Supercurrents in $\mathcal{N}$=1 Minimal Supergravity in the Superconformal Formalism

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Outline

- Review of Callan-Coleman-Jackiw (CCJ) improved currents
- Non-linear form of Ferrara-Zumino equations from 1975
- Generalized expressions for Ward identities of Einstein tensor and Supercurrent multiplets using superconformal approach
  - Old Minimal supergravity
  - Old Minimal with FI term
  - New Minimal supergravity
- Conclusions

Currents

- *In general for every rigid symmetry there is a conserved current upon using equations of motion*
- *Currents from gauge couplings*
  For example $S = \int d^4x[-\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + A^\mu J_\mu + ...] \quad \rightarrow$
  $\partial^\mu F_{\mu\nu} = J_\nu \quad \partial^\mu J_\mu \approx 0$
  Pure gravity $G_{\mu\nu} \approx 0$
- *Symmetric energy momentum tensor is due to Lorentz rotation invariance:* $T_{\mu\nu} = T_{\nu\mu}$
- *When there is a conformal symmetry C. G. Callan, Jr., S. R. Coleman and R. Jackiw, Annals Phys. 59 (1970) 42-73 proved that there is a traceless improved current* $\Theta_{\mu\nu} = T_{\mu\nu} +$ *improvement term*

Example:

$$S = \int d^4x\sqrt{-g}(\tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi))$$

has a symmetric and conserved, but not traceless canonical energy-momentum tensor

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi + g_{\mu\nu}\mathcal{L}_M,\ \partial^\mu T_{\mu\nu} = 0,\ T_\mu{}^\mu = -\partial^\mu\varphi\partial_\mu\varphi + 4V$$

The improved energy momentum tensor

$$\Theta_{\mu\nu} = T_{\mu\nu} - \frac{1}{6}(\partial_\mu\partial_\nu - g_{\mu\nu}\Box)\varphi^2$$
$$\partial^\mu\Theta_{\mu\nu} = 0, \quad \Theta_\mu{}^\mu \approx 0 \quad \textit{quartic } V$$

$\approx$ *means equation of motions* $\Box\varphi \approx -V_\varphi$ *is satisfied.*

The improved energy-momentum tensor can also be obtained from the conformal invariant gravity-coupled action.
A conventional gravity theory

$$S = \int d^4x\sqrt{-g}\left[\tfrac{1}{2}\kappa^{-2}R + \mathcal{L}_M\right] \qquad G_{\mu\nu} \approx \kappa^2 T_{\mu\nu}$$

To have to conformal and Weyl symmetry, replace $\kappa^{-2} \to \kappa^{-2} - \frac{1}{6}\varphi^2$. *For the action*

$$\mathcal{S} = \int d^4x\sqrt{-g}\left[\tfrac{1}{2}(\kappa^{-2} - \tfrac{1}{6}\varphi^2)R + \tfrac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right]$$

$$G_{\mu\nu} \approx \kappa^2\Theta^{\mathrm{c}}_{\mu\nu}\,, \qquad \Theta^{\mathrm{c}}_{\mu\nu} = T_{\mu\nu} - \tfrac{1}{6}(\nabla_\mu\partial_\nu - g_{\mu\nu}\nabla^\rho\partial_\rho)\varphi^2 + \tfrac{1}{6}\varphi^2 G_{\mu\nu}$$

$$\nabla^\mu\Theta^{\mathrm{c}}_{\mu\nu} \approx 0, \qquad \Theta^{\mathrm{c}}_{\mu}{}^{\mu} \approx 0 \qquad \textit{for quartic } V(\varphi).$$

Rigid Poincaré $T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi + g_{\mu\nu}L$

Rigid conformal $\Theta_{\mu\nu} = T_{\mu\nu} - \dfrac{1}{6}(\partial_\mu\partial_\nu - g_{\mu\nu}\Box)\varphi^2$

Local conformal $\Theta^{\mathrm{c}}_{\mu\nu} = T_{\mu\nu} - \frac{1}{6}(\nabla_\mu\partial_\nu - g_{\mu\nu}\nabla^\rho\partial_\rho)\varphi^2 + \frac{1}{6}\varphi^2 G_{\mu\nu}$

