Cosmology in Horndeski Gravity

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Lovelock Gravity

GR equations of motion in vacua should satisfy:

$$(a)\ E^{ij} = E^{ji}m \quad (b)\ E^{ij} = E^{ij}(g_{lk}; \partial_m g_{lk}; \partial_m \partial_n g_{lk}), \quad (c)\ \nabla_i E^{ij} = 0.$$

If also (d): equations linear in second derivatives of g_{ab} then $E^{ij} = G^{ij} + \Lambda g^{ij}$ regardless of spacetime dimension, D.
Without (d) depends on D, (Lovelock, 1971):

$$E^i_j = \sum_{p=1}^{[(D+1)/2]-1} a_p \delta^{ih_1..h_{2p}}_{jk_1..k_{2p}} R^{j_1 j_2}_{h_1 h_2} \cdot \ldots \cdot R^{j_{2p-1} j_{2p}}_{h_{2p-1} h_{2p}} + a\delta^j_i.$$

The Lagrangian read as

$$L = \sum_{p=1}^{[(D+1)/2]-1} a_p \delta^{h_1..h_{2p}}_{k_1..k_{2p}} R^{j_1 j_2}_{h_1 h_2} \cdot \ldots \cdot R^{j_{2p-1} j_{2p}}_{h_{2p-1} h_{2p}} + a.$$

Horndeski theory (in $D = 4$)

(Horndeski, 1974) The most general scalar-tensor interaction such that equations of motion depend only on $(g_{lk}; \partial_m c g_{lk}; \partial_m \partial_n g_{lk}; \phi, \partial_m \phi, \partial_m \partial_n \phi)$.

$$\begin{aligned}
\mathscr{L} = {} & \sqrt{(g)}\,\mathscr{K}_1 \delta^{cde}_{hjk}\phi_{|c}{}^{|h}R_{de}{}^{jk} - \tfrac{4}{3}\sqrt{(g)}\,\dot{\mathscr{K}}_1\delta^{cde}_{hjk}\phi_{|c}{}^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} \\
& + \sqrt{(g)}\,\mathscr{K}_3\delta^{cde}_{hjk}\phi_{|c}\phi^{|h}R_{de}{}^{jk} - 4\sqrt{(g)}\,\dot{\mathscr{K}}_3\delta^{cde}_{hjk}\phi_{|c}\phi^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} \\
& + \sqrt{(g)}(\mathscr{F}+2\mathscr{W})\delta^{cd}_{fh}R_{cd}{}^{fh} + 2\sqrt{(g)}(2\mathscr{K}_3 - 2\mathscr{K}'_1 + 4\rho\dot{\mathscr{K}}_3)\delta^{cd}_{fh}\phi_{|c}{}^{|f}\phi_{|d}{}^{|h} \\
& - 3\sqrt{(g)}(2\mathscr{F}' + 4\mathscr{W}' + \rho\mathscr{K}_8)\phi_{|c}{}^{|c} + 2\sqrt{(g)}\,\mathscr{K}_8\delta^{cd}_{fh}\phi_{|c}\phi^{|f}\phi_{|d}{}^{|h} \\
& + \sqrt{(g)}\{4\mathscr{K}_9 - \rho(2\mathscr{F}'' + 4\mathscr{W}'' + \rho\mathscr{K}'_8 + 2\dot{\mathscr{K}}_9)\}
\end{aligned}$$

(Horndeski, 1976) The gauge field-tensor interaction such that (a) equations of motion depend only on $(g_{ab}; \partial_m g_{ab}; \partial_m \partial_n g_{ab}; A_i, \partial_m A_i, \partial_m \partial_n A_i)$; b 'charge' conserves; (c) flat space limit is Maxwell theory:

$$L = L_{EH} + L_{Maxwell} + \tilde{R}FF$$

Revival of Interest

1990s: late-time cosmic accelerated expansion. Motivation for modifications of gravity: large at long range, negligible at solar system range.

