Finite temperature bosonization for the Thirring model.

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Plan

- Linearisation of HEq of TM in a weak sense
- Free Bose and Dirac fields in 1+1D
- Inequivalent representations and DM
- Different solutions, superselection rules and VEV
- TFD. Thermal bosonization. Hot and cold thermofields
- Thermofield solutions of TM

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Linearisation of HEq of TM in a weak sense

Despite a considerable age the 1+1 Dimensional Thirring model:

$$H[\Psi] = H_{0[\Psi]}(x^0) + H_{I[\Psi]}(x^0), \quad x = (x^0, x^1), \ \ x^0 = t, \tag{1}$$

$$H_{0[\Psi]}(x^0) = \int_{-\infty}^{\infty} dx^1 \Psi^\dagger(x) E(P^1) \Psi(x), \tag{2}$$

$$H_{I[\Psi]}(x^0) = \frac{g}{2} \int_{-\infty}^{\infty} dx^1 J_{(\Psi)\,\mu}(x) J^{\mu}_{(\Psi)}(x), \ \ \text{with currents:} \tag{3}$$

$$J^{\mu}_{(\Psi)}(x) \longmapsto \overline{\Psi}(x)\gamma^{\mu}\Psi(x), \ \ J^{5\mu}_{(\Psi)}(x) \longmapsto \overline{\Psi}(x)\gamma^{\mu}\gamma^{5}\Psi(x), \tag{4}$$

$$J^{\xi}_{(\Psi)}(x) = J^{0}_{(\Psi)}(x) + \xi J^{1}_{(\Psi)}(x) \longmapsto 2\Psi^{\dagger}_{\xi}(x)\Psi_{\xi}(x), \ \ \text{for:} \ \ \xi = \pm, \tag{5}$$

$$P^1 = -i\partial_1, \ \ E(P^1) = \gamma^5 P^1, \ \ \gamma^5 = \gamma^0\gamma^1 \Rightarrow \sigma_3, \ \ \gamma^{\mu}\gamma^{5} = -\epsilon^{\mu\nu}\gamma_{\nu},$$

is still remained as important touchstone for non-perturbative methods of quantum field theory. It turns out to be a two -dimensional analog of the well-known Nambu-Jona-Lasinio model and provides an important example of using the well-known bosonization procedure (BP). In the present work the BP for TM is considered as a particular case of dynamical mapping (DM), what for Schwinger model was previously done by Greenberg[1].

[1] O.W. Greenberg, *Found. Phys.* **30**, 2000, 383.

Linearisation of HEq of TM in a weak sense

In the framework of canonical quantization scheme

$$\text{with the CAR:} \quad \left\{\Psi_\xi(x), \Psi_{\xi'}(y)\right\}\Big|_{x^0=y^0} = 0, \;\; \xi, \xi' = \pm, \qquad (6)$$

$$\left\{\Psi_\xi(x), \Psi^\dagger_{\xi'}(y)\right\}\Big|_{x^0=y^0} = \delta_{\xi,\xi'}\delta\left(x^1 - y^1\right), \quad \left(\otimes Z_{(\Psi)}(x-y)\right), \qquad (7)$$

the DM method[2] consists in the construction of Heisenberg field (HF) $\Psi(x)$ as a solution of Heisenberg equations (HEq) of motion:

$$i\partial_0\Psi(x) = [\Psi(x), H[\Psi]] \Rightarrow :\left[E(P^1) + g\gamma^0\gamma_\nu J^\nu_{(\Psi)}(x)\right]\Psi(x):, \qquad (8)$$

in the form of Haag expansion built as a sum of normal products of free physical fields $\psi(x)$, whose representation space accords with *a priori* unknown physical states of the given field theory. The DM $\Psi(x) \overset{\mathrm{w}}{=} :\Upsilon[\psi^{in}(x)]:$, being generally speaking a weak equality "w", implies the choice of appropriate representation space as well as the initial conditions to HEq: $\lim_{t\to-\infty}\Psi(x^1,t) \overset{\mathrm{w}}{=} \Upsilon[\psi^{in}(x^1,-\infty)]$, with the appropriate asymptotic physical field $\psi^{in}(x)$.

[2] H. Umezawa, H. Matsumoto, M. Tachiki, Thermo-field dynamics and condensed states. NHPC, Amsterdam, 1982

Linearisation of HEq of TM in a weak sense

The existence of such a DM implies for the full Hamiltonian (1) $H = H_0(t) + H_I(t)$ a weak representation as a sum of some constant W_0 and some free H_0^{in}:

$$H \stackrel{\mathrm{w}}{=} H_0^{in} + W_0, \quad \text{in a sense:} \quad \langle a|H|b\rangle = \langle a|H_0^{in}|b\rangle + W_0\langle a|b\rangle, \quad (9)$$

However the asymptotic completeness and irreducibility are absent in the presense of bound states. In particular for **this** two-dimensional model the physical asymptotic states of propagated physical particles have nothing to do with massless free Dirac asymptotic fields. For such a case it is convenient[3] to use DM onto the Schrödinger -like physical field $\psi(x)$, associated with the HF at $t \to 0$: $\lim_{t\to 0} \Psi(x^1, t) \stackrel{\mathrm{w}}{=} \Upsilon[\psi(x^1, 0)]$, which is a generalization of the well-known interaction representation and is closely related to the above canonical quantization procedure (6), (7).

[3] L.D. Faddeev, *Sov. Phys. Doklady*, **8**, 1964, 881; A.N. Vall, S.E. Korenblit, V.M. Leviant, A.B. Tanaev, *J. Nonlin. Math. Phys.* 1997, **4**, 492; A.V. Shebeko, M.I.Shirokov, *Phys. Part. Nucl.* **32**, 2001, 31.

Linearisation of HEq of TM in a weak sense

An immediate consequence of HEq (8), is the conservation of both the currents (4), that determine their dynamics as a free one:

$$\partial_\mu J^\mu_{(\Psi)}(x) = 0, \;\; \partial_\mu J^{5\mu}_{(\Psi)}(x) = -\epsilon_{\mu\nu}\partial^\mu J^\nu_{(\Psi)}(x) = 0, \tag{10}$$

$$i\partial_0\gamma^0\gamma_\nu J^\nu_{(\Psi)}(x) - \left[\gamma^0\gamma_\nu J^\nu_{(\Psi)}(x), H_{0[\Psi]}(x^0)\right] = iI\,\partial_\mu J^\mu_{(\Psi)}(x) +$$

$$+i\gamma^5\,\epsilon_{\mu\nu}\partial^\mu J^\nu_{(\Psi)}(x) \equiv 0 = \left[\gamma^0\gamma_\nu J^\nu_{(\Psi)}(x), H_{I[\Psi]}(x^0)\right], \tag{11}$$

suggesting the linearization of HEq in a weak sense for free curent:

$$\gamma^0\gamma_\nu J^\nu_{(\Psi)}(x) \overset{\mathrm{w}}{\mapsto} \frac{\beta}{2\sqrt{\pi}}\gamma^0\gamma_\nu \widehat{J}^\nu_{(\chi)}(x) \Rightarrow \frac{\beta}{2\sqrt{\pi}}\gamma^0\gamma_\nu :J^\nu_{(\chi)}(x):. \tag{12}$$

$\chi(x)$ is a free massless Dirac field, $(\gamma\partial)\chi(x) = 0$, $Z_{(\chi)}(a) = 1$.
Note that Heisenberg current operators $J^\nu_{(\Psi)}(x)$ acquire precise operator meaning – with non-vanishing Schwinger term – only after the choice of the representation space of CAR (6), (7) and subsequent reduction in this representation to the normal-ordered form by means of renormalization via well-known Mandelstam -Schwinger point-splitting prescription on time-like and/or space-like vectors and subtraction of vacuum expectation value:

