Various facets of 2d/4d correspondemce

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Foreword

Based on RP-1006, Fucito, Morales, Pacifici, RP-1103, RP-1601, G. Poghosyan & RP-1602;

ongoing work: Functional relations for $N = 2$ SYM with gauge group $SU(3)$

Motivated by

Alexei Zamolodchikov's unpublished paper "Generalized Mathieu equation and Liouville TBA"-2000

& recent papers:

Alba Grassi, Jie Gu, Marcos Marino-arXiv:1908.07065

Davide Fioravanti & Daniele Gregory-arXiv:1908.08030

Plan

- Introduction to $\mathcal{N} = 2$ SYM, SW theory and Nekrasov partition function
- AGT relation
- Prepotential in the limit $\epsilon_2 \to 0$
- Deformed Seiberg-Witten curve
- From DSW to linear Differential equation
- A-cycles from differential equation
- Analogs of Baxter's T-Q equations, Relation to ODE/IM for $c = 98$, A_2-Toda
- Extension of Al.Zamolodchikov's conjecture
- Numerical checks, comparison with instanton calculus
- Conclusions

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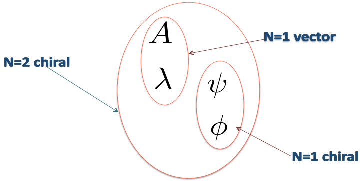
The field content and action



$$S = \int d^4x d^4\theta \, \Im\tau \, \mathrm{tr}\Psi^2$$

Scalar potential: $V \sim \mathrm{tr}[\phi, \phi^\dagger]^2$

Low energy effective action

Below Ψ includes only massless fields (i.e. those from the Cartan of the gauge group)

$$S_{eff} = \int d^4xd^4\theta \, \Im \mathcal{F}(\Psi)$$

$\mathcal{F}$ - the Seiberg Witten prepotential
In the case of $SU(2)$

$$\mathcal{F}(\Psi) = \frac{i}{2\pi}\Psi^2 \log \frac{2\Psi^2}{e^3\Lambda^2} - \frac{i}{\pi}\sum_{k=1}^{\infty} \mathcal{F}_k \left(\frac{\Lambda}{\Psi}\right)^{4k} \Psi^2$$

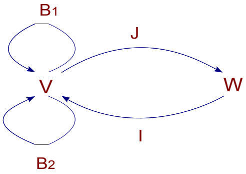
$\mathcal{F}_1 = \frac{1}{2}$, $\mathcal{F}_2 = \frac{5}{16}$, $\mathcal{F}_3 = \frac{3}{4}$, $\mathcal{F}_4 = \frac{1469}{512}, \ldots$

Moduli space of instantons, ADHM

gauge group: $U(N)$; instanton number: k; $V = \mathbb{C}^k$; $W = \mathbb{C}^N$
ADHM equations:

$$[B_1, B_2] + IJ = 0; \quad \left[B_1, B_1^\dagger\right] + \left[B_2, B_2^\dagger\right] + II^\dagger - J^\dagger J = \zeta$$

Equivalence relation: $(B_i, I, J) \sim (\phi B_i \phi^{-1}, \phi I, J\phi^{-1})$, $\phi \in U(k)$
Global gauge trans. : $(B_i, I, J) \to (B_i, Ig, g^{-1}J)$, $g \in U(N)$
Rotations of Euclidean space time: $(z_1, z_2) \to (e^{i\epsilon_1} z_1, e^{i\epsilon_1} z_2)$
$(B_i, I, J) \to (e^{i\epsilon_i} B_i, I, e^{i\epsilon_1 + i\epsilon_2} J)$,



The induced action

R.Flume, R.P., H.Storch 'arXiv:hep-th/0110240

$$\mathcal{F}_k \simeq \int_{\mathcal{M}'_k} e^{-d_x \omega},$$

$d_x \equiv d + i_x$ is an equivariant exterior derivative, i_x denotes contraction with the vector field x which generates the $U(1)$ subgroup of global gauge transformations selected by the choice of "Higgs" expectation values $\langle \phi \rangle_{cl} = diag(a_1, \ldots, a_N)$.
ω is the differential one-form

$$\omega = G(x, \bullet)$$

$G(\bullet, \bullet)$ is the natural induced metric on moduli space.

