Yerevan
SQS-2019

Evaluating the Chern-Simons Path Integral on a Genral Seifert Manifold

George Thompson
(Matthias Blau, Keita Mady & K. Narain)

ICTP, arXiv: 1812.10966

August 27, 2019

Witten's Chern-Simons Invariant

- P be a (trivial) principal G fibre bundle over M. Denote the space of connections by $\mathbb{A}$.
- The action at level k is

$$I(\mathbb{A}) = i\frac{k}{4\pi}\int_M \mathrm{Tr}\left(\mathbb{A}\wedge d\mathbb{A} + \frac{2}{3}\mathbb{A}\wedge\mathbb{A}\wedge\mathbb{A}\right)$$

  and Tr is normalized so that under large gauge transformations $I(\mathbb{A}^g) = I(\mathbb{A}) + 2\pi i n$.
- The invariant

$$Z_{k,G}[M] = \int_{\mathbb{A}} \exp\left(I(\mathbb{A})\right)$$

On any Seifert 3-Manifold The Answer Is Known

- Lawrence and Rozansky, Rozansky, Mariño, Hansen, Hansen and Takata find,

$$\sum_{\mathbf{n}_0 \in \mathbb{Z}} \left(\prod_{i=1}^{N} \sum_{\mathbf{n}_i=1}^{a_i-1} \right) \int_{\mathfrak{t}} d\phi \ \sqrt{T_M(\phi; \mathbf{n}_i)}. \ \exp\left(4\pi i \Phi(\mathcal{L}_M) + i k_{\mathfrak{g}} I(\phi, \mathbf{n})\right)$$

  where $k_{\mathfrak{g}} = k + c_{\mathfrak{g}}$ where $c_{\mathfrak{g}}$ is the dual Coxeter number for the group G.

-
$$\Phi(\mathcal{L}_M) = -\frac{\dim G}{48} \left(c_1(\mathcal{L}_M) - 12 \sum_{i=1}^{N} s(b_i, a_i) \right)$$

So Why Bother?

- The formula come from an application of the Reshetikhin-Turaev invariant which combines quantum groups and surgery presentations of the manifold. (CFT and surgery).
- But this is a gauge theory problem and it should be doable. The pay-off would be an application to other theories with no known CFT interpretation.
- There is also an extra benefit in that we get a concrete relationship with the intersection pairings on certain moduli spaces of vector bundles on Riemann surface with marked points.

Here Goes

- Going to be very brief with precious few details... first the Seifert spaces themselves.

Orbifolds & Line V-bundles

- Σ a smooth genus g Riemann surface with N orbifold points p_i: locally around each point the neighborhood is $D^2/\mathbb{Z}_{a_i}$

$$z \longrightarrow \zeta.z, \quad \zeta = \exp\left(2\pi i/a_i\right)$$

- We consider V-line bundles on Σ with the local description $D^2 \times \mathbb{C}/\mathbb{Z}_{a_i}$ with the action on local coordinates as

$$(z,\, s) \longrightarrow (\zeta.z,\, \zeta^{b_i}.s)$$

  with integers $0 \leq b_i < a_i$.
- The first Chern class of such a line V-bundle is (in $\mathbb{Q}$)

$$c_1(\mathcal{L}) = \deg\left(\mathcal{L}\right) + \sum_{i=1}^{N} \frac{b_i}{a_i}$$

The underlying manifolds of interest

- $M[\deg(\mathcal{L}), g, (a_i,\ b_i)] = S(\mathcal{L})$
  The circle V-bundle associated to the line V-bundle $\mathcal{L}$.
- Such an M is smooth if $\gcd(a_i,\ b_i) = 1$ for $i = 1, \ldots, N$.
- These are integral homology spheres if $g = 0$ and $\gcd(a_i,\ a_j) = 1,\ \forall\, i \neq j$.

Structure on M

- Let κ be a connection on M (a globally defined real 1-form) and ξ the fundamental vector field on M,

$$\iota_\xi \kappa = 1 \quad L_\xi \kappa = 0$$

  Locally, $\kappa = d\theta + \beta$ (θ a fibre direction $0 \leq \theta < 1$).

Back to the Gauge Theory

- We will take advantage of the principal bundle structure on M to simplify our lives ...