Linearisation of HEq of TM in a weak sense

$\varepsilon^2 = -\widetilde{\varepsilon}^2 \geq 0$, so that $\widetilde{\varepsilon}^0 = \varepsilon^1 \to 0$, with fixed $\widetilde{\varepsilon}^1 = \varepsilon^0$:

$$J^0_{(\Psi)}(x) \mapsto \lim_{\widetilde{\varepsilon}\to 0} \widehat{J}^0_{(\Psi)}(x;\widetilde{\varepsilon}) = \widehat{J}^0_{(\Psi)}(x), \text{ the same for } \widehat{J}^{\nu}_{(\chi)}(x;\widetilde{\varepsilon}(\varepsilon)),$$

$$J^1_{(\Psi)}(x) \mapsto \lim_{\varepsilon\to 0} \widehat{J}^1_{(\Psi)}(x;\varepsilon) = \widehat{J}^1_{(\Psi)}(x), \text{ with: } \widehat{J}^{\nu}_{(\Psi)}(x;a) = \tag{13}$$

$$= Z^{-1}_{(\Psi)}(a)\left[\overline{\Psi}(x+a)\gamma^{\nu}\Psi(x) - \langle 0|\overline{\Psi}(x+a)\gamma^{\nu}\Psi(x)|0\rangle\right]. \tag{14}$$

The constant β, multiplicative renormalization $Z_{(\Psi)}(a)$ for $a \to 0$, depending on ultraviolet cut-off Λ, and dynamical dimension $\mathrm{d}_{(\Psi)}$ will be defined dynamically by means of self-consistent calculation

$$\text{as: } Z_{(\Psi)}(a) \Rightarrow \left[-\Lambda^2 a^2\right]^{-\overline{\beta}^2/4\pi}, \quad \mathrm{d}_{(\Psi)} = 1/2 + \overline{\beta}^2/4\pi. \tag{15}$$

According to (10), (12) now it is enough only a free current's bosonization rules. There $\exists$ a free massless pseudoscalar field $\phi(x)$:

$$\widehat{J}^{\mu}_{(\chi)}(x) = -\frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_{\nu}\phi(x), \quad \widehat{J}^{-\xi}_{(\chi)}(x) = \frac{2}{\sqrt{\pi}}\partial_{\xi}\varphi^{\xi}\left(x^{\xi}\right), \tag{16}$$

$$x^{\xi} = x^0 + \xi x^1, \quad \phi(x) = -\sum_{\xi=\pm}\xi\varphi^{\xi}\left(x^{\xi}\right), \quad \partial^2\phi(x) = 0. \tag{17}$$

Free Bose and Dirac fields in 1+1D

The right ($\xi = -$) and left ($\xi = +$) moving fields $\varphi^{\xi}(x^{\xi})$ and their charges Q^{ξ} are defined in the space $c(k^1)|0\rangle = 0$ of $\phi(x)$, as:

$$\mathcal{P}c(k^1)\mathcal{P}^{-1} = -c(-k^1), \quad [c(k^1), c^{\dagger}(q^1)] = 4\pi k^0 \delta(k^1 - q^1), \quad (18)$$

$$\phi(x) = \int_{-\infty}^{\infty} \frac{dk^1}{4\pi k^0} \left[c\left(k^1\right) e^{-i(kx)} + c^{\dagger}\left(k^1\right) e^{i(kx)} \right], \quad (19)$$

$$\varphi^{\xi(+)}(s) = -\int_{-\infty}^{\infty} \frac{dk^1}{2k^0} \frac{\xi\theta(-\xi k^1)}{2\pi} c(k^1) e^{-ik^0 s}, \quad k^0 = |k^1|, \quad (20)$$

$$\varphi^{\xi(-)}(s) = \left[\varphi^{\xi(+)}(s)\right]^{\dagger}, \quad \varphi^{\xi}(s) = \varphi^{\xi(+)}(s) + \varphi^{\xi(-)}(s), \quad (21)$$

$$Q^{\xi(+)}(\widehat{x}^0) = \lim_{L\to\infty} i\xi \int_{-\infty}^{\infty} dk^1 \theta\left(-\xi k^1\right) c(k^1) e^{-ik^0 \widehat{x}^0} \delta_L(k^1)/2, \quad (22)$$

$$Q^{\xi(-)} = \left[Q^{\xi(+)}\right]^{\dagger}, \quad Q^{\xi} = Q^{\xi(+)} + Q^{\xi(-)}, \quad \delta_L(k^1) \underset{L\to\infty}{\longmapsto} \delta(k^1), \quad (23)$$

$$\left.\begin{matrix} O \\ O_5 \end{matrix}\right\} = \lim_{L\to\infty} \int_{-\infty}^{\infty} dy^1 \Delta\left(\frac{y^1}{L}\right) \left\{\begin{matrix} -\partial_1 \\ \partial_0 \end{matrix}\right\} \phi(y^1, x^0) = \sum_{\xi=\pm} \left\{\begin{matrix} Q^{\xi}, \\ -\xi Q^{\xi}. \end{matrix}\right. \quad (24)$$

$\delta_L(k^1)$ is Fourier image of charge's regularization function $\Delta(y^1/L)$.

Free Bose and Dirac fields in 1+1D

Using the main CCR:

$$\left[\varphi^{\xi(\pm)}(s),\varphi^{\xi'(\mp)}(\tau)\right]=\mp\frac{\delta_{\xi,\xi'}}{4\pi}\ln\Big(i\overline{\mu}\Big\{\pm(s-\tau)-i0\Big\}\Big),\quad (25)$$

$$\left[\varphi^{\xi(\pm)}(s),Q^{\xi'(\mp)}\right]=\frac{i}{4}\delta_{\xi,\xi'},\quad \left[Q^{\xi(\pm)},Q^{\xi'(\mp)}\right]=\pm a_0\delta_{\xi,\xi'},\quad (26)$$

$$a_0=a_0(L)=\pi\int_0^{\infty}dk^1k^0\left(\delta_L(k^1)\right)^2,\quad (27)$$

one can construct a variety of different inequivalent representations of the solutions of free Dirac equation $\partial_\xi\chi_\xi(x)=0$. The simplest[4], which for $Z_{(\chi)}(a)=1$ leads exactly to the bosonization rules (16) is:

$$\chi_\xi\left(x^{-\xi}\right)=\mathcal{N}_\varphi\left\{\exp\left(-i2\sqrt{\pi}\left[\varphi^{-\xi}(x^{-\xi})+\frac{\xi}{4}Q^{\xi}\right]\right)\right\}u_\xi,\quad (28)$$

$$u_\xi=\left(\frac{\overline{\mu}}{2\pi}\right)^{1/2}e^{i\varpi-i\xi\Theta/4}\exp\left\{-a_0\frac{\pi}{8}\right\}\equiv u_\xi^{Ok}\exp\left\{-a_0\frac{\pi}{8}\right\}.\quad (29)$$

It gives the formal solution of HEq (8), (12) as exponentials

[4] N.N. Bogoliubov, A.A. Logunov, A.I. Oksak, I. T. Todorov, General principles of quantum field theory. KAP, Boston, 1990.