Localization to the zero locus of the vector field x

The coefficient $\mathcal{F}_k$ may be deformed into

$$\mathcal{F}_k(t) \equiv \int_{\mathcal{M}'_k} e^{-\frac{1}{t} d_x \omega}$$

Compute

$$\frac{d}{dt}\mathcal{F}_k(t) = -\frac{1}{t^2}\int_{\mathcal{M}'_k} d_x \left(\omega e^{-\frac{1}{t} d_x \omega}\right) = -\frac{1}{t^2}\int_{\mathcal{M}'_k} d \left(\omega e^{-\frac{1}{t} d_x \omega}\right).$$

The saddle point approximation is exact! There are contributions only from the points where $x = 0$. Unfortunately they are too many: in fact union of sub-manifolds of dimensions $2Nk - 4$ (c.f. $\dim \mathcal{M}'_k = 4Nk - 4$)

Incorporating space-time rotations

A wonderful way out: modify the vector field x incorporating (Euclidean) space-time rotations (parametrized by ϵ_1, ϵ_2) with the global gauge transformations (parametrized by the expectations values $a_1, \ldots, a_N$)

$$Z_k\left(a_u, \epsilon_1, \epsilon_2\right) \equiv \int_{\mathcal{M}_k} e^{-d_{\tilde{x}}\tilde{\omega}},$$

$\tilde{x}$ is the modified vector field and

$$\tilde{\omega} = G(\tilde{x}, \bullet)$$

Now we are lucky: the vector field $\tilde{x}$ has finitely many zeros!

Generalized partition function

complete localization!

$$Z_k(a_u, \epsilon_1, \epsilon_2) = \sum_{i \in fixed\ points} \frac{1}{\det \mathcal{L}_{\tilde{x}}} \Bigg|_i .$$

How this is related to SW prepotential? Introduce the partition function Nekrasov 'arXiv:hep-th/0206161

$$Z(a_u, \epsilon_1, \epsilon_2, q) \equiv 1 + \sum_{k=1}^{\infty} Z_k(a, \epsilon_1, \epsilon_2) q^k = e^{\frac{1}{\epsilon_1 \epsilon_2} \mathcal{F}(a_u, \epsilon_1, \epsilon_2, q)}$$

$\frac{1}{\epsilon_1 \epsilon_2}$ is the "volume factor" and $\mathcal{F}(a_u, 0, 0, q)$ coincides with the instanton part of SW prepotential.

$\mathcal{N} = 2$ SYM in Ω bacground

From the point of view of the initial theory above modification boils down to the consideration of the $\mathcal{N} = 2$ SYM in a specific background commonly referred as Ω-background. The two parameters ϵ_1, ϵ_2 specifying the general Ω- background are introduced in [Moor,Nekrasov,Shatashvili 'arXiv:hep-th/9712241], Losev,Nekrasov,Shatashvili 'arXiv:hep-th/9801061 to regularize the integrals over moduli space of instantons.

- It is clarified in Nekrasov 'arXiv:hep-th/0206161 how the partition function in this background is related to the Seiberg-Witten prepotential.
- In the same paper:
  calculation of the prepotential up to 5 instantons are performed choosing $h = \epsilon_1 = -\epsilon_2$ and it was demonstrated that at vanishing h one exactly recovers the results extracted from the Seiberg-Witten curve.
  AGT tells us that this case corresponds to $c = 1$ CFT if gauge group is $SU(2)$. For $SU(N)$ we get $c = N - 1$

Partition function with generic ϵ_1, ϵ_2

- In Flume, R.P. 'arXiv:hep-th/0208176 a closed combinatorial formula which allows to calculate the Nekrasov partition function for generic ϵ_1, ϵ_2 was found. The partition function is represented as a sum over arrays of Young diagrams with total number of boxes equal to the number of instantons.
- The partition function with generic ϵ_1, ϵ_2 is essential from AGT duality point of view relating partition function to the conformal blocks in 2d Conformal Field Theory Alday,Gaiotto,Tachikawa ' arXiv:0906.3219 . ϵ_1, ϵ_2 parametrize the Virasoro central charge.