From conformal action

These formulations can be obtained from a conformal action, containing apart from the physical field φ also a compensating scalar φ_0. Both have then Weyl weight 1

$$\mathcal{S} = \int d^4\sqrt{g}\left[-\tfrac{1}{2}\varphi_0\Box^C\varphi_0 + \tfrac{1}{2}\varphi\Box^C\varphi + \lambda\varphi^4\right]$$

$$= \int d^4\sqrt{g}\left[\tfrac{1}{2}\partial_\mu\varphi_0\partial^\mu\varphi_0 - \tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \tfrac{1}{12}(\varphi_0^2 - \varphi^2)R + \lambda\varphi^4\right]$$

The Einstein frame: $\varphi_0^2 = \varphi^2 + 6\kappa^{-2}$ $\rightarrow$ No conformal part in the action, not traceless energy-momentum tensor.
Conformal frame: $\varphi_0^2 = 6\kappa^{-2}$ $\rightarrow$ Conformal action, traceless energy-momentum tensor.

Supercurrents

$$\text{Old Minimal} \quad \bar{D}^{\dot{\alpha}} E_{\alpha\dot{\alpha}} = D_\alpha \mathcal{R}, \qquad \text{New Minimal} \quad \bar{D}^{\dot{\alpha}} E_{\alpha\dot{\alpha}} = W_\alpha$$

$$\nabla^\mu G_{\mu\nu} = 0$$

At the linearized level

$$E_{\alpha\dot{\alpha}} + \kappa^2 J_{\alpha\dot{\alpha}} \approx 0 \quad \rightarrow \quad G_{\mu\nu} + \kappa^2 T_{\mu\nu} \approx 0$$

S. Ferrara and B. Zumino, Nucl. Phys. B134 (1978) 301-326, Z. Komargodski and N. Seiberg, JHEP 1007 (2010) 017

$$\text{Old Minimal} \quad \bar{D}^{\dot{\alpha}} J_{\alpha\dot{\alpha}} \approx D_\alpha Y, \quad \text{New Minimal} \quad \bar{D}^{\dot{\alpha}} J_{\alpha\dot{\alpha}} \approx \omega_\alpha$$

$$\bar{D}^{\dot{\alpha}} J_{\alpha\dot{\alpha}} \approx D_\alpha Y + \omega_\alpha .$$

S. Ferrara and B. Zumino, Nucl. Phys. B134 (1978) 301-326, S. M. Kuzenko, Eur. Phys. J. C71 (2011) 1513.

The trace equations:

$$\mathcal{R} \approx -\kappa^2 Y, \quad W_\alpha \approx -\kappa^2 \omega_\alpha \quad \rightarrow \quad R \approx -\kappa^2 T_\mu{}^\mu$$

Supercurrent and Einstein tensor from superconformal approach. Action and field equations

The actions of chiral multiplets in the superconformal setup are symbolically obtained from

$$\mathcal{S} = \Big[N(X^I, \bar{X}^{\bar{I}})\Big]_D + \Big[\mathcal{W}(X^I)\Big]_F$$

X^I *are chiral multiplets with* $(1,1)$ *Weyl and chiral weights.* $I = 0, ..., n$. n *is the number of physical multiplets,* 0 *is for compensating multiplet.*
$N = (2,0)$ *is real.* $\mathcal{W} = (3,3)$ *is holomorphic.*
$S^0 = X^0\,, \quad S^i = \frac{X^i}{X^0}\,, \quad i = 1, \ldots, n\,,$

$$N(X, \bar{X}) = S^0 \bar{S}^{\bar{0}} \Phi(S, \bar{S})\,, \qquad \mathcal{W}(X) = (S^0)^3 W(S)$$

Pure supergravity is obtained for

$$N = -3S^0\bar{S}^{\bar{0}}\,, \quad \Phi = -3.$$

This allows us to define the matter coupling function Φ_M *as* $\Phi(S,\bar{S}) = -3 + 3\Phi_M(S,\bar{S})$ *such that*

$$N(X,\bar{X}) = S^0\bar{S}^{\bar{0}}(-3 + 3\Phi_{\mathrm{M}}(S,\bar{S})), \qquad \mathcal{W}(X) = (S^0)^3 W(S)$$