in the normal-ordered form similar to Klaiber[5] but is not the same:

$$\Psi_\xi(x) = \Psi_\xi(x^1, x^0) = e^{C^{\xi(-)}(x)} \Psi_\xi\left(x^1 - \xi x^0, 0\right) e^{C^{\xi(+)}(x)}, \qquad (30)$$

where the bos-ion rules (16) for the current of field $\chi(x)$ (28) gives:

$$C^{\xi(\pm)}(x) = -i\frac{\beta g}{2\sqrt{\pi}} \int_0^{x^0} dy^0 \widehat{J}^{-\xi(\pm)}_{(\chi)}\left(x^1 + \xi y^0 - \xi x^0, y^0\right) =$$

$$= -i\frac{\beta g}{2\pi}\left[\varphi^{\xi(\pm)}\left(x^\xi\right) - \varphi^{\xi(\pm)}\left(-x^{-\xi}\right)\right].$$

Remarkably, that the completely unknown initial HF appears here also as a solution $\Psi_\xi(x^1 - \xi x^0, 0) \Rightarrow \lambda_\xi(x^{-\xi})$ of free massless Dirac equation $\partial_\xi \lambda_\xi(x^{-\xi}) = 0$, but certainly unitarily inequivalent to the free field $\chi(x)$ according to Haag theorem. It should be taking also in the normal-ordered form $\mathcal{N}_\varphi$ with respect to the field $\varphi^{\xi(\pm)}(s)$, using appropriate Bogoliubov bosonic canonical transformation of this field $U_\eta = \exp F_\eta$ with constant parameters $\overline{\alpha} = 2\sqrt{\pi}\cosh\eta$ and $\overline{\beta} = 2\sqrt{\pi}\sinh\eta$, obeying $\overline{\alpha}^2 - \overline{\beta}^2 = 4\pi$,

[5] B. Klaiber, "Lectures in Theoretical Physics", Gordon and Breach, NY, 1968, **X**, Part A, pp. 141-176.

Inequivalent representations and DM

with the following new fields, charges and generator: $y^\xi = x^0 + \xi y^1$,

$$\omega^\xi\left(x^\xi\right) = U_\eta^{-1}\varphi^\xi\left(x^\xi\right)U_\eta = \frac{1}{2\sqrt{\pi}}\left[\overline{\alpha}\varphi^\xi\left(x^\xi\right) + \overline{\beta}\varphi^{-\xi}\left(-x^\xi\right)\right],$$

$$v_\xi = \exp\left\{-a_0\frac{\overline{\beta}^2}{16}\right\}\left(\frac{\overline{\mu}}{\Lambda}\right)^{\overline{\beta}^2/4\pi} u_\xi, \quad \lambda_\xi(x^{-\xi}) = U_\eta^{-1}\chi_\xi(x^{-\xi})U_\eta =$$

$$= \mathcal{N}_\varphi\left\{\exp\left(-i2\sqrt{\pi}\left[\omega^{-\xi}\left(x^{-\xi}\right) + \frac{\xi}{4}\mathcal{W}^\xi\right]\right)\right\}v_\xi, \qquad (31)$$

$$\mathcal{W}^\xi = U_\eta^{-1}Q^\xi U_\eta = \frac{1}{2\sqrt{\pi}}\left[\overline{\alpha}Q^\xi - \overline{\beta}Q^{-\xi}\right],$$

$$F_\eta = \frac{\eta}{2\pi}\int_{-\infty}^{\infty}\frac{dk^1}{2k^0}\theta(k^1)\left(c(k^1)c(-k^1) - c^\dagger(-k^1)c^\dagger(k^1)\right) =$$

$$= \frac{i\eta}{2}\int_{-\infty}^{\infty}dy^1\phi(y^1, -x^0)\overset{\leftrightarrow}{\partial_0}\phi(y^1, x^0) = 2i\eta\int_{-\infty}^{\infty}dy^1\varphi^\xi(y^\xi)\partial_0\varphi^{-\xi}(-y^\xi).$$

By imposing the conditions onto the parameters, that necessary to have a correct spin 1/2 Lorentz -transformation properties

Inequivalent representations and DM

and correct CAR (6), (7), respectively:

$$\overline{\alpha}^2 - \overline{\beta}^2 = 4\pi, \;\; \overline{\beta} - \frac{\beta g}{2\pi} = 0, \;\; e^{\eta} = \frac{2\sqrt{\pi}}{\beta} = \sqrt{1 + \frac{g}{\pi}}, \tag{32}$$

the following variant of Oksak solution[6] of TM is obtained:

$$\Psi_{\xi}^{Ok}(x) = \mathcal{N}_{\varphi}\left\{\exp\left(-i2\sqrt{\pi}\left[\varrho^{-\xi}(x) + \frac{\xi}{4}\mathcal{W}^{\xi}\right]\right)\right\} v_{\xi}, \tag{33}$$

$$2\sqrt{\pi}\,\varrho^{-\xi}(x) = \overline{\alpha}\varphi^{-\xi}\left(x^{-\xi}\right) + \overline{\beta}\varphi^{\xi}\left(x^{\xi}\right), \tag{34}$$

$$v_{\xi} = \widehat{\mathrm{v}}_{\xi}\exp\left\{-a_0\frac{\pi}{8}\cosh 2\eta\right\}, \;\; \widehat{\mathrm{v}}_{\xi} = \left(\frac{\overline{\mu}}{2\pi}\right)^{1/2}\left(\frac{\overline{\mu}}{\Lambda}\right)^{\overline{\beta}^2/4\pi} e^{i\varpi - i\xi\Theta/4}. \tag{35}$$

The observed weak linearization of HEq for TM together with the nonlinearity of DM (33) and the weak initial conditions at $x^0 = 0$ allows to overcome the restrictions of Haag theorem, by removing the problems again into the representation construction of physical fields: at first as reducible massless free Dirac fields $\chi(x)$, and then as irreducible free massless pseudo scalar field $\phi(x)$.

[6] Oksak A. I., *Teoret. Mat. Fiz.* 1981. **48**. 297.

Inequivalent representations and DM

This solution contains all Klein factors also under the normal form $\mathcal{N}_\varphi$, as is demanded for DM. The infrared regularization parameters $\overline{\mu}$ and a_0 appear for both the free and Oksak field (35) in the one and the same combination as: ($\eta = 0$, $\mathrm{d}_{(\chi)} = 1/2$)

$$\left[\overline{\mu}\exp\left\{-a_0\frac{\pi}{4}\right\}\right]^{\mathrm{d}_{(\Psi)}}, \quad \mathrm{d}_{(\Psi)} = \frac{\cosh 2\eta}{2}, \quad \overline{\alpha} \pm \overline{\beta} = 2\sqrt{\pi}e^{\pm\eta}. \quad (36)$$

Different solutions, superselection rules and VEV

In order to connect the solution (33), (35) with another known solutions of TM the unitary transformations of conformal shift for the fields φ^{ξ} are used. To this end we consider one- and two-parametric families of solutions for arbitrary real σ, ρ, defined by: $\mathrm{K}_{\sigma} = \exp \mathrm{X}_{\sigma}$, $\mathcal{L}_{\rho} = \exp \mathrm{Y}_{\rho}$, as $\Psi_{\xi}(x,\sigma) = \mathrm{K}_{\sigma}^{-1} \Psi_{\xi}^{Ok}(x) \mathrm{K}_{\sigma}$, with

$$\mathrm{X}_{\sigma} = i(\sigma\bar{\xi}/4)\left(Q^{-\bar{\xi}}Q^{-\bar{\xi}} - Q^{\bar{\xi}}Q^{\bar{\xi}}\right) = i\sigma O O_5/4, \tag{37}$$

$$\mathrm{Y}_{\rho} = -i\rho Q^{\bar{\xi}} Q^{-\bar{\xi}}/2 = -i\rho\left(O^2 - O_5^2\right)/8, \text{ and the field:} \tag{38}$$