Partition function with generic ϵ_1 and $\epsilon_2 = 0$

- In a parallel very interesting development Nekrasov and Shatashvily in 'arXiv:0908.4052 show that when $\epsilon_2 = 0$ the prepotential is related to the quantum integrable many body systems.
  In this case we are lead to the notion of "quantum" Seiberg-Witten curve R.P. 'arXiv:1006.4822.
- Note one more point which to my opinion makes the investigation of $\epsilon_2 = 0$ case even more interesting: namely, due to above mentioned AGT we get relation to the quasi-classical ($c \to \infty$) limit of conformal blocks, hence to the semiclassical Liouville (or Toda, if rank is greater than 1) field theory.
- There is a link DSW $\to$ ODE. The latter coincides with the one appearing in ODE/IM correspondence for the Liouville with $c = 25$ ($SU(2)$ case), A_2 -Toda with $c = 98$ ($SU(2)$ case),...

Linear quiver theory and CFT conformal blocks

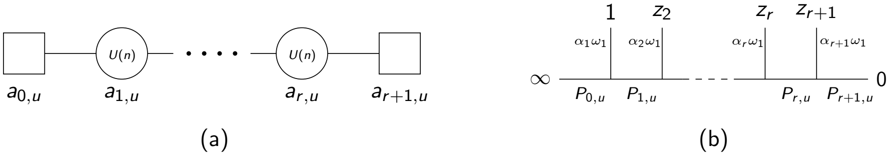


Figure: (a) The quiver diagram for the conformal linear quiver $U(n)$ gauge theory: r circles stand for gauge multiplets; two squares represent n anti-fundamental (on the left edge) and n fundamental (the right edge) hypermultiplets; the lines connecting adjacent circles are the bi-fundamentals. (b) The AGT dual conformal block of the Toda field theory.

Relation between couplings q and insertion points z

$$q_\alpha = \frac{z_{\alpha+1}}{z_\alpha}\,.$$

The masses of fundamental and anti-fundamental specify initial and final states:

$$p_{0u} = a_{r+1,u} - \bar{a}_r \quad \text{and} \quad p_{r+1,u} = a_{0,u} - \bar{a}_1$$

the "center of mass" quantities:

$$\bar{a}_i = \frac{1}{n}\sum_{u=1}^{n} a_{i,u}$$

the parameters of vertical legs are:

$$\alpha_i = \bar{a}_{i+1} - \bar{a}_i\,.$$

Toda central charge $c = (n-1)(1+n(n+1)Q^2)$ where $Q = b + 1/b$. Dimensions of primaries

$$h_{\vec{p}} = \frac{(n^3-n)Q^2 - 2\vec{p}^{\,2}}{4}\,.$$

special fields $V_{\lambda\omega_1}$:

$$h_{\lambda\omega_1} = \frac{\lambda(n-1)}{2}\left(q - \frac{\lambda}{n}\right)\,.$$

Partition function for pure $SU(N)$ theory

Consider pure $SU(N)$ theory without hypers in Ω-background. The instanton part of partition function is given by [Nekrasov: arXiv:hep-th/0206161]

$$Z_{inst}(\vec{a}, \epsilon_1, \epsilon_2, q) = \sum_{\vec{Y}} Z_{\vec{Y}} q^{|\vec{Y}|} ,$$

where sum runs over all N-tuples of Young diagrams $\vec{Y} = (Y_1, \cdots, Y_N)$, $|\vec{Y}|$ is the total number all boxes, $\vec{a} = (a_1, a_2, \cdots, a_N)$ are VEV's of adjoint scalar from $\mathcal{N} = 2$ vector multiplet, ϵ_1, ϵ_2, as already mentioned, parametrize the Ω-background and the instanton counting parameter $q = \exp 2\pi i \tau$, $\tau = \frac{i}{g^2} + \frac{\theta}{2\pi}$ is the (complexified) coupling constant. The coefficients $Z_{\vec{Y}}$ are factorized as