2000s: DGP model, massive gravity e.t.c. Galilean symmetry, $\pi(x) \to \pi(x) + b_\mu x^\mu + c$, is essential.
Galileon models in flat space: both Lagrangian and E.o.M. depend only on second derivatives of scalar field(s), p-forms.
In $D = 4$ 'covariantized' galileons, dimensional reductions of Lovelock gravity and Horndeski theory are almost the same.

2010: Horndeski-coupled Higgs field, $(g^{\mu\nu} + \xi G^{\mu\nu})\partial_\mu\varphi\partial_\nu\varphi$, provides nice inflation.

...Let us further investigate inflationary cosmology in Horndeski gravity!

Horndeski Yang–Mills

The key object is the dual of a Riemann tensor,
$\tilde{R}^{\alpha\beta\gamma\delta} = \frac{1}{4}\epsilon^{\alpha\beta\mu\nu}\epsilon^{\gamma\delta\rho\sigma}R_{\mu\nu\rho\sigma}$, where $\epsilon^{[\alpha\beta\mu\nu]}$ is the Levi-Civita tensor. It is divergent-free and thus can be used to provide safe coupling of the field strength to gravity:

$$L_{coupl} \sim \tilde{R}^{\alpha\beta\mu\nu}F^a_{\alpha\beta}F^a_{\mu\nu} \sim R_{\alpha\beta\mu\nu}\tilde{F}^{a\alpha\beta}\tilde{F}^{a\mu\nu}.$$

Both Riemann and field tensors satisfy the Bianchi identities which annihilate higher order derivatives in the equations of motion.

$$S = \int \left(\frac{M_{Pl}^2}{2}R - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{\mu^2\, l^2}{8}R_{\alpha\beta\mu\nu}\tilde{F}^{a\alpha\beta}\tilde{F}^{a\mu\nu} \right)\sqrt{-g}d^4x\,,$$

where the modified Planck mass is $M_{Pl} = 1/\sqrt{8\pi G}$, and the appropriate length scale of the theory is $l \equiv 1/(eM_{Pl})$

Equations of Motion

Stress-energy tensor:

$$T^{\rho\sigma} \equiv \frac{2}{\sqrt{-g}} \frac{\partial(L\sqrt{-g})}{\partial g_{\rho\sigma}} = F^{a\rho\alpha} F^{a\sigma}_{\ \ \alpha} - \frac{1}{4} g^{\rho\sigma} F^{a}_{\mu\nu} F^{a\mu\nu}$$

$$- \frac{\mu^2}{8} \left[-g^{\rho\sigma} R_{\alpha\beta\mu\nu} \tilde{F}^{a\alpha\beta} \tilde{F}^{a\mu\nu} + 2\, R^{\rho}_{\beta\mu\nu} \tilde{F}^{a\sigma\beta} \tilde{F}^{a\mu\nu} + 4\, \nabla_\beta \nabla_\mu \left(\tilde{F}^{a\rho\beta} \tilde{F}^{a\mu\sigma} \right) \right]$$

where

$$\nabla_\beta \nabla_\mu \left(\tilde{F}^{a\rho\beta} \tilde{F}^{a\mu\sigma} \right) = (D_\mu \tilde{F}^{\rho\beta})^a (D_\beta \tilde{F}^{\mu\sigma})^a + [F_{\beta\mu}, \tilde{F}^{\rho\beta}]^a \tilde{F}^{a\mu\sigma} + R^{\rho}_{\alpha\beta\mu} \tilde{F}^{a\alpha\beta} \tilde{F}^{a\mu\sigma} + R_{\alpha\mu} \tilde{F}^{a\rho\alpha} \tilde{F}^{a\mu\sigma} \,.$$

Notice that by virtue of Bianchi identities the term which is third-order in field tensor, $F\tilde{F}\tilde{F}$, arises instead of third-order derivatives. The field equation:

$$D_\rho \left(F^{\rho\sigma} + \frac{\mu^2}{2} \tilde{R}^{\rho\sigma\mu\nu} F_{\mu\nu} \right) = 0\,.$$