$$\Psi_{\xi}(x,\sigma,\rho) = \mathcal{L}_{\rho}^{-1}\Psi_{\xi}(x,\sigma)\mathcal{L}_{\rho} = \mathcal{N}_{\varphi}\left\{e^{R_{\xi}(x,\sigma,\rho)}\right\} v_{\xi}(\sigma,\rho), \tag{39}$$

$$R_{\xi}(x,\sigma,\rho) = -i2\sqrt{\pi}\left[\varrho^{-\xi}(x) + \frac{1}{8}\Sigma_{+}^{\xi}Q^{\xi} + \frac{1}{8}\Sigma_{-}^{\xi}Q^{-\xi}\right], \tag{40}$$

$$v_{\xi}(\sigma,\rho) = \widehat{\mathrm{v}}_{\xi}\exp\left\{-a_0\frac{\pi}{32}\left[\left(\Sigma_{-}^{\xi}\right)^2 + \left(\Sigma_{+}^{\xi}\right)^2\right]\right\}, \text{ with:} \tag{41}$$

$$\Sigma_{\pm}^{\xi} = e^{-\eta}\left[\xi(1-\sigma)+\rho\right] \pm e^{\eta}\left[\xi(1+\sigma)+\rho\right]. \tag{42}$$

This extension of Oksak solution (33) with any σ, ρ obeys again the same CAR (6), (7) and the bosonization rule (12)–(16),

Different solutions, superselection rules and VEV

with the same renormalization constant $Z_{(\Psi)}(a)$ (15).
For $\rho = 0$, by using the definition (24) both of charges O, O_5 it is a simple matter to check that $\sigma = \pm 1$ gives the two types of Mandelstam solution[7], for example:

$$\Psi_\xi(x,1) = \mathcal{N}_\varphi \left\{ e^{R_\xi(x,1)} \right\} \widehat{\mathrm{v}}_\xi \exp\left\{ -a_0 \frac{\pi}{4} e^{2\eta} \right\}, \quad \sigma = 1, \tag{43}$$

$$R_\xi(x,1) = -i\sqrt{\pi}\left[\xi e^{-\eta}\phi(x^1,x^0) - e^{\eta}\int_{-\infty}^{x^1} dy^1 \partial_0\phi(y^1,x^0) \right], \tag{44}$$

while $\sigma = -\coth 2\eta$ corresponds to normal form of solution of Morchio[8] et al.

[7] Mandelstam S., *Phys. Rev.* 1975. **D 11**. 3026.
[8] Morchio G., Pierotti D., Strocchi F., *J. Math. Phys.* 1992. **33**. 777.

Different solutions, superselection rules and VEV

The p -point Wightman function corresponding to vacuum expectation value (VEV) of the string of the fields (39) with: $l_i = +1$, for Ψ_i; $l_i = -1$, for $\Psi_i^\dagger$, acquires the multiplier:

$$\left(\Lambda^{\overline{\beta}^2/4\pi}\sqrt{2\pi}\right)^p \left\langle 0 \left| \prod_{i=1}^{p} \Psi_{\xi_i}^{(l_i)}(x_i,\sigma,\rho) \right| 0 \right\rangle \propto$$

$$\propto \exp\left\{ i\varpi\mathcal{S}_p - i\frac{\Theta}{4}\mathcal{S}_{p5} \right\} \exp\left\{ \frac{1}{4}\left[e^{2\eta}\mathcal{S}_p^2 + e^{-2\eta}\mathcal{S}_{p5}^2 \right] \ln\overline{\mu} \right\} \cdot$$

$$\cdot \exp\left\{ -a_0\frac{\pi}{16}\left(e^{2\eta}\Big[(1+\sigma)\mathcal{S}_p + \rho\mathcal{S}_{p5}\Big]^2 + e^{-2\eta}\Big[(1-\sigma)\mathcal{S}_{p5} + \rho\mathcal{S}_p\Big]^2 \right) \right\},$$

which absorbs all the infrared regularization parameters $\overline{\mu}$ and a_0 as well as σ, ρ dependence and disappears only when both of superselection rules (45) are fulfilled:

$$\mathcal{S}_p \equiv \sum_{i=1}^{p} l_i \Rightarrow 0, \quad \mathcal{S}_{p5} \equiv \sum_{i=1}^{p} l_i\xi_i \Rightarrow 0. \qquad (45)$$

TFD. Thermal bosonization. Hot and cold thermofields

At finite temperature $T > 0$, $\varsigma = (k_B T)^{-1}$, in the framework of thermofield dynamics[9] (TFD) it is necessary to double the number of degrees of freedom by providing all the fields Ψ with their tilde partners $\widetilde{\Psi}$ according to antilinear homomorphism

$$(AB)^{\sim} = \widetilde{A}\widetilde{B}, \quad (\alpha A + \beta B)^{\sim} = \alpha^* \widetilde{A} + \beta^* \widetilde{B}, \tag{46}$$

$$(A^{\dagger})^{\sim} = (\widetilde{A})^{\dagger}, \quad \text{with the condition :} \quad (\widetilde{b}(\varsigma))^{\sim} = b(\varsigma). \tag{47}$$

The resulting theory will be determined by the Hamiltonian $\widehat{H}[\Psi, \widetilde{\Psi}] = H[\Psi] - \widetilde{H}[\widetilde{\Psi}]$, with $\widetilde{H}[\widetilde{\Psi}] = H^*[\widetilde{\Psi}^*]$, $H[\Psi] = H_{0[\Psi]} + H_{I[\Psi]}$. The kinematic independence of tilde-conjugate fields $\widetilde{\Psi}$ means that:

$$\left\{ \Psi_{\xi}(x), \widetilde{\Psi}^{\#}_{\xi'}(y) \right\}\Big|_{x^0=y^0} = 0, \quad \left\{ \Psi_{\xi}(x), \widetilde{\Psi}^{\#}_{\xi'}(y) \right\}\Big|_{(x-y)^2<0} = 0, \tag{48}$$

and corresponds to independence of their Hamiltonians and their HEqs. This allows to consider a solution only for the one of them. Since the thermal transformations $\mathcal{V}_{\vartheta(F)}$, $\mathcal{V}_{\vartheta(B)}$ are not depend on coordinates and time, they can be applied directly to (48) and HEq of TM, leading again to the same equations for the new HF $\Psi(x, \varsigma)$.

[9] H. Umezawa, H. Matsumoto, M. Tachiki, Amsterdam, 1982

TFD. Thermal bosonization. Hot and cold thermofields

The thermal transformation is given by infinite product of one mode operators and the transformed vacuum state is a coherent state: $|0(\varsigma)\rangle = \mathcal{V}^{-1}_{\vartheta(B)}|0\widetilde{0}\rangle$. For Bose fields: $\tanh^2\vartheta(k^1,\varsigma) = e^{-\varsigma k^0}$,
$L \Rightarrow 2\pi\delta(0) \to \infty$, $a_{k^1}\sqrt{2k^0 L} \Rightarrow c_{k^1}$, $\mathcal{K}_-(k^1) = \widetilde{c}_{k^1}c_{k^1}$,
$\mathcal{K}_+(k^1) = c^\dagger_{k^1}\widetilde{c}^\dagger_{k^1}$, $\mathcal{K}_0(k^1) = (c^\dagger_{k^1}c_{k^1} + \widetilde{c}_{k^1}\widetilde{c}^\dagger_{k^1})/2$,

$$\prod_{k^1=-\infty}^{\infty} \mathrm{V}^{-1}_{\vartheta(k^1)(B)} \underset{L\to\infty}{\Longrightarrow} \mathcal{V}^{-1}_{\vartheta(B)} = \exp\{-\mathcal{X}_\vartheta\} = \mathcal{V}^{\dagger}_{\vartheta(B)}, \tag{49}$$

$$\mathcal{X}_\vartheta = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{dk^1}{2k^0}\vartheta(k^1,\varsigma)\left[\mathcal{K}_-(k^1) - \mathcal{K}_+(k^1)\right] = \widetilde{\mathcal{X}}_\vartheta, \tag{50}$$

$$\mathcal{V}^{-1}_{\vartheta(B)} = \exp\left\{\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{dk^1}{2k^0}\tanh\vartheta(k^1,\varsigma)\mathcal{K}_+(k^1)\right\}\cdot$$

$$\cdot\exp\left\{-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{dk^1}{2k^0}\ln\left(\cosh^2\vartheta(k^1,\varsigma)\right)\mathcal{K}_0(k^1)\right\}\cdot$$

$$\cdot\exp\left\{-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{dk^1}{2k^0}\tanh\vartheta(k^1,\varsigma)\mathcal{K}_-(k^1)\right\}, \tag{51}$$