$$Z_{\vec{Y}} = \prod_{i,j=1}^{N} \frac{1}{P(Y_i, a_i | Y_j, a_j)} ,$$

where the factor $P(\lambda, a, \mu, b)$ for arbitrary pair of Young diagrams λ, μ and associated VEV parameters a, b explicitly are given by the formula [FP: hep-th/0208176]

$$P(\lambda, a|\mu, b) = \prod_{s\in\lambda}(a - b + \epsilon_1(1 + L_\mu(s)) - \epsilon_2 A_\lambda(s)) \prod_{s\in\mu}(a - b - \epsilon_1 L_\lambda(s) + (1 + \epsilon_2 A_\lambda(s)))$$

If one specifies location of a box s by its horizontal and vertical coordinates (i, j), so that $(1, 1)$ corresponds to the corner box, its leg length $L_\lambda(s)$ and arm length $A_\lambda(s)$ with respect to the diagram λ (s does not necessarily belong to λ) are defined as

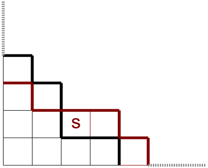
$$A_\lambda(s) = \lambda_i - j; \qquad L_\lambda(s) = \lambda'_j - i\,,$$

where λ_i (λ'_j) is i-th column (j-th row) of diagram λ with convention that when i exceeds the number of columns (j exceeds the number of rows) of λ, one simply sets $\lambda_i = 0$ ($\lambda'_j = 0$).

Demonstration: Arm and Leg lengths

Let λ be the black diagram and μ the red one, (the box $s \in \mu$)

$A_\lambda(s) = -1$
$A_\mu(s) = 0$
$L_\lambda(s) = -1$
$L_\mu(s) = 1$



Bethe ansatz equation for NS limit

- Though the total number of boxes $\to \infty$, in $\epsilon_2 \to 0$ limit the rescaled column lengths $\epsilon_2 Y_{u,i}^{(cr)}$, converge to finite values

$$\xi_{u,i} = \lim_{\epsilon_2 \to 0} \epsilon_2 Y_{u,i}^{(cr)}$$

- The rescaled column lengths at small q behave as $\xi_{u,i} \sim O(q^i)$
- Up to arbitrary order $\sim O(q^{L+1})$ the quantities

$$x_{u,i} = a_u + \epsilon_1 (i-1) + \xi_{u,i}$$

satisfy the Bethe-ansatz equations (for each $u = 1, 2, \cdots N$)

$$-q \prod_{v,j}^{N,L} \frac{(x_{u,i} - x_{v,j} - \epsilon_1)(x_{u,i} - x_{v,j}^0 + \epsilon_1)}{(x_{u,i} - x_{v,j} + \epsilon_1)(x_{u,i} - x_{v,j}^0 - \epsilon_1)} = \prod_{v=1}^{N} (x_{u,i} - a_v + \epsilon_1)(x_{u,i} - a_v)\,,$$

where, by definition

$$x_{u,i}^0 = a_u + \epsilon_1 (i-1)$$

Baxter's difference equation and Deformed SW "curve"

The BA equations can be transformed into a difference equation

$$Y(z+\epsilon_1)+\frac{q}{\epsilon_1^{2N}}Y(z-\epsilon_1)=\epsilon_1^{-N}P_N(z+\epsilon_1)\,Y(z)\,,$$

where $Y(z)$ is an entire functions with zeros located at $z=x_{u,i}$:

$$Y(z)=\prod_{u=1}^{N}e^{\frac{z}{\epsilon_1}\psi(\frac{a_u}{\epsilon_1})}\prod_{i=1}^{\infty}\left(1-\frac{z}{x_{u,i}}\right)e^{z/x_{u,i}^0}\,,$$

and

$$\psi(x)=\frac{d}{dx}\log\Gamma(x)$$

is the logarithmic derivative of Gauss' gamma-function. Finally $P_N(z)$ is an N-th order polynomial which parametrizes the Coulomb branch of the theory. Explicit expressions of coefficients of this polynomial in terms of VEV's

$$u_J\equiv\langle \mathbf{t}r\phi^J\rangle$$

will be presented later for de case of our current interest $N=3$. For more general cases one can see e.g. [RP-1601].

