Irrelevant Operators and the Local Renormalization Group

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with Adam Schwimmer [1902.04473]

What is the problem that we have addressed?

The structure of the Weyl anomaly of a CFT in the presence of sources for irrelevant operators poses a puzzle. This puzzle is related to the non-existence of conformally covariant completions of higher powers of the Laplacian in even dimensions.

In the following I will explain this puzzle and its resolution.

Consider a CFT:

It has various local operators $\mathcal{O}_i$ with scaling dimensions Δ_i.

We include also the energy-momentum tensor $T_{\mu\nu}$ in the list of operators of interest.

Introduce space-time dependent non-dynamical sources

$$J^i(x) \quad \text{for} \quad \mathcal{O}_i \qquad \text{and} \qquad g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \quad \text{for} \quad T_{\mu\nu}$$

Collect the non-local correlation functions into the generating functional

$$W[J_i, h] = \int dx\, dy\, \langle \mathcal{O}_i(x)\, \mathcal{O}_j(y) \rangle\, J^i(x)\, J^j(y) + \int dx\, dy\, \langle T_{\mu\nu}(x)\, T_{\alpha\beta}(y) \rangle\, h^{\mu\nu}(x)\, h^{\alpha\beta}(y)$$
$$+ \int dx\, dy\, dz\, \langle\, T_{\mu\nu}(x)\, \mathcal{O}_i(y)\, \mathcal{O}_j(z) \rangle\, h^{\mu\nu}(x)\, J^i(y)\, J^j(y) + \dots$$

In the absence of an anomaly W is a diffeo and Weyl invariant functional.

But the anomaly is unavoidable. By a choice of counter terms we can move it to Weyl, i.e. diffeo invariance is a good symmetry throughout.

We then have

$$\delta_\sigma W \equiv \int dx \left\{ \delta_\sigma g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} + \sum_i \delta_\sigma J^i \frac{\delta}{\delta J^i} \right\} W = \int dx \sqrt{g}\, \mathcal{A}$$

This is the local Callan-Symanzik equation, also called Osborn equation.

It is very non-trivial: while W on the l.h.s. is very non-local (e.g. multiple logs), the local CS equation asserts that the r.h.s. is local. We will have to come back to this.

But let us first look at a simple example and its contribution to W:

Scalar operator $\mathcal{O}$ of dimension $\Delta = d/2 + n \quad n = 0, 1, 2, \ldots$

$$\langle \mathcal{O}(x)\, \mathcal{O}(y) \rangle = \frac{1}{|x-y|^{2\Delta}}$$

up to normalization fixed by conformal symmetry. It is singular and requires regularization. This is most easily seen in momentum space:

$$\langle \mathcal{O}(p)\, \mathcal{O}(-p) \rangle \sim p^{2n} \, \log(p^2/\mu^2)$$

- a scale μ has appeared in a CFT !
- it transforms inhomogeneously under dilatations: under $x^\mu \to e^\lambda x^\mu$

$$\delta_\lambda \langle \mathcal{O}(x)\, \mathcal{O}(y) \rangle \;\sim\; \Box^n \delta(x-y)$$

This reflects a Weyl anomaly $\qquad (\partial^\mu j^D_\mu = \eta^{\mu\nu} T_{\mu\nu})$

$$\eta^{\mu\nu} \langle T_{\mu\nu}\, \mathcal{O}\, \mathcal{O} \rangle \neq 0$$

If we introduce the source J for operator $\mathcal{O}$, this implies a term in the generating functional (schematically)

$$W = \int dx\, J\, \Box^n \log\left(\Box/\mu^2\right) J + \mathcal{O}(J^3)$$

and the anomalous transformation under Weyl transformations

$$\delta_\sigma W = \int d^d x\, \sigma\, J\, \Box^n J + \mathcal{O}(J^3)$$

If we also introduce a source $g_{\mu\nu}$ for the energy-momentum tensor, then we expect this to become

$$\delta_\sigma W = \int d^d x\, \sqrt{g}\, \sigma\, J\, \Delta_c J + \mathcal{O}(J^3)$$

where

$$\Delta_c = \Box^n + \text{terms which vanish in flat space}$$

is a conformally covariant power of the Laplacian.