FLRW Cosmology

Unlike Maxwell field, the Yang–Mills $SU(2)$ configuration is compatible with FLRW metrics,

$$ds^2 = -N^2 dt^2 + a^2 \left[d\chi^2 + \Sigma_k^2(\chi)(d\theta^2 + \sin^2\theta d\varphi^2) \right] ,$$

where $\Sigma_k(\chi) = \{\sin\chi, \chi, \sinh\chi\}$.
The most general cosmological ansatz preserving the isotropy and homogeneity of the metrics can be written in terms of a single function $f(t)$:

$$\begin{aligned} A = & f(t) T_\chi d\chi + \left[f(t)\Sigma_k T_\theta + (\Sigma_k' - 1) T_\varphi \right] d\theta + \\ & \left[f(t)\Sigma_k T_\varphi - (\Sigma_k' - 1) T_\theta \right] \sin\theta d\varphi \,. \end{aligned}$$

The group generators, T_a, are the Pauli matrices, $\tau_k/(2i)$, contracted with spherical unit vectors, $n^k_{(\chi,\theta,\varphi)}$.

Effective Lagrangian

Let us introduce 'electric', $\mathcal{E} = \dot{f}/Na$, and 'magnetic', $\mathcal{H} = (k - f^2)/a^2$, components of the YM field tensor. The pure YM Lagrangian then read as:

$$L_{YM} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} = \frac{3}{2}(\mathcal{E}^2 - \mathcal{H}^2)\,.$$

The standard Einstein-Hilbert term (in the gauge $N = 1$) is:

$$L_{EH} = 3\left[\frac{\dot{a}^2 + k}{a^2} + \frac{\ddot{a}}{a}\right],$$

The coupling term looks like a 'safe' combination of above:

$$L_{coupl}\sqrt{-g} = -\frac{3\mu^2}{2}\left[\frac{\dot{a}^2 + k}{a^2}\mathcal{E}^2 - \frac{\ddot{a}}{a}\mathcal{H}^2\right]\,.$$

Equations of Motion

In the inflationary cosmology, the usual notations are: $\psi \equiv f/a$, $H \equiv \dot{a}/a$, and assume $k = 0$. The energy density and pressure read as:

$$\rho_{ym} = \frac{3}{2}\left(\dot{\psi}^2 + 2H\psi\dot{\psi} + H^2\psi^2 + \psi^4\right)\,, \quad p_{ym} = \frac{\rho_{ym}}{3}\,,$$

$$\rho_c = -\frac{3\mu^2}{2}\left[H^2(3\dot{\psi}^2 + 3H^2\psi^2 + 2\psi^4) + 2H\psi\dot{\psi}(3H^2 + 2\psi^2)\right]\,,$$

$$p_c = \frac{\mu^2}{2}\Big[3\dot{\psi}^2(3H^2 + 4\psi^2) + 2H\psi\dot{\psi}(7H^2 + 8\psi^2) + H^2\psi^2(5H^2 + 2\psi^2) + 4\ddot{\psi}(H^2\psi + H\dot{\psi} + \psi^3) + 2\dot{H}(\dot{\psi}^2 + 4H\psi\dot{\psi} + 3H^2\psi^2)\Big]\,.$$

Mention that the energy density corresponding to coupling term, ρ_c, is not positive-defined. The gauge field equation takes the following form:

$$(1-\mu^2H^2)\left(\dot{\psi} + H\psi\right)^{\dot{}} + 2\left[1 - \mu^2(\dot{H} + H^2)\right]\left(H\dot{\psi} + H^2\psi + \psi^3\right) = 0\,.$$

De Sitter Space

Mention that the Riemann tensor in de Sitter space read as

$$R_{\alpha\beta\mu\nu} = -\tilde{R}_{\alpha\beta\mu\nu} = H^2(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}) \, .$$

Since $\tilde{F}^2 = -F^2$, the full field Lagrangian is proportional to the conventional F^2 term:

$$L_{coupl} + L_{YM} = -\frac{1}{4}(1 - \mu^2 H^2) F^a_{\mu\nu} F^{a\mu\nu} \, .$$

And the equation of motion for the gauge field also acquires the same factor:

$$(1 - \mu^2 H^2) D_\rho F^{\rho\sigma} = 0 \, .$$

De Sitter space with $H = H_c \equiv \mu^{-1}$ is a special case for Horndeski model. However, the energy density and pressure are not vanishing!