The difference equation is related to the SW "curve".
Introducing function

$$y(z) = \epsilon_1^N \ \frac{Y(z)}{Y(z-\epsilon_1)}$$

we get

$$y(z) + \frac{q}{y(z-\epsilon_1)} = P_N(z)$$

At large z the function $y(z)$ behaves as

$$y(z) = z^N(1 + O(1/z)) .$$

Notice that setting $\epsilon_1 = 0$ one obtains an equation of hyperelliptic curve, which is just the Seiberg-Witten curve. When $\epsilon_1 \neq 0$, everything goes surprisingly similar to original Seiberg-Witten theory. For example the role of Seiberg-Witten differential plays the quantity

$$\lambda_{SW} = z \frac{d}{dz} \log y(z)$$

and, as in undeformed theory the expectation values are given by the contour integral

$$\langle \mathbf{tr}\, \phi^J \rangle = \oint_{\mathcal{C}} \frac{dz}{2\pi i} z^J \partial_z \log y(z)$$

where $\mathcal{C}$ is a large contour, enclosing all zeros and poles of $y(z)$.

Details on $SU(3)$ theory

Without any essential loss of generality, from now on we'll assume that

$$u_1 \equiv \langle \mathbf{tr}\, \phi \rangle = a_1 + a_2 + a_3 = 0$$

Representing $y(z)$ as a power series in $1/z$

$$y(z) = z^3(1 + c_1 z^{-1} + c_2 z^{-2} + c_3 z^{-3} + \cdots)$$

calculating the contour integral one easily finds the relations

$$c_1 = 0; \qquad c_2 = -\frac{u_2}{2}; \qquad c_3 = -\frac{u_3}{3}$$

Now, the difference equation immediately specifies the polynomial $P_3(z)$ (we omit the subscript 3, since only the case $N = 3$ will be considered later on)

$$P(z) = z^3 - \frac{u_2}{2}\, z - \frac{u_3}{3}$$

The differential equation

To keep expressions simple, from now on we will set $\epsilon_1 = 1$. In fact, at any stage the ϵ_1 dependence can be easily restored on the dimensional grounds. Taking the results of previous subsection, the difference equation for $N = 3$ case can be rewritten as

$$Y(z) - \left(z^3 - \frac{u_2}{2} z - \frac{u_3}{3}\right) Y(z-1) + q\, Y(z-2) = 0\,,$$

Now by means of an inverse Fourier transform, starting from above difference equation we'll derive a third order linear differential equation for function

$$f(x) = \sum_{z \in a+\mathbb{Z}} e^{x(z+1)} Y(z)$$

At least when $|q|$ is sufficiently small, it is expected that the series is convergent for finite x, provided a takes one of the three possible values a_1, a_2 or a_3. Taking into account the difference relation one can easily check that $f(x)$ solves the differential equation

$$-f'''(x) + \frac{u_2}{2} f'(x) + \left(e^{-x} + q\, e^{x} + \frac{u_3}{3} \right) f(x) = 0\,.$$

Denoting

$$q = \Lambda^6$$

and shifting the variable

$$x \to x - \log \Lambda^3$$

the differential equation may be cast into a more symmetric form

$$-f'''(x) + \frac{u_2}{2} f'(x) + \left(\Lambda^3 (e^{x} + e^{-x}) + \frac{u_3}{3} \right) f(x) = 0\,.$$

A-cycles from differential equation

Consider the basis of solutions $u_1(x)$, $u_2(x)$, $u_3(x)$ with standard initial conditions $(n, k = 1, 2, 3)$

$$u_n^{(k-1)}(x)\Big|_{x=0} = \delta_{n,k}$$

Since the functions $u_n(x + 2\pi i)$ are solutions too, we can define the monodromy matrix M as

$$u_n(x + 2\pi) = \sum_{k=1}^{3} u_k(x) M_{k,n}$$

Evidently

$$M_{k,n} = u_n^{(k-1)}(2\pi i)$$

For any fixed values of parameters Λ, p_n it is easy to integrate numerically the diff. eq. with above boundary conditions and find the matrix M and then its eigenvalues $\exp 2\pi i a_n$. Taking into account generalized Matone relation [FFMP-0403], this opens up a nonperturbative access to deformed prepotential.