Its Weyl transformation properties are such that $\sqrt{g}\, J\, \Delta_c J$ is Weyl invariant.

Examples (all in $d = 4$):

- $\Delta = 2$ (mass term)

$$\Delta_c = 1$$

- $\Delta = 3$

$$\Delta_c = \Box - \frac{1}{6} R$$

'conformally coupled scalar'

- $\Delta = 4$ (marginal perturbation)

$$\Delta_c = \Box^2 + 2\nabla_\mu \left(R^{\mu\nu} - \frac{1}{3} g^{\mu\nu} R \right) \nabla_\nu$$

(critical) Fradkin-Tseytlin-Riegert-Paneitz operator

- $\Delta = 5, 6, \ldots$ (irrelevant)

Theorem: Δ_c does not exist.

More generally (Gover & Hirachi)

in d even, conformally covariant powers of the Laplacian do not exist for $n > \frac{d}{2}$.

On the other hand: the generating functional for correlation functions of operators of arbitrary dimensions does exits. This is the puzzle.

The resolution of the puzzle follows from a closer look at the local CS equation and its workings:

$$\delta_\sigma W \equiv \int dx \left\{ \delta_\sigma g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} + \sum_i \delta_\sigma J^i \frac{\delta}{\delta J^i} \right\} W = \int dx \sqrt{g}\, \mathcal{A} \tag{1}$$

I said before that it is very non-trivial: while W on the l.h.s. is very non-local (e.g. multiple logs), the local CS equation asserts that the r.h.s. is local. We now come back to this.

Consider again the scalar operator . . .

...but now the OPE

$$\mathcal{O}(x)\,\mathcal{O}(y) = \frac{c}{|x-y|^{\Delta}}\mathcal{O}(y) + \ldots$$

If $\Delta = d + n$ this has a logarithmic divergence. Together with the logarithm in

$$\langle \mathcal{O}(x)\,\mathcal{O}(y)\rangle$$

the three-point function

$$\langle \mathcal{O}(x)\,\mathcal{O}(y)\,\mathcal{O}(z)\rangle$$

has a double logarithm for $c \neq 0$. The appropriate term in the generating functional is (very schematically, dropping e.g. $\Box^n$)

$$W = J\,\log(\Box/\mu^2)J + c\,J^2\big(\log(\Box/\mu^2)\big)^2\,J + \ldots$$

The action of the Weyl variation of the metric in the covariant $\Box$ in the second term cancels the variation of the first term if

$$\delta_\sigma J \equiv \beta_J = (\Delta - d)\sigma\,J + c\,\sigma\,J^2 + \ldots$$

where the first term is the canonical scaling dimension and the second a non-trivial beta-function.

Therefore, the single logs are cancelled due to non-trivial beta function for the source J. This continues recursively for all higher logs: they are cancelled and a local expression remains.

There are other terms in the OPE of two scalar operators, again largely fixed by conformal symmetry

$$\mathcal{O}(x)\,\mathcal{O}(0) \sim \cdots + \frac{a}{c_T}\frac{1}{x^{2\Delta+2-d}}T_{\alpha\beta}(0)\,x^\alpha\,x^\beta + \frac{a}{2\,c_T}\frac{1}{x^{2\Delta+2-d}}\partial_\gamma T_{\alpha\beta}(0)x^\alpha x^\beta x^\gamma + \dots$$

where ... are higher derivatives of T and contributions from other fields.

From this we extract two pieces of information (cf. the discussion above)

1. The coefficient functions in front of the energy momentum tensors are log singular for integer dimension irrelevant operators ($\Delta > d$) and the correlator

$$\langle T_{\mu\nu}\,\mathcal{O}\,\mathcal{O}\rangle$$

   which contributes to the anomaly, will have a double logarithm (the second log coming from $\langle TT\rangle$). This is a priori incompatible with the locality of the anomaly.

2. The singularity in the OPE implies a beta-function for the coupling of the energy momentum tensor, i.e. for the metric.