Exact Solutions

With the ansatz $H = H_c$ for the metrics, the gauge field equation is identically satisfied. One has to solve only the Friedmann equation for $\dot{\psi}$:

$$\dot{\psi}_{\pm} = -\frac{1}{2H_c}\left(\psi^3 + H_c^2\psi \pm \sqrt{\psi^6 + (3/2)\psi^4 H_c^2 - H_c^4}\right) .$$

With any additional matter, ρ_m, in a state of a perfect fluid:

$$\dot{\psi}_{\pm} = -\frac{1}{2H_c}\left(\psi^3 + H_c^2\psi \pm \sqrt{\psi^6 + 2\psi^4 H_c^2/3 - H_c^4 + H_c^2\rho_m/3}\right) .$$

And the second Friedmann equation then holds if

$$\dot{\rho_m} + 3H_c(\rho_m + p_m) = 0 .$$

Properties of Exact Solution

dominating gauge field		dominating matter
$\dot{\psi}_+ \simeq -\frac{2\psi^3}{H_c}$	$\dot{\psi}_- \simeq -\frac{H_c\psi}{4}$	$\dot{\psi}_\pm \simeq \mp\sqrt{\frac{\rho_m}{3}}$
$\psi_+ \simeq \sqrt{\frac{H_c}{4(t-t_0)}}$	$\psi_- \simeq \psi_0 \exp\left(-H_c\, t/4\right)$	$\psi_\pm \simeq \psi_0 \mp \sqrt{\frac{\rho_m}{3}}\, t$

Condensate solution: $\rho_g \simeq \rho_m$, $\psi_c^4 \simeq \frac{2\rho_m}{3}$.

Solution	Eigenvalues
$\psi = \sqrt{\frac{H_c}{4(t-t_0)}}$	$\frac{12\psi^2}{H_c}, \frac{2\sqrt{15}\psi^2}{H_c}, -\frac{2\sqrt{15}\psi^2}{H_c}$
$\psi = \psi_0 \exp\left(-H_c\, t/4\right)$	$-2H_c, -\frac{H_c}{4}, -\frac{5H_c}{4}$
$\psi = \psi_c$	$-H_c, -2H_c, -2H_c$
$\psi = \psi_0 \mp \sqrt{\frac{\rho_m}{3}}\, t$	$\frac{3\psi^2}{H_c} - 2H_c, -\frac{3}{2H_c}(H_c^2 + \psi^2) \pm \sqrt{\frac{\pm 2\psi}{H_c}}(3\rho_m)^{1/4}$

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Gauge inflation

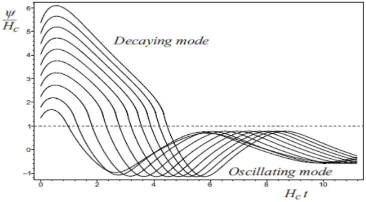


FIG. 1: The solutions for the gauge field, ψ. A non-minimal coupling scale, H_c, divides exponentially decaying solutions and oscillations.

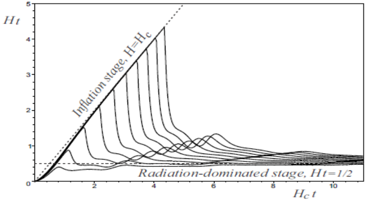


FIG. 2: The solution for the metrics, Ht represnets the inflationary stage with Hubble parameter value $H = H_c$, and radiation-dominated universe, $Ht = 1/2$.