The field equation for the compensating multiplet, $I = 0$ *can be written as*

$$\mathcal{R} + \frac{Y}{(S^0)^2} \approx 0 \quad \textit{with} \quad Y \equiv -2(S^0)^3\Delta W + S^0 T(\bar{S}^{\bar{0}}\Delta K)$$

$\mathcal{R} \equiv \frac{1}{S^0}T(\bar{S}^{\bar{0}})$ *T. Kugo and S. Uehara, Prog.Theor.Phys. 73 (1985) 235. The operation* T *is the superconformal version of the superspace operation* $\bar{D}^2$*. In the conformal frame* $S^0 = \kappa^{-1}$*, we have* $\mathcal{R} + \kappa^2 Y \approx 0.$

Defined

$$\Delta K \equiv -\frac{1}{3\bar{S}^{\bar{0}}}\left(N_0 + 3\bar{S}^{\bar{0}}\right) = S^i \Phi_{\mathrm{M}\,i} - \Phi_{\mathrm{M}}$$
$$\Delta W \equiv \frac{1}{3(S^0)^2}\mathcal{W}_0 = W - \frac{1}{3}S^i W_i \,.$$

Conformal case: $\Delta K = \Delta W = 0$. *Thus* W *is homogeneous of rank* 3 *and* Φ_M *is homogeneous of rank 1 both in* S^i *and* $\bar{S}^i$, $\Phi_{Mi\bar{j}}$ *has degree zero. One field* $\Phi_M = S\bar{S}$ *corresponds to the conformally coupled scalar of CCJ.*

Non-linear form of Ferrara-Zumino equations 1975

We want to generalize the FZ equation $\overline{D}^{\dot{\alpha}} E_{\alpha\dot{\alpha}} = D_\alpha \mathcal{R}$ *to the non linear level. In the superconformal formulation nonlinearities come from two sources: compensator dependence, coupling to matter. The non linear version of* $E_{\alpha\dot{\alpha}}$ *we denote as* $\mathcal{E}_{\alpha\dot{\alpha}}$.

$$\overline{D}^{\dot{\alpha}} \mathcal{E}_{\alpha\dot{\alpha}} = (S^0)^{k_1} (\bar{S}^{\bar{0}})^{k_2} D_\alpha \left(\frac{\mathcal{R}}{S^0} \right)$$

Matching the Weyl and chiral weights on both sides we find
$k_1 + k_2 = w, \quad k_1 - k_2 = 3$
We found that the non-linear version is the tensor $\mathcal{E}_{\alpha\dot{\alpha}}$ *and* $\mathcal{R}$ *satisfy*

$$\overline{\mathcal{D}}^{\dot{\alpha}} \mathcal{E}_{\alpha\dot{\alpha}} = (S^0)^3 \mathcal{D}_\alpha \left(\frac{\mathcal{R}}{S^0} \right)$$

Generalized Bianchi identity. The scalar curvature multiplet $\mathcal{R}$, *with chiral and Weyl weights* $(1, 1)$

$$\mathcal{E}_{\alpha\dot{\alpha}} = -4i \bar{S}^0 \overleftrightarrow{\partial}_{\alpha\dot{\alpha}} S^0 - 2(D_\alpha S^0)(\bar{D}_{\dot{\alpha}} \bar{S}^{\bar{0}})$$

$\mathcal{R} \equiv (S^0)^{-1} T(\bar{S}^{\bar{0}})$.