   Explicitly one finds for a dimension five operator in $d = 4$

$$\delta_\sigma g_{\mu\nu} \equiv \beta^g_{\mu\nu} = 2\,\sigma\, g_{\mu\nu} + \frac{a\,\pi^2}{24\, c_T}\sigma\, J\,\partial_\mu\partial_\nu J + \dots$$

   where possible other contributions to $\mathcal{O}(J^2)$ would not contribute to this OPE (cf. below how to obtain them).

These two facts guarantee that the Weyl variation of the generating functional is local (at least to $\mathcal{O}(J^2)$ in the source for the irrelevant operator).

I will now present a systematic treatment of the metric beta-function and show how the puzzle raised by the Gover-Hirachi theorem is resolved;

i.e. I will construct the Weyl anomaly which has all the required properties (e.g. correct limit for $g_{\mu\nu} = \eta_{\mu\nu}$) as required by the $\langle \mathcal{O}\mathcal{O}\rangle$ two-point function, as discussed before.

There are two cohomological structures underlying the local CS equation, which are related to the abelian Weyl symmetry:

$$[\delta_{\sigma_1}, \delta_{\sigma_2}] = 0$$

1. the transformation of the sources implied by the local CS equation (I include the metric among the J_i)

   $$\delta_\sigma J_i = \beta_i(J_j; \sigma)$$

   should obey the integrability condition

   $$\delta_{\sigma_2}\beta_i(\sigma_1) = \delta_{\sigma_1}\beta_i(\sigma_2)$$

   modulo trivial solutions corresponding to local redefinitions of the sources

   $$J_i' = \gamma_i(J_j)$$

   s.t. that J_i' transform just following their dimension

2. The Weyl anomaly $\mathcal{A}(J_i)$ is the solution of a second cohomology problem

$$\delta_{\sigma_2} \int d^d x \sqrt{g}\, \sigma_1(x) \mathcal{A} = \delta_{\sigma_1} \int d^d x \sqrt{g}\, \sigma_2(x) \mathcal{A}$$

modulo variations of local diffeo-invariant functionals of the sources $\{J_i\}$.

This is the Wess-Zumino consistency condition.

We can now solve these cohomology problems.

Of course there is no general solution: we have to specify the space-time dimension and the set of operators involved.

Examples

a.) $d = 2$, only source for the energy-momentum tensor, $g_{\mu\nu}$:

The solutions for the two cohomology problems are

$$\delta_\sigma g_{\mu\nu} = 2\,\sigma\,g_{\mu\nu}$$

$$\delta_\sigma W = \frac{c}{96\pi}\int d^2x\,\sqrt{g}\,\sigma\,R$$

which integrates to

$$W[g] = -\frac{c}{24\pi}\int d^2x\,\sqrt{g}R\frac{1}{\Box}R$$

In higher dimensions, while $\delta_\sigma W$ is known (solution of the cohomology problem) the explicit closed form of W is generally not known.

b.) In $d = 4$ with sources for the energy-momentum tensor and for a dimension four (exactly marginal) operator, $g_{\mu\nu}$ and J:

The first cohomology problem has the solution

$$\delta_\sigma g_{\mu\nu} = 2\,\sigma\, g_{\mu\nu}\,, \qquad\qquad \delta J = 0 \qquad \text{(no } \beta \text{ function!)}$$

The cohomological analysis for the generating functional requires the anomaly to have the form

$$\delta_\sigma W = \int d^d x\, \sqrt{g}\, \sigma \left(a\, E_4 + c\, C^2 + J \Delta_c J \right)$$

where

- E_4 and C^2 are the Euler density and the square of the Weyl-tensor, respectively and a and c are the usual Weyl anomaly coefficients
- Δ_c is the FTRP-operator discussed before
- The normalization of the last term is fixed by the $\langle \mathcal{O}\mathcal{O} \rangle$ two-point function.