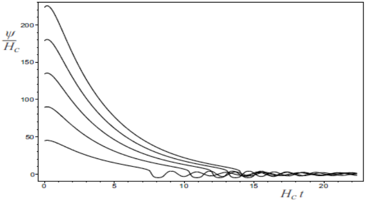


FIG. 3: The solutions for the gauge field, ψ, with initial state $\psi_i \gg H_c$ demonstrate a continuous exponentially decaying mode.

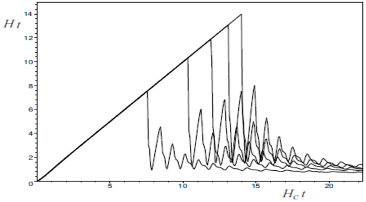


FIG. 4: The decaying mode of gauge field corresponds to de Sitter metrics with Hubble parameter value $H = H_c$; the oscillating gauge field gives rise to the radiation-dominated universe, $Ht = 1/2$.

Gauge Inflation

Horndeski theory is expected to be valid in the range $\psi_i \sim 1$, $\psi_e \sim H_c$.

For the inflating mode, $\psi \simeq \psi_i e^{(-H_c\, t/4)}$, one finds:

$$N_{e-\mathrm{folds}} \simeq H_c\, t \simeq 4 \ln \psi_i/\psi_e \,.$$

With $H \sim 10^{-6}..10^{-5}$ (from Planck) one obtains $N_{e-\mathrm{folds}} = 50..60$.

I.e. gauge inflation could play a significant role during the *observed* inflation stage.

Gauge Inflation with Matter

The duration of inflation is approximately

$$\Delta t \approx \int_{\rho_m^{(i)}}^{\rho_m^{(e)}} \frac{d\rho_m}{\dot{\rho}_m} \, .$$

Then with e.o.s. for matter in the background $H = H_c$ one has:

$$N_{e-\mathrm{folds}} \simeq H_c \Delta t \approx -\frac{1}{3} \int_{\rho_m^{(i)}}^{\rho_m^{(e)}} \frac{d\rho_m}{\rho_m + p_m} \, .$$

For the matter with equation of state $p_m = w\rho_m$ therefore:

$$N_{e-\mathrm{folds}} = \frac{1}{3(1+w)} \ln(\rho_m^{(i)}/\rho_m^{(e)}) \approx -\frac{4}{3(1+w)} \ln(H_c) \, .$$

With dust or radiation, $w = 0, 1/3$, one finds $N_{e-\mathrm{folds}} \approx -\ln H_c$.
Too weak for inflation. Another inflaton?

Slow-roll Inflation

Let us consider a system with dominating inflaton field, a slow-rolling scalar field:

$$L_m = \frac{\dot{\varphi}^2}{2} - V(\varphi)\,.$$

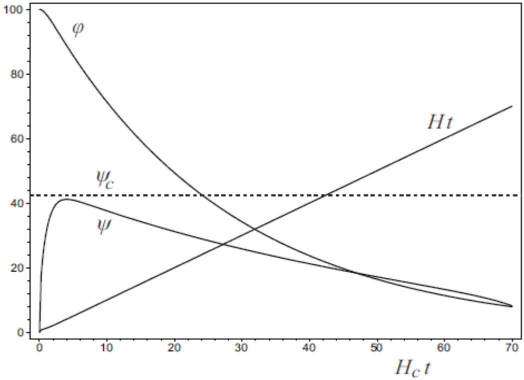
By assumption, $\rho_m \gg 3H_c^2 \simeq 3H^2$, while normally for the slow-roll model one has $\rho_m = 3H^2$. Therefore the stronger slow-roll conditions should be imposed on scalar field:

$$\begin{cases} \ddot{\varphi} \ll 3H_c\,\dot{\varphi}\,, \\ \dfrac{\dot{\varphi}^2}{2} \ll V\,, \\ 3H_c^2 \ll V\,, \end{cases} \quad \Rightarrow \quad \begin{cases} V'' \ll 9H_c^2\,, \\ \dfrac{V'^2}{V} \ll 18H_c^2 \\ 3H_c^2 \ll V\,. \end{cases}$$

