String Two-point Amplitude Revisited by Operator Formalism

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String Two-point Amplitudes

So far …

Let us consider a string two-point tree-level amplitude.
It is evaluated by a correlation function of two vertex operators with conformal weight one.

$$\langle V_1 V_2 \rangle$$

For open strings, the automorphism of upper half plane, $\mathrm{PSL}(2,\mathbb{R})$, is partially fixed by the two points of vertex operators. But the gauge volume of residual symmetry is infinity. So the amplitude vanishes as

$$\mathcal{A} = \frac{\text{finite}}{\infty} = 0$$

Two-point amplitude does not vanish

[H. Erbin, J. Maldacena and D. Skliros, “Two-Point String Amplitudes”, arXiv:1906.06051]

Naively there is another infinity coming from on-shell energy-momentum conservation.

$$\delta(p_1^0 - p_2^0)\delta^{D-1}(\boldsymbol{p}_1 - \boldsymbol{p}_2) \xrightarrow[\text{on-shell}]{} \delta(0)\delta^{D-1}(\boldsymbol{p}_1 - \boldsymbol{p}_2)$$

$$p_i^0 = \sqrt{\boldsymbol{p}_i^2 + m^2}$$

This $\delta(0)$ cancels the infinite gauge volume.

$$\mathcal{A} = \frac{\infty}{\infty} = 2p^0(2\pi)^{D-1}\delta^{D-1}(\boldsymbol{p}_1 - \boldsymbol{p}_2)$$

Fadeev-Popov trick

For open strings, 3 gauge fixing functions are necessary due to $\mathrm{PSL}(2,\mathbb{R})$.

The Fadeev-Popov determinant is given by

$$1 = \Delta_{\mathrm{FP}} \int Dg \prod_{i=0}^{2} \delta(f_i(x^g))$$

For the gauge fixing functions

$$f_0 = X^0(\hat{z}_0, \hat{\bar{z}}_0)\,, \quad f_1 = z_1 - \hat{z}_1\,, \quad f_2 = z_2 - \hat{z}_2$$

the Fadeev-Popov determinant becomes

$$\Delta_{\mathrm{FP}} = \hat{z}_{01}\hat{z}_{02}\hat{z}_{12}\partial X^0 + \hat{\bar{z}}_{01}\hat{\bar{z}}_{02}\hat{\bar{z}}_{12}\bar{\partial} X^0\,, \quad \hat{z}_{12} = \hat{\bar{z}}_{12}$$

Finally we obtain the two-point amplitude

$$\begin{aligned}\mathcal{A} &= \langle \Delta_{\rm FP}\, \delta(X^0(z_0, \bar{z}_0))\, V_1(z_1)\, V_2(z_2) \rangle \\ &= 2p^0 \hat{z}_{12}^2 \langle V_1 V_2 \rangle' \\ &= 2p^0 (2\pi)^{D-1} \delta^{D-1}(\boldsymbol{p}_1 - \boldsymbol{p}_2)\end{aligned}$$

where $\langle\ \rangle'$means that only spacial integration remains, *i.e.*,

$$\langle V_1 V_2 \rangle' = (2\pi)^{D-1} \delta^{D-1}(\boldsymbol{p}_1 - \boldsymbol{p}_2) \frac{1}{\hat{z}_{12}^2}$$

The key point: in the calculation of the amplitude there appears

$$\langle {\color{red}\partial X^0(z_0)} V_1(z_1) V_2(z_2) \rangle \sim {\color{red}p^0} \frac{z_{12}}{z_{01} z_{02}} \langle V_1 V_2 \rangle$$

Revisited by Operator Formalism

[SS and T. Takahashi, to appear]

Let us consider the two-point amplitudes in terms of ghost fields in the operator formalism.

In this formalism, BRST symmetry can be manifest. For example, it is valuable for constructing string field theory.

Open string

Energy conservation

$$\langle 0|p_1^2 - p_2^0\rangle = 2\pi\delta(p_1^0 - p_2^0)$$

$\langle 0|$: $\mathrm{SL}(2,\mathbb{R})$ invariant vacuum

leads to $\delta(0)$. We want to avoid this divergence.

When we introduce the eigenstate $|x^0\rangle$ of the zero-mode of the world-sheet field X^0, the delta function changes to one.

$$\langle x^0 = 0|p_1^0 - p_2^0\rangle = e^{i0\cdot(p_1^0 - p_2^0)} = 1$$

The operator corresponding to $|x^0 = 0\rangle$ is

$$\int_{-\infty}^{\infty} \frac{dq}{2\pi}\, e^{iqX^0(z,\bar{z})}$$

We want to rewrite this operator as the BRST invariant operator with ghost number one.

If we can find such an operator $\mathcal{V}_0$, the open string two-point amplitude on a disk is given by

$$\langle \mathcal{V}_0 \mathcal{V}_1 \mathcal{V}_2 \rangle\,, \quad \mathcal{V}_i := cV_i \quad (i = 1, 2)$$

(The total ghost number is three.)

The answer is

$$\mathcal{V}_0(z) := \int_{-\infty}^{\infty} \frac{dq}{2\pi} \left(c\partial X^0 e^{iqX^0} + i\alpha' q(\partial c) e^{iqX^0} \right)$$

This operator can be rewritten as

$$\mathcal{V}_0(z) = \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{1}{iq} [Q_B, e^{iqX^0}]$$

Q_B : BRST charge

$\mathcal{V}_0(z)$ is mostly BRST exact, because the integrand is singular at $q = 0$.

$$\mathcal{A} = \langle \mathcal{V}_0(z_0)\mathcal{V}_1(z_1)\mathcal{V}_2(z_2)\rangle\,, \quad \mathcal{V}_i(z) := cV_i(z) \quad (i=1,2)$$

All z_i are on the real axis. V_i are matter vertex operators with conformal weight one and momentum p_i^μ.