TFD. Thermal bosonization. Hot and cold thermofields

Bogoliubov transformations for "hot" [+], and "cold" [−] operators, their CCR, condensate density, and vacuum overlaping read:

$$c_{k^1}([\pm]\varsigma) = \mathcal{V}^{\mp 1}_{\vartheta(B)} c_{k^1} \mathcal{V}^{\pm 1}_{\vartheta(B)} = c_{k^1} \cosh\vartheta \mp \widetilde{c}^{\dagger}_{k^1} \sinh\vartheta,$$
$$\widetilde{c}_{k^1}([\pm]\varsigma) = \mathcal{V}^{\mp 1}_{\vartheta(B)} \widetilde{c}_{k^1} \mathcal{V}^{\pm 1}_{\vartheta(B)} = \widetilde{c}_{k^1} \cosh\vartheta \mp c^{\dagger}_{k^1} \sinh\vartheta, \tag{52}$$

$$\left[c_{k^1}([\pm]\varsigma), c^{\dagger}_{q^1}([\pm]\varsigma)\right] = (2\pi)\left(2k^0\right)\delta\left(k^1 - q^1\right),$$
$$\left[\widetilde{c}_{k^1}([\pm]\varsigma), \widetilde{c}^{\dagger}_{q^1}([\pm]\varsigma)\right] = (2\pi)\left(2k^0\right)\delta\left(k^1 - q^1\right), \tag{53}$$

$$c_{k^1}([+]\varsigma)|0(\varsigma)\rangle = 0, \quad \widetilde{c}_{k^1}([+]\varsigma)|0(\varsigma)\rangle = 0, \quad k^0 = \omega_{k^1}, \tag{54}$$

$$\langle 0(\varsigma)|c^{\dagger}_{k^1} c_{k^1}|0(\varsigma)\rangle = \langle 0\widetilde{0}|c^{\dagger}_{k^1}([+]\varsigma) c_{k^1}([+]\varsigma)|0\widetilde{0}\rangle =$$
$$= 4\pi k^0 \delta(0) \sinh^2\vartheta = 2k^0 L(e^{\varsigma k^0} - 1)^{-1}, \quad L = 2\pi\delta(0), \tag{55}$$

$$\langle 0\widetilde{0}|0(\varsigma)\rangle = \langle 0\widetilde{0}|\mathcal{V}^{-1}_{\vartheta(B)}|0\widetilde{0}\rangle \Rightarrow \exp\left\{-(L/\varsigma)(\pi/12)\right\} \Leftarrow \tag{56}$$

$$\Leftarrow \exp\left\{-\delta(0)\int_{-\infty}^{+\infty} dk^1 \ln\left(\cosh\vartheta(k^1,\varsigma)\right)\right\} \underset{L\to\infty}{\Longrightarrow} 0, \tag{57}$$

what means the unitarily inequivalense of different representations of QFT at different temperature

TFD. Thermal bosonization. Hot and cold thermofields

About motivation of the hot and cold thermofields.
For any functional $\mathcal{F}[\Psi]$ of HF in the given representation of physical fields $\psi(x)$, i.e. for given DM $\Psi(x) = \Upsilon[\psi(x)]$ at zero temperature, being interested in the matrix elements on the thermal vacuum of the type:

$$\langle 0(\varsigma)|\mathcal{F}[\Psi(x)]|0(\varsigma)\rangle = \langle 0\widetilde{0}|\mathcal{V}_\vartheta \mathcal{F}[\Psi(x)]\mathcal{V}_\vartheta^{-1}|0\widetilde{0}\rangle = \tag{58}$$

$$= \langle 0\widetilde{0}|\mathcal{F}\left[\mathcal{V}_\vartheta \Psi(x)\mathcal{V}_\vartheta^{-1}\right]|0\widetilde{0}\rangle \equiv \langle 0\widetilde{0}|\mathcal{F}[\Psi(x,[-]\varsigma)]|0\widetilde{0}\rangle,$$

we come to formal mapping:

$$\Psi(x,[-]\varsigma) = \mathcal{V}_\vartheta \Psi(x)\mathcal{V}_\vartheta^{-1} = \Upsilon\left[\mathcal{V}_\vartheta \psi(x)\mathcal{V}_\vartheta^{-1}\right] = \Upsilon[\psi(x,[-]\varsigma)], \tag{59}$$

onto the "cold" physical thermofield:

$$\psi(x,[-]\varsigma) = \mathcal{V}_\vartheta \psi(x)\mathcal{V}_\vartheta^{-1}, \text{ with the same coefficient} \tag{60}$$

functions, as for the initial DM, that transferring so all the temperature dependence from the vacuum state $|0(\varsigma)\rangle$ onto these "cold" physical thermofields.

TFD. Thermal bosonization. Hot and cold thermofields

To compute matrix element (58) it is necessary to substitute into the r.h.s. of (58), (59) this cold physical thermofields (60) again in terms of the initial physical fields $\psi(x)$ (52) $[-]$, and reorder again the so obtained operator with respect to this initial fields to obtain:

$$\langle 0(\varsigma)|\mathcal{F}\left[\Psi\right]|0(\varsigma)\rangle \Rightarrow \langle 0\widetilde{0}|\mathcal{F}\left[\widehat{\Upsilon}\left[[-]\varsigma;\psi(x)\right]\right]|0\widetilde{0}\rangle. \tag{61}$$

The standard computation way implies the substitution of the inverse to (52) $[+]$ linear expressions of physical fields $\psi(x) = \mathcal{V}_{\vartheta}\psi(x,[+]\varsigma)\mathcal{V}_{\vartheta}^{-1}$ in terms of the "hot" physical thermofields into the l.h.s. of (58) and reordering the so obtained operator with respect to **this** hot physical thermofield over the thermal ("hot") vacuum $|0(\varsigma)\rangle$, to obtain:

$$\langle 0(\varsigma)|\mathcal{F}\left[\Psi\right]|0(\varsigma)\rangle \Rightarrow \langle 0(\varsigma)|\mathcal{F}\left[\widehat{\Upsilon}\left[[+]\varsigma;\psi(x,[+]\varsigma)\right]\right]|0(\varsigma)\rangle.$$

"Hot" pseudoscalar field with respect to the "hot" vacuum $|0(\varsigma)\rangle$

reads: $\phi(x;[+]\varsigma) = \mathcal{V}_{\vartheta(B)}^{-1}\phi(x)\mathcal{V}_{\vartheta(B)} =$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dk^1}{2k^0}\left[c_{k^1}([+]\varsigma)e^{-i(kx)} + c_{k^1}^{\dagger}([+]\varsigma)e^{+i(kx)}\right]. \tag{62}$$