Solutions at $x \to \pm\infty$ and Q function

It is convenient to introduce parameters $p_1,\ p_2,\ p_3$ satisfying $p_1 + p_3 + p_3 = 0$ such that

$$u_2 = p_1^2 + p_2^2 + p_3^2 = 2(p_1^2 + p_2^2 + p_1 p_2); \quad u_3 = p_1^3 + p_2^3 + p_3^3 = -3p_1 p_2(p_1 + p_2)$$

In the $\Lambda \to 0$ limit the parameters p_n and a_i coincide.
At large positive values $x \gg 0$ the term e^{-x} in diff. eq. can be neglected. In this region the differential equation can be solved in terms of hypergeometric function ${}_0F_2(a, b; z)$ defined by the power series

$${}_0F_2(a, b; z) = \sum_{k=0}^{\infty} \frac{z^k}{(a)_k (b)_k k!} \,,$$

where

$$(x)_k = x(x+1) \cdots (x+k-1)$$

is the Pochhammer symbol.

The three linearly independent solutions can be chosen as

$$U_n(x) \approx e^{(x+3\theta)p_n} \; {}_0F_2(1 + p_n - p_j), 1 + p_n - p_k; e^{x+3\theta}) \; ,$$

where by definition $\exp\theta = \Lambda$ and the indices (n, k, l) are cyclic permutations of $(1, 2, 3)$. We used the symbol $\approx$ to emphasize that the solutions are valid only asymptotically at $x \gg 3\theta$.
Wronskian of these tree functions

$$\begin{vmatrix} U_1(x) & U_2(x) & U_3(x) \\ U_1'(x) & U_2'(x) & U_3'(x) \\ U_1''(x) & U_2''(x) & U_3''(x) \end{vmatrix} = (p_1 - p_2)(p_2 - p_3)(p_3 - p_1)$$

is nonzero provided the parameters p_n are pairwise different. This confirms that $U_n(x)$ are independent and constitute a basis in the space of all solutions.

Similarly in region $x \ll -3\theta$ the term $\Lambda^3 e^x$ of diff. eq. becomes negligible and one gets solutions

$$V_n(x) \approx e^{(x-3\theta)p_n} \, {}_0F_2(1-p_n+p_j), 1-p_n+p_k; -e^{-x+3\theta}) \ ,$$

For the Wronskian we get the same answer

$$\begin{vmatrix} V_1(x) & V_2(x) & V_3(x) \\ V_1'(x) & V_2'(x) & V_3'(x) \\ V_1''(x) & V_2''(x) & V_3''(x) \end{vmatrix} = (p_1-p_2)(p_2-p_3)(p_3-p_1)$$

All three solutions are increasing at $x \to -\infty$, but there is a unique (apart from a trivial rescaling) decreasing combination:

$$\chi(x) = \sum_{n=1}^{3} \frac{\Gamma(p_{nj})\Gamma(p_{nk})}{4\pi^2} V_n(x)$$

In terms of $v = \exp(-x+3\theta)$,

$$\chi(v) \approx \frac{v^{-1/3} e^{-3v^{1/3}}}{2\pi\sqrt{3}}$$

Since $U_n(x)$ are complete one can expand $\chi(x)$ as

$$\chi(x,\theta) = \sum_{n=1}^{3} Q_n(\theta)\Gamma(p_{nj})\Gamma(p_{nk})e^{-3p_n\theta}U_n(x,\theta)$$