In the components $\{X^0,\ \Omega^0,\ F^0\}$, *using* $\mathcal{E}_{\alpha\dot{\alpha}} = \frac{1}{4} i (\gamma^\mu)_{\alpha\dot{\alpha}} \mathcal{E}_\mu$

$$\mathcal{E}_\mu = 4iX^0 \mathcal{D}_\mu \bar{X}^{\bar{0}} - 4i\bar{X}^{\bar{0}} \mathcal{D}_\mu X^0 + 2i\overline{\Omega}^0 P_L \gamma_\mu \Omega^{\bar{0}}$$

$$\mathcal{D}_\mu X^I = (\partial_\mu - b_\mu - iA_\mu)\, X^I - \frac{1}{\sqrt{2}} \bar{\psi}_\mu \Omega^I \,.$$

$$\begin{aligned} \mathcal{E}_\mu &= -8A_\mu \bar{X}^{\bar{0}} X^0 - 4i\bar{X}^{\bar{0}} \overset{\leftrightarrow}{\partial}_\mu X^0 \\ &+ 2i\bar{\Omega}^0 P_L \gamma_\mu \Omega^{\bar{0}} + 2i\sqrt{2}\bar{\psi}_\mu \left(\bar{X}^{\bar{0}} \Omega^0 - X^0 \Omega^{\bar{0}}\right) \end{aligned}$$

Since we have used an explicit splitting in the action we can write

$$[N(X,\bar{X})]_D = S^0\bar{S}^{\bar{0}}(1-\Phi_M(S,\bar{S}))$$

using the relation between $\mathcal{R}$ *and* Y *and* $\mathcal{E}_{\alpha\dot{\alpha}} + J_{\alpha\dot{\alpha}} \approx 0$ *one finds the conservation law for the supercurrent*

$$\begin{aligned}\overline{\mathcal{D}}^{\dot{\alpha}}\mathcal{J}_{\alpha\dot{\alpha}} &\approx (S^0)^3\mathcal{D}_\alpha\left(\frac{Y}{(X^0)^3}\right)\\ &\approx -(S^0)^3\mathcal{D}_\alpha\left(2\Delta W-(S^0)^{-2}T\left(\bar{S}^{\bar{0}}\Delta K\right)\right)\end{aligned}$$

$\overline{\mathcal{D}}^{\dot{\alpha}}\mathcal{J}_{\alpha\dot{\alpha}} \approx 0$ *in the conformal frame.*

For the compensator $S^0 = \{X^0, \chi^0, F^0\}$ *and for the physical multiplets* $S^i = \{S^i, \chi^i, F^i\}$

$$
\begin{aligned}
J_\mu &= -\Phi_M \mathcal{E}_\mu \\
&+2i\bar{X}^{\bar{0}}\Phi_{M\,\bar{i}}\bar{\chi}^{\bar{0}}\gamma_\mu\chi^{\bar{i}} + 2iX^0\Phi_{M\,i}\bar{\chi}^i\gamma_\mu\chi^0 \\
&+2iX^0\bar{X}^{\bar{0}}\left[2(\Phi_{M\,i}\mathcal{D}_\mu S^i - \Phi_{M\,\bar{i}}\mathcal{D}_\mu\bar{S}^{\bar{i}}) - \Phi_{M\,i\bar{j}}\bar{\chi}^i\gamma_\mu\chi^{\bar{j}}\right] \\
\mathcal{E}_\mu &= 4iX^0\mathcal{D}_\mu\bar{X}^{\bar{0}} - 4i\bar{X}^{\bar{0}}\mathcal{D}_\mu X^0 + 2i\bar{\chi}^0\gamma_\mu\chi^{\bar{0}}
\end{aligned}
$$

Choosing a frame:

In the Einstein frame : $-3\kappa^{-2} = N = 3X^0\bar{X}^{\bar{0}}\left(-1+\Phi_M\right)$,

$$0 = N_I\Omega^I = 3\bar{X}^{\bar{0}}\left[(-1+\Phi_{\rm M})\chi^0 + \Phi_{{\rm M}\,i}\chi^i\right]$$

Supergravity and matter fields get mixed.