TFD. Thermal bosonization. Hot and cold thermofields

To construct the solution of HEq for TM in the doubling space of pseudoscalar fields $c(k^1)|0\widetilde{0}\rangle = \widetilde{c}(k^1)|0\widetilde{0}\rangle = 0$, all previous $T = 0$ steps should be repeated again for $\phi(x;[\pm]\varsigma) = \mathcal{V}^{\mp 1}_{\vartheta(B)}\phi(x)\mathcal{V}^{\pm 1}_{\vartheta(B)}$:

$$\varphi^{\xi(+)}\left(x^{\xi};[\pm]\varsigma\right) = -\int_{-\infty}^{\infty}\frac{dk^1}{2k^0}\frac{\xi\theta(-\xi k^1)}{2\pi}\left[\cosh\vartheta c(k^1)e^{-ik^0x^{\xi}} \mp \mp\sinh\vartheta\widetilde{c}(k^1)e^{ik^0x^{\xi}}\right], \quad \varphi^{\xi(-)}\left(x^{\xi};[\pm]\varsigma\right) = \left\{\varphi^{\xi(+)}\left(x^{\xi};[\pm]\varsigma\right)\right\}^{\dagger}, \tag{63}$$

$$Q^{\xi(+)}([\pm]\varsigma) = \lim_{L\to\infty}\int_{-\infty}^{\infty}dk^1\frac{i\xi\theta(-\xi k^1)}{2}\left[\cosh\vartheta c(k^1)e^{-ik^0\widehat{x}^0} \pm \pm\sinh\vartheta\widetilde{c}(k^1)e^{ik^0\widehat{x}^0}\right]\delta_L\left(k^1\right), \quad Q^{\xi(-)}([\pm]\varsigma) = \left\{Q^{\xi(+)}([\pm]\varsigma)\right\}^{\dagger}. \tag{64}$$

with corresponding expressins for the tilde-partners, $\vartheta = \vartheta(k^1;\varsigma)$. Here the $\widehat{x}^0$ – dependence of charge frequency parts is fictitious. It is the artifact of space regularization (24) of charges and should be eliminated at the end of calculation.

TFD. Thermal bosonization. Hot and cold thermofields

Putting corresponding $\pm$ into respective brackets, the main CCR read, exhibiting very dangerouse coordinate dependence:

$$\left[\varphi^{\xi(\pm)}\left(s;[\pm]\varsigma\right),\varphi^{\xi'(\mp)}\left(\tau;[\pm]\varsigma\right)\right]=$$

$$=(\mp 1)\frac{\delta_{\xi,\xi'}}{4\pi}\left\{\ln\left(i\overline{\mu}\frac{\varsigma}{\pi}\sinh\left(\frac{\pi}{\varsigma}(\pm(s-\tau)-i0)\right)\right)-g\left(\varsigma,\mu_1\right)\right\},\tag{65}$$

$$\left[\varphi^{\xi(\pm)}(s;[\pm]\varsigma),Q^{\xi'(\mp)}([\pm]\varsigma)\right]=\delta_{\xi,\xi'}\left[\frac{i}{4}-(\pm 1)\left(\frac{\widehat{x}^0-s}{2\varsigma}\right)\right],\tag{66}$$

$$\left[\varphi^{\xi(\pm)}(s;[\pm]\varsigma),\widetilde{Q}^{\xi'(\mp)}([\pm]\varsigma)\right]=(\pm 1)[\pm 1]\delta_{\xi,\xi'}\left(\frac{\widehat{x}^0-s}{2\varsigma}\right),\tag{67}$$

$$\left[Q^{\xi(\pm)}([\pm]\varsigma),Q^{\xi'(\mp)}([\pm]\varsigma)\right]=(\pm 1)a_1\delta_{\xi,\xi'},\tag{68}$$

$$\left[Q^{\xi(\pm)}([\pm]\varsigma),\widetilde{Q}^{\xi'(\mp)}([\pm]\varsigma)\right]=(\pm 1)[\mp 1]a_2\delta_{\xi,\xi'}.\tag{69}$$

And for the tilde-conjugate relations correspondingly.

TFD. Thermal bosonization. Hot and cold thermofields

Here the following quantities are defined, which control the additional infrared divergences: $\overline{\mu} = \mu e^{C_{\ni}}, \overline{\mu}_1 = \mu_1 e^{C_{\ni}} \to 0$

$$g(\varsigma, \mu_1) = \int_{\mu_1}^{\infty} \frac{dk^1}{k^0} \left(\frac{2}{e^{\varsigma k^0} - 1} \right) \to \frac{2}{\varsigma \mu_1} - \ln \left(\frac{2\pi}{\varsigma \overline{\mu}_1} \right), \tag{70}$$

$$\text{if: } \delta_L(k^1) \Rightarrow L\overline{\Delta}(k^1 L), \quad I_n^{\Delta} = \int_0^{\infty} dt t^n \left(\overline{\Delta}(t) \right)^2, \text{ then: } a_0 \Rightarrow \pi I_1^{\Delta}, \tag{71}$$

$$a_1 = a_0 + 2\pi \int_0^{\infty} dt t \frac{(\overline{\Delta}(t))^2}{e^{\varsigma t/L} - 1} \Rightarrow 2\pi I_0^{\Delta} \frac{L}{\varsigma} + \frac{\pi}{6} I_2^{\Delta} \frac{\varsigma}{L} + O\left(\left(\frac{\varsigma}{L} \right)^3 \right), \tag{72}$$

$$a_2 = \pi \int_0^{\infty} dt t \frac{(\overline{\Delta}(t))^2}{\sinh(t\varsigma/2L)} \Rightarrow 2\pi I_0^{\Delta} \frac{L}{\varsigma} - \frac{\pi}{12} I_2^{\Delta} \frac{\varsigma}{L} + O\left(\left(\frac{\varsigma}{L} \right)^3 \right), \tag{73}$$

for: $L \to \infty$; whence: $a_1 - a_2 \to 0$, if: $a_0 < \infty$,

$$\text{but: } \lim_{\varsigma \to \infty} g(\varsigma, \mu_1) = 0, \quad \lim_{\varsigma \to \infty} a_1 = a_0, \quad \lim_{\varsigma \to \infty} a_2 = 0, \tag{74}$$

Thermofield solutions of TM

Now one can construct again a variety of different inequivalent representations for solutions of the free massless Dirac equation for the physical field at finite temperature, $\partial_\xi \chi_\xi \left(x^{-\xi}, \varsigma\right) = 0$ in the form of local normal ordered exponentials of the left and right bosonic thermofields $\varphi^\xi(x^\xi, \varsigma)$, and their charges. Due to above coordinate dependence of commutators its appear, that kinematic independence (48) of the tilde-partners can be achieved only by admixing all the Klein factors coming from both of the charges $Q^\xi(\varsigma)$, $Q^{-\xi}(\varsigma)$ and $\widetilde{Q}^\xi(\varsigma)$, $\widetilde{Q}^{-\xi}(\varsigma)$ into the same field by using the new charges, with simple commutation relations:

$$\mathrm{G}^\xi([\pm]\varsigma) = Q^\xi([\pm]\varsigma) + [\pm 1]\widetilde{Q}^\xi([\pm]\varsigma), \quad \text{with:} \tag{75}$$

$$\left[\varphi^{\xi(\pm)}(s;[\pm]\varsigma), \mathrm{G}^{\xi'(\mp)}([\pm]\varsigma)\right] = \frac{i}{4}\,\delta_{\xi,\xi'}, \tag{76}$$

$$\left[\varphi^{\xi(\pm)}(s;[\pm]\varsigma), \widetilde{\mathrm{G}}^{\xi'(\mp)}([\pm]\varsigma)\right] = [\pm 1]\frac{i}{4}\,\delta_{\xi,\xi'}, \tag{77}$$

$$\left[\mathrm{G}^{\xi(\pm)}([\pm]\varsigma), \mathrm{G}^{\xi'(\mp)}([\pm]\varsigma)\right] = (\pm 1)2(a_1 - a_2)\delta_{\xi,\xi'}, \tag{78}$$

$$\left[\mathrm{G}^{\xi(\pm)}([\pm]\varsigma), \widetilde{\mathrm{G}}^{\xi'(\mp)}([\pm]\varsigma)\right] = (\pm 1)[\pm 1]2(a_1 - a_2)\delta_{\xi,\xi'}. \tag{79}$$