An important property: Wronskian of two solutions solves the "dual" diff. eq, i.e. the one obtained by reversing the signs $p_n \to -p_n$ and $\Lambda^3 \to -\Lambda^3$. Exploring this property we get the relation

$$Wr\left[\chi(x,\theta+\frac{i\pi}{3}),\chi(x,\theta-\frac{i\pi}{3})\right] = -\frac{i}{2\pi}\bar{\chi}(x,\theta)$$

where $\bar{\chi}(\theta) = \chi(\theta,-\mathbf{p})$. This is equivalent to the functional relations

$$\frac{\sin(\pi p_{jk})}{2i\pi^2}\bar{Q}_n(\theta) = Q_j\left(\theta+\frac{i\pi}{3}\right)Q_k\left(\theta-\frac{i\pi}{3}\right) - Q_j\left(\theta-\frac{i\pi}{3}\right)Q_k\left(\theta+\frac{i\pi}{3}\right)$$

$SU(3)$ version of T-Q relation

Using above functional relations it is easy to get the $SU(3)$ analog off Baxter's $T-Q$ equations (for any pair $j \neq k$):

$$T(\theta)Q_j(\theta-\frac{\pi i}{6})\bar{Q}_k(\theta+\frac{\pi i}{6}) =$$
$$Q_j(\theta-\frac{5\pi i}{6})\bar{Q}_k(\theta+\frac{\pi i}{6}) + Q_j(\theta+\frac{\pi i}{2})\bar{Q}_k(\theta-\frac{\pi i}{2}) + Q_j(\theta-\frac{\pi i}{6})\bar{Q}_k(\theta+\frac{5\pi i}{6})$$

The functions $T(\theta)$, $Q(\theta)$ are entire. These functional relations emerge in ODE/IM context when a 2d CFT with extra spin 3 current (W_3 symmetry)

[Dorey,Tateo: hep-th/9910102], [Bazhanov,Hibberd,Khoroshkin: hep-th/0105177].

It should be possible to derive corresponding TBA equations.

Conjecture (the $SU(3)$ analog of Al.Zamolodchikov's conjecture for Mathieu):

$$T(\theta) = \sum_{n=1}^{3} e^{2\pi i a_n}$$

Numerical comparisons with instanton calculus

Instanton counting up to $q^3 = \Lambda^{18}$ gives

$$\begin{aligned}\langle \mathbf{tr}\,\phi^2 \rangle &= a_1^2 + a_2^2 + a_3^2 + \frac{6(a_1^2 + a_2^2 + a_3^2 - 2)q}{(a_{12}^2 - 1)(a_{13}^2 - 1)(a_{23}^2 - 1)} \\ &+ \frac{P_{14}(a_1, a_2, a_3)q^2}{(a_{12}^2 - 4)(a_{13}^2 - 4)(a_{23}^2 - 4)(a_{12}^2 - 1)^3(a_{13}^2 - 1)^3(a_{23}^2 - 1)^3} + O(q)^4\end{aligned}$$

We made numerical integrations of diff. eq. for the choice of parameters $(p_1, p_2, p_3) = (0.12, 0.17, -0.29)$ and $\Lambda = 0.07$ and calculated

- The eigenvalues of monodromy matrix

$$\begin{aligned}&e^{2\pi i a_n} = \{0.48179 + 0.876287i, 0.728948 + 0.684569i, -0.24868 - 0.968586i\} \\ &a_n = \{0.169993, 0.120005, -0.289998\}\end{aligned}$$

  Inserting a_n in above formula ve get $\langle \mathbf{tr}\,\phi^2 \rangle = 0.127396$, while $p_1^2 + p_2^2 + p_3^2 = 0.1274$
- We calculated the numerical values of Q_n, $\bar{Q}_n$, and checked the validity of functional relations
- from T-Q relation we get: $T(\theta) = 0.9621 + 0.5925i$ while $\sum_{n=1}^{3} e^{2\pi i a_n} = 0.96205 + 0.59227i$

Conclusions

- ODE/IM and TBA can be efficiently applied for $\mathcal{N}=2$ SYM in NS limit of Ω-background (at least in $\mathcal{N}_f=0$) case, which allows to sum up entire instanton series.
- Extension of this method for the cases $\mathcal{N}_f \neq 0$ will lead to better, nonperturbative understanding of semiclassical conformal blocks.
- Alternatively, application of DSW method could lead to new insights in understanding of integrable structure of CFT
- It would be interesting to see if it is possible to generalize this method for generic Ω-background
- Purely mathematical applications: theory of differential equations, special functions ...

THANKS