In the conformal frame : $\quad X^0 = \kappa^{-1}\,, \qquad \Omega^0 = 0$

In this frame $\mathcal{E}_\mu$ *does not depend on matter fields, and we have*

$$\mathcal{E}_\mu = -8\kappa^{-2} A_\mu\,,$$
$$J_\mu = 8\kappa^{-2}\Phi_M A_\mu + 2i\kappa^{-2}\left[2(\Phi_{M\,i}\mathcal{D}_\mu S^i - \Phi_{M\,\bar{i}}\mathcal{D}_\mu \bar{S}^{\bar{i}}) - \Phi_{M\,i\bar{j}}\bar{\chi}^i\gamma_\mu\chi^{\bar{j}}\right]$$

This is the generalization of the CCJ.
The bosonic part: improved currents and a modified Einstein equation with a matter energy-momentum tensor that contains the gravity part $G_{\mu\nu}\Phi_M$ *and a* $U(1)$ *part, such that it is conserved and traceless due to the equations of motions.*

A_μ field equation

We observe that

$$-\frac{3}{4}(\mathcal{E}_\mu + J_\mu) = iN_{\bar{I}}\mathcal{D}_\mu \bar{X}^{\bar{I}} - iN_I \mathcal{D}_\mu X^I + \frac{1}{2} iN_{I\bar{J}} \overline{\Omega}^I \gamma_\mu \Omega^{\bar{J}}$$

Since $[\mathcal{W}]_F$ does not involve A_μ (the gauge field of the R-symmetry in the conformal approach, and is the auxiliary field in the super-Poincaré action) the A_μ field equation

$$e^{-1}\frac{\delta}{\delta A^\mu}[N(X,\bar{X})]_D = iN_{\bar{I}}\mathcal{D}_\mu \bar{X}^{\bar{I}} - iN_I \mathcal{D}_\mu X^I + \frac{1}{2} iN_{I\bar{J}} \overline{\Omega}^I \gamma_\mu \Omega^{\bar{J}}$$

$$\mathcal{E}_\mu + J_\mu \approx 0 \qquad \overline{\mathcal{D}}^{\dot{\alpha}} (\mathcal{E}_{\alpha\dot{\alpha}} + J_{\alpha\dot{\alpha}}) \approx 0$$

This expression is a superconformal primary, and can be used as first component of a superconformal multiplet.

Old minimum supergravity with a FI term

$$\mathcal{L} = \left[-3S^0 e^{\xi V}\bar{S}^0\right]_D$$

ξ is the (dimensionless) FI constant. S^0 transforms under an abelian gauge group gauged by a real multiplet V.

$$S^0 \to S^0 e^{-\xi V}, \quad V \to V + \Lambda + \bar{\Lambda}$$

$$\bar{\mathcal{D}}^{\dot{\alpha}}\mathcal{E}_{\alpha\dot{\alpha}} = (e^{\xi V}S^0)^3\mathcal{D}_\alpha\left[\frac{e^{-3\xi V}T\left(e^{\xi V}\bar{S}^0\right)}{(S^0)^2}\right] + 3\xi S^0 e^{\xi V}\bar{S}^0 W_\alpha\,.$$

Chirally extended supergravity in B. de Wit and P. van Nieuwenhuizen, Nucl. Phys. B139 (1978) 216-220.
$\xi \to 0$ gives the pure old minimal.

Adding Matter

$$\mathcal{L} = \left[N(X^I, \bar{X}^I)e^{\xi V}\right]_D$$

$$N(X^I, \bar{X}^I) = S^0\bar{S}^0\Phi(S^i, \bar{S}^i) = S^0\bar{S}^0(-3 + \Phi_M)$$

The super-Einstein equations are obtained by variation of the auxiliary field A_μ

$$\tfrac{1}{4}i\gamma^\mu_{\alpha\dot{\alpha}}e^{-1}\frac{\delta\mathcal{L}}{\delta A^\mu} = \mathcal{E}_{\alpha\dot{\alpha}}(S^0, V) + J_{\alpha\dot{\alpha}}(S^0, V, S^i) \approx 0$$

from which

$$\bar{\mathcal{D}}^{\dot{\alpha}}\mathcal{E}_{\alpha\dot{\alpha}} + \bar{\mathcal{D}}^{\dot{\alpha}}J_{\alpha\dot{\alpha}} \approx 0$$

$$\bar{\mathcal{D}}^{\dot{\alpha}}J_{\alpha\dot{\alpha}} \approx \tfrac{1}{3}(e^{\xi V}S^0)^3\mathcal{D}_\alpha\left[\frac{e^{-3\xi V}T\left(e^{\xi V}N_0^M\right)}{(S^0)^2}\right] - \xi W_\alpha N^M e^{\xi V}$$

which is the generalization of $R \approx -\kappa^2 T_\mu{}^\mu$ *of general relativity.*

