^{(i)}}^{\rho_m^{(e)}} \frac{d \rho_m}{\dot{\rho}_m}.$$ 

Then with e.o.s. for matter in the background $H = H_c$ one has:

$$N_{e-\text{folds}} \simeq H_c \Delta t \approx -\frac{1}{3} \int_{\rho_m^{(i)}}^{\rho_m^{(e)}} \frac{d \rho_m}{\rho_m + p_m}.$$ 

For the matter with equation of state $p_m = w \rho_m$ therefore:

$$N_{e-\text{folds}} = \frac{1}{3(1 + w)} \ln\left(\frac{\rho_m^{(i)}}{\rho_m^{(e)}}\right) \approx -\frac{4}{3(1 + w)} \ln(H_c).$$

With dust or radiation, $w = 0, 1/3$, one finds $N_{e-\text{folds}} \approx -\ln H_c$.

Too weak for inflation. Another inflaton?
Slow-roll Inflation

Let us consider a system with dominating inflaton field, a slow-rolling scalar field:

\[ L_m = \frac{\dot{\varphi}^2}{2} - V(\varphi). \]

By assumption, \( \rho_m \gg 3H_c^2 \simeq 3H^2 \), while normally for the slow-roll model one has \( \rho_m = 3H^2 \). Therefore the stronger slow-roll conditions should be imposed on scalar field:

\[
\begin{aligned}
\ddot{\varphi} &\ll 3H_c \dot{\varphi}, \\
\frac{\dot{\varphi}^2}{2} &\ll V, \\
3H_c^2 &\ll V,
\end{aligned}
\]

\[
\Rightarrow
\begin{aligned}
V'' &\ll 9H_c^2, \\
\frac{V'}{V} &\ll 18H_c^2, \\
3H_c^2 &\ll V.
\end{aligned}
\]
Ghost-modified Inflation

Two attractors:

- Normal inflation (vanishing gauge field)
- Ghost-modified inflation (ghost condensate of the gauge field)
Regularization of Chaotic Inflation

During slow-roll, one has \( d\rho_m \approx dV = V'd\varphi \), 
\[
\rho_m + p_m = \dot{\varphi}^2 \approx V'/9H_c^2,
\]
so that:

\[
N_{e-\text{folds}} \approx 3H_c^2 \int_{\varphi_i}^{\varphi_e} \frac{d\varphi}{V'}.
\]

With power-like potential, \( V = g\varphi^n \) the slow-roll conditions imply 
\[
gn(n-1)\varphi^{n-2} \ll 9H_c^2 \ll 3g\varphi^n.
\]

\[
N_{e-\text{folds}} \approx \begin{cases} 
\frac{1}{n(n-2)} \left( \frac{3H_c^2}{g} \right)^\frac{2}{n}, & n > 2, \\
\frac{3H_c^2}{4g} \ln \frac{g\varphi_i^2}{3H_c^2}, & n = 2.
\end{cases}
\]

Compare this to the value derived in common slow-roll inflation:

\[
N_{e-\text{folds}} \approx \frac{\varphi_i^2}{2n}.
\]
Planck2015 Data

![Planck2015 Data Diagram](image-url)
Hilltop Inflation

\[ V(\varphi) \sim 1 - (\varphi/a)^p + \ldots \] For example, a Higgs mechanism in GUT models.

\[ L_H = -\frac{1}{2} (D^\mu \Phi)^\dagger D_\mu \Phi - \frac{\beta^2}{4} (\Phi^\dagger \Phi - \alpha^2). \]

**FIG. 9:** The gauge, \( \psi \), and Higgs, \( h \), fields evolution during the compound inflation scenario.

**FIG. 10:** The Hubble parameter evolution during the compound inflation scenario indicates two inflation stages with \( H = H_c \) and \( H = H_s \), i.e. driven by gauge and Higgs fields in turn.