This can be easily generalized to many marginal operators where the J_i are then local coordinates on the 'conformal manifold'.

c.) In $d = 4$ with sources for the energy-momentum tensor and for a dimension five operator, $g_{\mu\nu}$ and J:

The first cohomology problem has the solution (to $\mathcal{O}(J^2)$):

$$\delta_\sigma J = \sigma\, J$$

and

$$\delta_\sigma g_{\mu\nu} = 2\,\sigma\, g_{\mu\nu} + \sigma \frac{a\,\pi^2}{48\, c_T} \Big(R_{\mu\nu}\, J^2 + 2\, J\, \nabla_\mu \nabla_\nu J - 3\, g_{\mu\nu} (\nabla J)^2 + g_{\mu\nu}\, J \Box J \Big)$$

where the term in parantheses is the Ricci tensor evaluated for the Weyl invariant metric $\hat{g}_{\mu\nu} \equiv \frac{1}{J^2} g_{\mu\nu}$.

The second term is the one we had previously determined.

One then finds precisely one solution of the second cohomology problem, the WZ consistency condition for the anomaly, which reduces in flat space to $\int \sigma\, J\, \Box^3 J$:

$$\mathcal{A} = c\, C^2 + \frac{\pi^2}{9 \cdot 1024}\Big(J\,\Box^3 J + \text{31 terms which vanish in flat space}\Big) + \mathcal{O}(J^4)$$

The normalization is fixed by conformal symmetry via the various OPEs involved in the argument and it, of course, reproduces the anomaly in $\langle T_{\mu\nu}\mathcal{O}\mathcal{O}\rangle$.

Note that it is 'seeded' by one of the purely gravitational Weyl anomalies, $c\, C^2$

There are other non-trivial solutions to the second cohomology problem. One is seeded by E_4, but the others are independent. They all vanish in flat space.

We worked to quadratic order in the source J but to all orders in the metric $g_{\mu\nu}$. There are higher order terms in J, but at higher orders they mix with sources of dimension six, etc. operators. They are relevant for higher point correlators.

The complete expression is

$$
\begin{aligned}
\mathcal{A} = c\, C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \alpha\, c \Big\{ & \Box J\, \Box^2 J - \tfrac{13}{8} R\, R^{\mu\nu} R_{\mu\nu}\, J^2 + \tfrac{53}{162} R^3\, J^2 + \tfrac{4}{3} R^{\mu\nu} R^{\rho\sigma} R_{\mu\rho\nu\sigma}\, J^2 \\
& - \tfrac{1}{8} R\, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}\, J^2 + \tfrac{43}{72} R_{\mu\nu\rho\sigma} R^{\mu\nu\alpha\beta} R_{\alpha\beta}{}^{\rho\sigma}\, J^2 - \tfrac{35}{72} R^2\, J\, \Box J + \tfrac{25}{24} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}\, J\, \Box J \\
& - \tfrac{1}{36} \nabla^\mu R\, \nabla_\mu R\, J^2 + \tfrac{167}{12} R^{\mu\nu} R_{\mu\nu}\, \nabla^\alpha J\, \nabla_\alpha J - \tfrac{101}{24} R^2\, \nabla^\alpha J\, \nabla_\alpha J \\
& - \tfrac{79}{24} R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} \nabla^\alpha J\, \nabla_\alpha J - \tfrac{1}{3} R\, \Box \nabla^\mu J\, \nabla_\mu J - \tfrac{10}{9} R^{\mu\nu} \nabla_\mu \nabla_\nu R\, J^2 + \tfrac{7}{9} R^{\mu\nu} R\, J\, \nabla_\mu \nabla_\nu J \\
& + \tfrac{1}{36} \Box R\, J\, \Box J - \tfrac{16}{9} R\, (\Box J)^2 + \nabla^\mu R\, \nabla_\mu J\, \Box J + \tfrac{1}{6} R\, J\, \Box^2 J - 4\, R^{\mu\nu}\, \nabla_\mu J\, \Box \nabla_\nu J \\
& - \tfrac{37}{18} R_{\mu\nu} \nabla^\mu R\, J\, \nabla_\nu J - 22\, R_\mu{}^\alpha R_{\nu\alpha}\, \nabla^\mu J\, \nabla^\nu J + \tfrac{116}{9} R^{\mu\nu}\, R\, \nabla^\mu J\, \nabla^\nu J \\
& - 13\, R^{\alpha\beta} R_{\mu\alpha\nu\beta}\, \nabla^\mu J\, \nabla^\nu J - \tfrac{5}{18} \nabla^\mu \nabla^\nu R\, J\, \nabla_\mu \nabla_\nu J - \tfrac{5}{9} R\, \nabla_\mu \nabla_\nu J\, \nabla^\mu \nabla^\nu J \\
& - 5\, R^{\beta\gamma}\, \nabla_\gamma R_{\alpha\beta}\, J\, \nabla^\alpha J - \tfrac{8}{3} R_\alpha{}^\gamma R^{\alpha\beta}\, J\, \nabla_\beta \nabla_\gamma J + \tfrac{10}{3} R^{\beta\gamma}\, \nabla^\alpha J\, \nabla_\gamma \nabla_\beta \nabla_\alpha J \\
& + \tfrac{5}{6} \Box R^{\mu\nu}\, J\, \nabla_\mu \nabla_\nu J + \tfrac{22}{3} R^{\mu\nu}\, \nabla_\mu \nabla_\nu J\, \Box J - \tfrac{5}{3} \nabla^\mu R^{\alpha\beta}\, \nabla_\mu R_{\alpha\beta}\, J^2 \Big\} + \mathcal{O}(J^4)
\end{aligned}
$$