Ghost-modified Inflation

Two attractors:

- Normal inflation (vanishing gauge field)
- Ghost-modified inflation (ghost condensate of the gauge field)



Regularization of Chaotic Inflation

During slow-roll, one has $d\rho_m \simeq dV = V' d\varphi$, $\rho_m + p_m = \dot{\varphi}^2 \simeq V'^2/9H_c^2$, so that:

$$N_{e-\text{folds}} \approx 3H_c^2 \int_{\varphi_e}^{\varphi_i} \frac{d\varphi}{V'} \,.$$

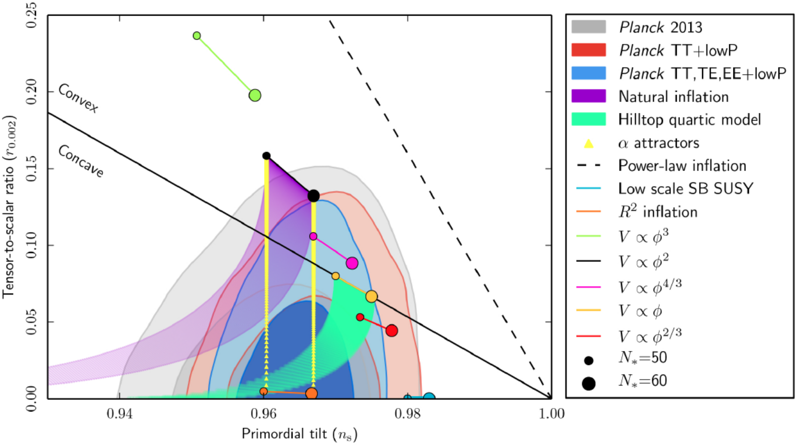
With power-like potential, $V = g\varphi^n$ the slow-roll conditions imply $gn(n-1)\varphi^{n-2} \ll 9H_c^2 \ll 3g\varphi^n$.

$$N_{e-\text{folds}} \approx \begin{cases} \dfrac{1}{n(n-2)} \left(\dfrac{3H_c^2}{g}\right)^{\frac{2}{n}} , & n > 2 \,, \\ \dfrac{3H_c^2}{4g} \ln \dfrac{g\varphi_i^2}{3H_c^2} \,, & n = 2 \,. \end{cases}$$

Compare this to the value derived in common slow-roll inflation:

$$N_{e-\text{folds}} \approx \frac{\varphi_i^2}{2n} \,.$$

Planck2015 Data



Hilltop Inflation

$V(\varphi) \sim 1 - (\varphi/a)^p + \ldots$. For example, a Higgs mechanism in GUT models.

$$L_H = -\frac{1}{2}(D^\mu \Phi)^\dagger D_\mu \Phi - \frac{\beta^2}{4}(\Phi^\dagger \Phi - \alpha^2)\,.$$

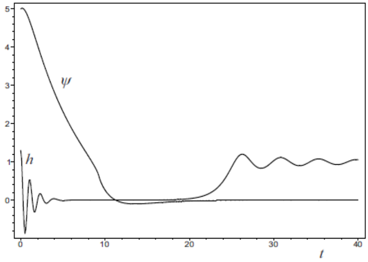


FIG. 9: The gauge, ψ, and Higgs, h, fields evolution during the compound inflation scenario.

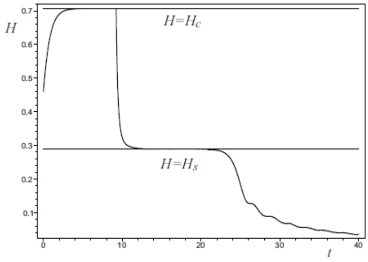


FIG. 10: The Hubble parameter evolution during the compound inflation scenario indicates two inflation stages with $H = H_c$ and $H = H_s$, i.e. driven by gauge and Higgs fields in turn.