New Minimal

$$\mathcal{L}^{nm} = \left[3L\ln\frac{L}{S^0\bar{S}^0}\right]_D$$

$$\bar{\mathcal{D}}^{\dot{\alpha}}\mathcal{E}^L_{\alpha\dot{\alpha}} = L\,W^L_\alpha$$

Adding Matter

$$\mathcal{L}^{nm} = \left[3L\ln\frac{L}{S^0\bar{S}^0}\right]_D + [L\,K]_D$$

K *is the matter Kähler potential. The field equation of* χ_α*, the fermionic component of the linear multiplet, which defines* W^L_α, W^K_α

$$W^L_\alpha + W^K_\alpha \approx 0$$

The A_μ *field equation is split as*

$$\mathcal{E}^L_a + J_a \approx 0$$

$$\overline{\mathcal{D}}^{\dot{\alpha}}J_{\alpha\dot{\alpha}} \approx L\,W^K_\alpha$$

Conclusions

- *We have generalized FZ multiplet for any* $\mathcal{N} = 1$ $d = 4$ *curved background using superconformal approach*
- *Generalized Ward identities for Supercurrent and Einstein tensor multiplet from superconformal approach*
  - *Pure Old Minimal*
  - *+ matter*
  - *Pure Old Minimal + FI*
  - *Pure New Minimal*
  - *+ matter*
- *Important for*
  - *Nonlinear realization of supersymmetry, nilpotent fields,...*
  - *Cosmological applications*
  - *Supergravity backgrounds for localization techniques*
  - *...*

Thank you

The auxiliary fields are

old minimal: $(A_a,\ u = \kappa \bar{F}^0)$,

new minimal: $(A_\mu,\ a_{\mu\nu})$, $\quad H^\mu \equiv e^{-1}\varepsilon^{\mu\nu\rho\sigma}\left(\kappa^2\partial_\nu a_{\rho\sigma} - \frac{1}{4}\bar{\psi}_\nu\gamma_\rho\psi_\sigma\right)$

The actions for pure Poincaré supergravity are

$$\begin{aligned}\kappa^2 e^{-1}\mathcal{L}^{\text{om}} =& \frac{1}{2}R - \frac{1}{2}\bar{\psi}^\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho - \frac{1}{3}u\,\bar{u} + 3A^a A_a \\ \kappa^2 e^{-1}\mathcal{L}^{\text{nm}} =& \frac{1}{2}R - \frac{1}{2}\bar{\psi}^\mu\gamma^{\mu\nu\rho}(D_\nu - \tfrac{3}{2}i\gamma_* A_\nu)\psi_\rho + \frac{3}{4}H_a H^a \\ &+ \frac{3}{2}\kappa^2\varepsilon^{\mu\nu\rho\sigma}(2A_\mu - H_\mu)\partial_\nu a_{\rho\sigma}\end{aligned}$$

the Weyl multiplet: $\{e^a_\mu,\, \psi_\mu,\, b_\mu,\, A_\mu\}$.

chiral multiplets: $\{X^I,\, \Omega^I,\, F^I\}$.

a real vector multiplet: $\{V,\, \zeta,\, \mathcal{H},\, W_a,\, \lambda,\, D\}$*, which in Wess–Zumino (WZ) gauge is reduced to* $\{W_\mu,\, \lambda,\, D\}$.

a real linear multiplet: $\{L,\, a_{\mu\nu},\, \chi\}$.

$$\overline{\mathcal{D}}^{\dot\alpha}\mathcal{E}_{\alpha\dot\alpha} = \tfrac{1}{6}\sqrt{2}\left[X^I\Theta(\Omega)_{I\alpha} + 2\Omega^I_\alpha\Theta(F)_I\right] - i\zeta_\alpha\Theta(\mathcal{H}) - 2W_\alpha\Theta(D) + W^\chi_\alpha\, L$$