Make CS look like Yang-Mills on Σ

- Use the $U(1)$ bundle structure and the associated nowhere vanishing vector field to decompose connections as

$$\mathbb{A} = A + \kappa\,\phi, \quad \iota_\xi A = 0$$

- The Chern-Simons action is now

$$I(A,\phi) = i\frac{k}{4\pi}\int_M \left(\kappa\,\mathrm{Tr}\,A\iota_\kappa dA + \kappa\,\mathrm{Tr}\,\phi F_A + \kappa d\kappa\ \mathrm{Tr}\,\phi^2\right)$$

and this has some resemblance to the YM action.

Gauge Choices

- Impose the gauge condition that ϕ is constant in the fibre direction,

$$\iota_\xi . d\phi = 0$$

  So ϕ is a $U(1)$ invariant section of $\mathrm{ad}\,(P)$. Equivalently it is a section of the trivial adjoint V-bundle V over Σ.
- Gauge transformations which also do not depend on the fibre still act, $\phi \longrightarrow g^{-1}.\phi.g$
- Decompose the Lie algebra as $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{k}$ with $\mathfrak{t}$ the Cartan sub-algebra and set $\phi^{\mathfrak{k}} = 0$. However, there is a price to be paid ... (more on this soon).

So now where are we?

- The action is

$$I(A,\phi) = i\frac{k}{4\pi}\int_M \mathrm{Tr}\left(\kappa A\, L_\phi\, A + \kappa\phi . F_A^{\mathfrak{t}} + \kappa d\kappa\, \phi^2\right)$$

  together with a ghost action

$$\int_M \mathrm{Tr}\left(\overline{c} * L_\phi c\right)$$

- Integrating out the charged A and the ghosts (they are also charged) leaves us with
-

$$\det\left(L_\phi\right)^{\Omega^0(\Sigma,\mathfrak{t})} / \det\left(L_\phi\right)^{\Omega^1(\Sigma,\mathfrak{t})/2}$$

So We are left with an Abelian Theory on Σ

- The path integral of ineterst is

$$\int_{(A,\phi)} \frac{\det\left(L_{\phi}\right)^{\Omega^{0}(\Sigma,\mathfrak{t})}}{\det\left(L_{\phi}\right)^{\Omega^{1}(\Sigma,\mathfrak{t})/2}} \exp\left(I(A,\phi)\right)$$

- The path integral over A imposes the condition that

$$d_{\Sigma}\phi = 0 \quad \text{so that } \phi \text{ is constant}$$

I Lied

- On Abelianizing $\phi^{\mathfrak{g}} \longrightarrow \phi^{\mathfrak{k}}$ you pay a price, namely: even though the bundle you started with was trivial you have 'liberated' nontrivial abelian bundles. Which ones? In this case all possible line V-bundles.

Correct Derivation

- So the previous is an outline of a derivation. To turn it into a derviation we need to substitute

$$\mathbb{A} \to \mathbb{A} + \mathbb{A}_B$$

where $\mathbb{A}_B$ is a background field taking into account the non-trivial bundles that we should sum over.

Correct Derivation Continued

- The ratio determinants that we needed to calcualte are of operators that are sections of non-trivial bundles. The ratio can be evaluated by the Holomorphic Lefschetz fixed point formula (which goes into the Kawasaki index theorem).
- And that is it.

The Answer Explained

- Recall the answer is

$$\sum_{\mathbf{n}_0 \in \mathbb{Z}} \left(\prod_{i=1}^{N} \sum_{\mathbf{n}_i=1}^{a_i-1} \right) \int_{\mathfrak{t}} d\phi \ \sqrt{T_M(\phi;\, \mathbf{n}_i)}.\ \exp\left(4\pi i \Phi(\mathcal{L}_M) + i k_{\mathfrak{g}} I(\phi, \mathbf{n})\right)$$

- The $\mathbf{n}_i$ label the possible non-trivial line V-bundles at the i'th orbifold point. $\mathbf{n}_0$ is the possible line bundle at some regular point.
- $\sqrt{T_M(\phi;\, \mathbf{n}_i)}$ is the absolute value of the ratio of determinants, while $\exp\left(4\pi i \Phi(\mathcal{L}_M)\right)$ is its phase.

Thanks To:

- Everyone who helped in the organisation of this meeting.