Comments

- While relevant and marginal operators can be use to deform the CFT, irrelevant operators would render it non-renormalizable. Therefore all correlation functions considered here are in the undeformed CFT.
- The example given here was in $d = 4$. The situation in $d = 2$ is somewhat different. There it is not possible to restrict to a dimension three operator. To get a consistent solution for the first cohomology problem one needs to introduce also a source for the dimension four $T\bar{T}$ operator of Zamolodchikov. It will, however, drop out of the solution of the second cohomology problem. $\Rightarrow$
- Anomalies quadratic in the sources are only possible in even dimensions, precisely when the Gover-Hirachi non-existence theorem holds.

Details of $d = 2$:

It turns out that there is no solution to the first cohomology problem if we only include a source J for a $\Delta = 3$ operator. If we also include a source $\tilde{J}$ for a $\Delta = 4$ operator, one finds

$$\begin{aligned}
\delta_\sigma J &= \sigma\, J \\
\delta_\sigma \tilde{J} &= 2\,\sigma\,\tilde{J} + \alpha\,\sigma\,J^2 \qquad\qquad\qquad \alpha \sim c_{\mathcal{O}\mathcal{O}\tilde{\mathcal{O}}} \\
\delta_\sigma g_{\mu\nu} &= 2\,\sigma\,g_{\mu\nu} + \tilde{J}\,\nabla_\mu\nabla_\nu\sigma + \alpha\,\sigma\big(J\,\nabla_\mu\nabla_\nu J - \frac{1}{2}g_{\mu\nu}(\nabla J)^2\big)
\end{aligned}$$

However the solution of the second cohomology problem is independent of $\tilde{J}$:

$$\delta_\sigma W = \int d^2x\sqrt{g}\,\sigma\,c\left(R + \frac{1}{16}\alpha R^2\,J^2 + \frac{1}{4}\alpha\,R\,J\,\Box J - \frac{1}{2}\alpha\,R\,(\nabla J)^2 + \frac{1}{4}\,\alpha\,J\,\Box J\right)$$

A more detailed analysis shows that

$$\tilde{O} = T\bar{T}$$

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Conclusions

- The presence of irrelevant operators requires the introduction of a metric beta-function in the local CS equation
- The Weyl anomalies are deformed and include the universal information contained in the beta-function
- Deforming the CFT by irrelevant operators to a certain order in PT produces a nontrivial "running" of the metric when the theory is formulated on a compact manifold
- Integrating the anomalies one can possibly relate the operator mixings and the partition function on a compact manifold (e.g. for the chiral ring of a SCFT)
- Is there a holographic computation of the deformed anomalies?

Thank you !