By the energy-momentum conservation,

$$\mathcal{A} = \int_{-\infty}^{\infty} dq\,(2\pi)^{D-1}\delta(q - p_1^0 + p_2^0)\delta^{D-1}(\boldsymbol{p}_1 - \boldsymbol{p}_2) \times \cdots$$

On-shell condition: $p_i^0 = \sqrt{\boldsymbol{p}_i^2 + m_i^2}$

If $m_1^2 \neq m_2^2$, q is replaced by non-zero value after q integration because of $p_1^0 - p_2^0 \neq 0$.
Since the integrand of $\mathcal{V}_0(z)$ is BRST exact for $q \neq 0$, the amplitude $\mathcal{A}$ vanishes.

If $m_1^2 = m_2^2$, the delta function for the energy conservation becomes $\delta(q)$. Therefore the amplitude becomes

$$\begin{aligned}\mathcal{A} &= \langle c\partial X^0(z_0)\, cV_1(z_1)\, cV_2(z_2)\rangle' \\ &= 2p^0(2\pi)^{D-1}\delta^{D-1}(\boldsymbol{p}_1 - \boldsymbol{p}_2)\end{aligned}$$

$\langle\ \rangle'$ means that the delta function of energy conservation is not included.

Only at $q = 0$ the BRST exactness is broken. So the amplitudes has non-zero value.

Lorentz invariance

For the infinitesimal Lorentz transformation; $X^{\mu} \to X^{\mu} + \epsilon^{\mu}_{\nu} X^{\nu}$

$$\delta \mathcal{V}_0 = [Q_B, \int_{-\infty}^{\infty} \frac{dq}{2\pi} \epsilon^{0}_{\nu} X^{\nu} e^{iqX^0}]$$

Since it does not have $1/q$ singularity, the finite Lorentz transformation leads to

$$\mathcal{V}_0 \to \mathcal{V}_0 + [Q_B, \bullet]$$

The BRST exact term does not contribute to the correlation function for physical states. So the two-point amplitude satisfies Lorentz invariance.

Closed string

The application of open string amplitude to closed string one is not straightforward.
Anyway, in a similar way to open string,

$$\mathcal{V}_0 = \int \frac{dq}{2\pi}\frac{1}{iq}[\mathcal{Q}_B, e^{iqX^0}] \quad (\mathcal{Q}_B = Q_B + \tilde{Q}_B\text{: BRST charge})$$

$$= \int \frac{dq}{2\pi}\left\{(c\partial X^0 + \tilde{c}\overline{\partial} X^0)e^{iqX^0} + \frac{i\alpha' q}{4}(\partial c + \overline{\partial}\tilde{c})e^{iqX^0}\right\}$$

$\langle \mathcal{V}_0(z_0)\, c\tilde{c}V_1(z_1)\, c\tilde{c}V_2(z_2)\rangle$ does not work. Because each term in it has ghost number ± 1.

A relevant correlation function on a sphere should have ghost number 0.

A prescription is

$$\mathcal{V}_0 = \int \frac{dq}{2\pi}\,\frac{1}{iq}[\mathcal{Q}_B, e^{iqX^0}]\,, \quad \mathcal{V}_1 = (c\partial c)\tilde{c}V_1\,, \quad \mathcal{V}_2 = c\tilde{c}V_2$$

(V_i : matter vertex operator with conformal weight (1,1))

Then one can calculates

$$\begin{aligned}
&\langle \mathcal{V}_0(z_0,\bar{z}_0)\mathcal{V}_1(z_1,\bar{z}_1)\mathcal{V}_2(z_2,\bar{z}_2)\rangle \\
&= \langle \tilde{c}\bar{\partial}X^0(\bar{z}_0)\,(c\partial c)\tilde{c}V_1(z_1,\bar{z}_1)\,c\tilde{c}V_2(z_2,\bar{z}_2)\rangle' \\
&= (2\pi)^{D-1}\delta^{D-1}(\boldsymbol{p}_1-\boldsymbol{p}_2)\,z_{12}^2\bar{z}_{01}\bar{z}_{02}\bar{z}_{12}\,\frac{2p^0}{z_{12}^2\bar{z}_{01}\bar{z}_{02}\bar{z}_{12}} \\
&= 2p^0(2\pi)^{D-1}\delta^{D-1}(\boldsymbol{p}_1-\boldsymbol{p}_2)
\end{aligned}$$

Summary

String tree-level two-point function

$$\langle \mathcal{V}_0 \mathcal{V}_1 \mathcal{V}_2 \rangle = 2p^0 (2\pi)^{D-1} \delta^{D-1}(\boldsymbol{p}_1 - \boldsymbol{p}_2)$$

Open string

(mostly) BRST exact operator

$$\mathcal{V}_0 = \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{1}{iq} [Q_B, e^{iqX^0}]$$

$$\mathcal{V}_1 = cV_1 \,, \quad \mathcal{V}_2 = cV_2$$

Closed string

$$\mathcal{V}_0 = \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{1}{iq} [\mathcal{Q}_B, e^{iqX^0}]$$

$$\mathcal{V}_1 = (c\partial c)\tilde{c}V_1 \,, \quad \mathcal{V}_2 = c\tilde{c}V_2$$

but we want a more beautiful prescription.