Thermofield solutions of TM

Moreover, according to the meaning of L as macroscopic parameter, the wanted thermofields should have a correct thermodynamic limit $L \to \infty$ for the finite temperature. The most simple case, which obeys again the same CAR (6), (7) and the bosonization rules (12)–(16), again with $Z_{(\chi)}(a) = 1$ reads as:

$$\chi_\xi(x^{-\xi};[\pm]\varsigma) = \mathcal{N}_\varphi\left(\exp\left\{R_\xi(x^{-\xi};[\pm]\varsigma)\right\}\right) u_\xi\left(\mu_1,[\pm]\varsigma\right), \qquad (80)$$

$$R_\xi(x^{-\xi};[\pm]\varsigma) = -i2\sqrt{\pi}\left[\varphi^{-\xi}\left(x^{-\xi};[\pm]\varsigma\right) + \frac{\sigma_0^\xi}{4}\mathrm{G}^{-\xi}([\pm]\varsigma) + \qquad (81)\right.$$

$$\left. + \frac{\sigma_1^\xi}{4}\mathrm{G}^{\xi}([\pm]\varsigma)\right], \quad \text{with:} \quad \sigma_0^\xi = -\xi\sigma, \quad \sigma_1^\xi = \xi 1 + \rho, \qquad (82)$$

$$u_\xi\left(\mu_1,[\pm]\varsigma\right)\Big|_{L\to\infty} = \left(\frac{\overline{\mu}}{2\pi}\right)^{1/2} e^{i\varpi - i\xi\Theta/4}\exp\left\{-\frac{g(\varsigma,\mu_1)}{2}\right\}, \qquad (83)$$

where the σ and ρ are the same parameters as above, but they become fixed below by the condition (48).

Thermofield solutions of TM

Integrating again the HEq of TM as above for $T = 0$ one finds:

$$\Psi_\xi(x;[\pm]\varsigma) = \mathcal{N}_\varphi \left(\exp\left\{\Re_\xi(x;[\pm]\varsigma)\right\}\right) w_\xi\left(\mu_1,\varsigma\right),$$

$$\Re_\xi(x;[\pm]\varsigma) = -i\left[\mathcal{B}^{-\xi}(x;[\pm]\varsigma) + \frac{\Sigma_0^\xi}{4}\mathrm{G}^{-\xi}([\pm]\varsigma) + \frac{\Sigma_1^\xi}{4}\mathrm{G}^{\xi}([\pm]\varsigma)\right],$$

with: $\mathcal{B}^{-\xi}(x;[\pm]\varsigma) = \overline{\alpha}\varphi^{-\xi}(x^{-\xi};[\pm]\varsigma)) + \overline{\beta}\varphi^{\xi}(x^{\xi};[\pm]\varsigma)),$

$$w_\xi(\mu_1,\varsigma)\Big|_{L\to\infty} = \left(\frac{\Lambda}{2\pi}\right)^{1/2}\left[\frac{\overline{\mu}}{\Lambda}\exp\left\{-g(\varsigma,\mu_1)\right\}\right]^{\mathrm{d}_{(\Psi)}} e^{i\varpi - i\xi\Theta/4},$$

$$\Sigma_0^\xi = \overline{\alpha}\sigma_0^\xi - \overline{\beta}\sigma_1^\xi,\quad \Sigma_1^\xi = \overline{\alpha}\sigma_1^\xi - \overline{\beta}\sigma_0^\xi,\quad \ell, n \text{ - integers},$$

$$\sigma_0^\xi = -\xi\sigma \Rightarrow \xi(2\ell+1),\quad \sigma_1^\xi = \xi 1 + \rho \Rightarrow \xi 1 + (2n+1), \tag{84}$$

with the same values of parameters (32) and renormalization constant $Z_{(\Psi)}(a)$ (15). Remarkably, that the obtained conditions (84) provide the anticommutation, locality and kinematic independence relations (48) for both the free and the Thirring fields and their tilde-partners simultaneously. Note that here $\sigma, \rho \neq 0$.

Conclusions

0. The correct HF should be only a fully normal ordered operator in the sense of DM onto the irreducible set of physical fields. Only this form clarifies and assures correct renormalization, commutation and symmetry properties and give simple connections between different types of solutions with finite and zero temperature.
1. The general solutions for HF (30) of TM keeps the Klaiber's normal form, but with distinct unitarily inequivalent representation of the free massless Dirac field sandwiched the simple dynamical factors. This elucidates an important dynamical role of inequivalent representations of these fields to built the solution of HEq.
2. The obtained thermofield solution of TM, at $T \to 0$ ($\varsigma \to \infty$) with finite L and omitted tilde charges, come back to above general solution $\Psi_{\xi}(x, \sigma, \rho)$ (39), defined for arbitrary σ, ρ, instead of only odd values (84) for $T > 0$. So the naive thermofield transformation of Mandelstam (or other's) solution[10] is not correct.
3.The notion of "hot" and "cold" physical thermofields is found to be convenient to distinguish different thermofield representations with respect to different vacua.

[10] Amaral R.L.P.G., Belvedere L.V., Rothe K.D., *Annals of Physics*, **320**, 2005, 399.

Conclusions

4. Only if the both superselection rules are fulfilled the thermodynamic limit $L \to \infty$ and the zero temperature limit $\varsigma \to \infty$ may be interchanged with the one and the same result for any p-point VEV (Wightman functions) that become independent of all infrared regularizations similarly to the case $T = 0$.

5. Due to automatical elimination of zero mode's contributions, the chosen here representation space of free massless pseudoscalar field relaxes the problem of non-positivity of the inner product induced by its two-point Wightman function.

Appendix: Motivation of TFD

Попытка записать статистическое среднее в виде некоторого вакуумного среднего немедленно приводит к удвоению числа степеней свободы: термополевой вакуум “живет” в прямом тензорном произведении пространств: системы и как бы ее “зеркального” отражения в термостате: $Z(\varsigma) = Tr\{e^{-\varsigma H}\}$, $\varsigma = (k_B T)^{-1}$, $H|n\rangle = E_n|n\rangle$, $\langle m|n\rangle = \delta_{mn}$,

$$\langle\!\langle A\rangle\!\rangle = Z^{-1} Tr\{e^{-\varsigma H} A\} \overset{?}{\Longrightarrow} \langle 0(\varsigma)|A|0(\varsigma)\rangle, \quad \text{или:} \tag{85}$$

$$Z^{-1}\sum_n \langle n|A|n\rangle e^{-\varsigma E_n} \overset{?}{\Longrightarrow} \langle 0(\varsigma)|A|0(\varsigma)\rangle, \quad \text{ищем в виде:} \tag{86}$$

$$|0(\varsigma)\rangle = \sum_n |n\rangle f_n(\varsigma), \quad f_n^*(\varsigma) f_m(\varsigma) = Z^{-1} e^{-\varsigma E_n}\delta_{nm}, \tag{87}$$

$$f_n(\varsigma) = Z^{-1/2} e^{-\varsigma E_n/2}|\widetilde{n}\rangle, \quad \widetilde{H}|\widetilde{n}\rangle = E_n|\widetilde{n}\rangle, \quad \langle\widetilde{m}|\widetilde{n}\rangle = \delta_{mn}, \tag{88}$$

$$|0(\varsigma)\rangle = Z^{-1/2}\sum_n e^{-\varsigma E_n/2}|n,\widetilde{n}\rangle, \quad |n,\widetilde{n}\rangle \equiv |n\rangle\otimes|\widetilde{n}\rangle = \begin{pmatrix} |n\rangle \\ |\widetilde{n}\rangle \end{pmatrix}, \tag{89}$$

$$\text{Очевидно: } \widehat{\boldsymbol{H}}|0(\varsigma)\rangle = 0, \quad \widehat{\boldsymbol{H}} = H - \widetilde{H}, \text{ при } H, \widetilde{H}|0(\varsigma)\rangle \neq 0, \tag{90}$$

$$\langle 0(\varsigma)|0(\varsigma)\rangle = 1, \quad b|0\rangle = \widetilde{b}|\widetilde{0}\rangle = 0, \quad b(\varsigma)|0(\varsigma)\rangle \overset{?}{=} 0 \overset{?}{=} \widetilde{b}(\varsigma)|0(\varsigma)\rangle. \tag{91}$$

Appendix: Bogoliubov transformations

Simplest fermionic oscillator: one fixed mode k^1; only two normalized states $|0\rangle$ and $|1\rangle$, with $E_0 = 0$, $E_1 = \omega$, annihilated or created by: $b|0\rangle = 0$, $|1\rangle = b^\dagger|0\rangle$, $\{b, b^\dagger\} = 1$, $\{b, b\} = 0$. Thermal vacuum appears as a normalized sum of tensor products of two independent copies of these states: $|0\widetilde{0}\rangle = |0\rangle \otimes |\widetilde{0}\rangle$, $|1\widetilde{1}\rangle = |1\rangle \otimes |\widetilde{1}\rangle$, weighted with corresponding Gibbs and relative phase factors: for $\{b, \widetilde{b}^\#\} = 0$, $(\widetilde{b}^\# = \widetilde{b}, \widetilde{b}^\dagger)$, $\tan^2\vartheta(k^1, \varsigma) = e^{-\varsigma\omega}$, $\omega = \omega_{k^1}$:

$$|0(\varsigma)\rangle_{(F)} = \left[|0\widetilde{0}\rangle + e^{i\Phi} e^{-\varsigma\omega/2}|1\widetilde{1}\rangle\right]\left[\langle 0\widetilde{0}|0\widetilde{0}\rangle + e^{-\varsigma\omega}\langle 1\widetilde{1}|1\widetilde{1}\rangle\right]^{-1/2} \equiv \quad (92)$$

$$\equiv \cos\vartheta \left(1 + e^{i\Phi}\tan\vartheta\, b^\dagger \widetilde{b}^\dagger\right)|0\widetilde{0}\rangle = \mathrm{V}^{-1}_{\vartheta(F)}|0\widetilde{0}\rangle, \quad \text{where:} \quad (93)$$

$$G_+ = b^\dagger \widetilde{b}^\dagger, \quad G_- = \widetilde{b} b = (G_+)^\dagger, \quad G_3 = (b^\dagger b - \widetilde{b}\widetilde{b}^\dagger)/2, \quad (94)$$

$$[G_+, G_-] = 2G_3, \quad [G_3, G_\pm] = \pm G_\pm, \quad G_\pm = G_1 \pm iG_2, \quad (95)$$

$$\mathrm{V}^{-1}_{\vartheta(F)} = \exp\left\{\vartheta\left[e^{i\Phi}G_+ - e^{-i\Phi}G_-\right]\right\} = \mathrm{V}_{-\vartheta(F)} = \mathrm{V}^\dagger_{\vartheta(F)} = \quad (96)$$

$$= \exp\left\{e^{i\Phi}\tan\vartheta\, G_+\right\}\exp\left\{-\ln(\cos^2\vartheta)\, G_3\right\} \cdot$$

$$\cdot \exp\left\{-e^{-i\Phi}\tan\vartheta\, G_-\right\}, \quad (97)$$

Appendix: Bogoliubov transformations

– is a standard form of operator of the coherent state for group $SU(2)$ allows to identify the algebra (95) as "quasispin" algebra, with the "cold" vacuum as its lowest state $|0\widetilde{0}\rangle \Rightarrow |s,-s\rangle$, for representation with "quasispin" $s=1/2$, and the state $|1\widetilde{1}\rangle \Rightarrow |s,s\rangle$, as the highest one, with: $G_3|s,\pm s\rangle = \pm s|s,\pm s\rangle$, $G_\pm|s,\pm s\rangle = 0$. The unique arisen arbitrary relative phase Φ reflects now the fact that the quantum state is not the vector, rather the ray. The thermal vacuum (93), as a coherent state, is annihilated (91) by operators: $\mathrm{V}^{-1}_{\vartheta(F)} G_- \mathrm{V}_{\vartheta(F)} = \cos^2\vartheta\, G_- + e^{i\Phi}\sin 2\vartheta\, G_3 - e^{2i\Phi}\sin^2\vartheta\, G_+ = \underset{\sim}{b}(\varsigma) b(\varsigma)$,

$$\begin{aligned} b(\varsigma) &= \mathrm{V}^{-1}_{\vartheta(F)}\, b\, \mathrm{V}_{\vartheta(F)} = b\cos\vartheta - \widetilde{b}^\dagger e^{i\Phi}\sin\vartheta, \\ \underset{\sim}{b}(\varsigma) &= \mathrm{V}^{-1}_{\vartheta(F)}\, \widetilde{b}\, \mathrm{V}_{\vartheta(F)} = \widetilde{b}\cos\vartheta + b^\dagger e^{i\Phi}\sin\vartheta. \end{aligned} \tag{98}$$

Up to now $\widetilde{b}^{\#}$ is only notation that does not define any operation. In order to fix it as an operation: $\underset{\sim}{b}(\varsigma) \mapsto \widetilde{b}(\varsigma)$, one should choose the phase Φ, which can not be removed by self consistent redefinition of operators $b, \widetilde{b}$.

Appendix: Bogoliubov transformations

The popular choice $\Phi = 0$ leads to complicated tilde conjugation rules for the fermionic case, different from the bosonic one. The Ojima choice $\Phi = -\pi/2$ gives the same rules for both bosonic and fermionic cases. We see now that the choice $\Phi = \pi/2$ is also good and, as well as the original Ojima's one, satisfies the properties of antilinear homomorphism and the condition $\widetilde{\widetilde{b}}(\varsigma) = b(\varsigma)$. It seems very convenient for the purposes of bosonization that the tilde operation has the same properties for both Fermi and Bose cases. As a byproduct, we observe a useful interpretation of the thermal vacuum, defined by Bogoliubov transformation (93), as a coherent state, obtained by coherent $SU(2)$ rotation of vacuum states for all Fermi oscillators $|0_{k^1}\widetilde{0}_{k^1}\rangle$ as a lowest quasispin states, around one and the same unit vector $\mathbf{u} = (\sin\Phi, \cos\Phi, 0)$ onto the different angles $= -2\vartheta(k^1)$: $\mathcal{V}^{-1}_{\vartheta(F)} = \exp\left[i2\vartheta\left(\mathbf{u}\cdot\boldsymbol{G}\right)\right]$.