“Supersymmetries and Quantum Symmetries - SQS’19”
Yerevan, 26-31 August 2019

Analytic expression for the octagon form factor in $\mathcal{N} = 4$ SYM theory

(The Octagon as a Determinant)

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based on work with Ivan Kostov and Didina Serban,

arXiv:1903.05038 [PRL **122** (2019) 23601]; arXiv:1905.11467

- **Octagon** - building block for the evaluation of a class of 4-point functions of single trace 1/2 BPS operators in $\mathcal{N} = 4$ planar SYM 4d theory - in the integrability (effectively 2d) approach

$$\mathcal{O}^{(K)}(x;y) = \mathrm{Tr}[(y \cdot \Phi(x))^K], \quad \Phi_i, \; i = 1,2,\ldots 6, \; y^2 = 0$$

$$\lambda_{su(4)} = (0, K, 0)$$

2- and 3-point functions **protected** $\quad \triangle = K$

$$< \mathcal{O}^{(K)}(x_1;y_1)\mathcal{O}^{(K)}(x_2;y_2) > \;\sim\; \left(\frac{y_1 \cdot y_2}{x_{12}^2}\right)^K$$

OPE - **nontrivial,** contains also fields with anomalous dims

heavy charges $K_j \gg 1$ and *special polarisations*

Example:

F. Coronado, arXiv:1811.xxxxx

$$\langle \mathcal{O}_1^{(K)}\mathcal{O}_2^{(K)}\mathcal{O}_3^{(K)}\mathcal{O}_4^{(K)} \rangle \underset{K\to\infty}{=} \frac{\mathbb{O}_0(z,\bar{z})^2}{(x_{12}^2 x_{34}^2 x_{13}^2 x_{24}^2)^{K/2}}$$

$$(y_1 \cdot y_4) = 0 = (y_2 \cdot y_3)$$

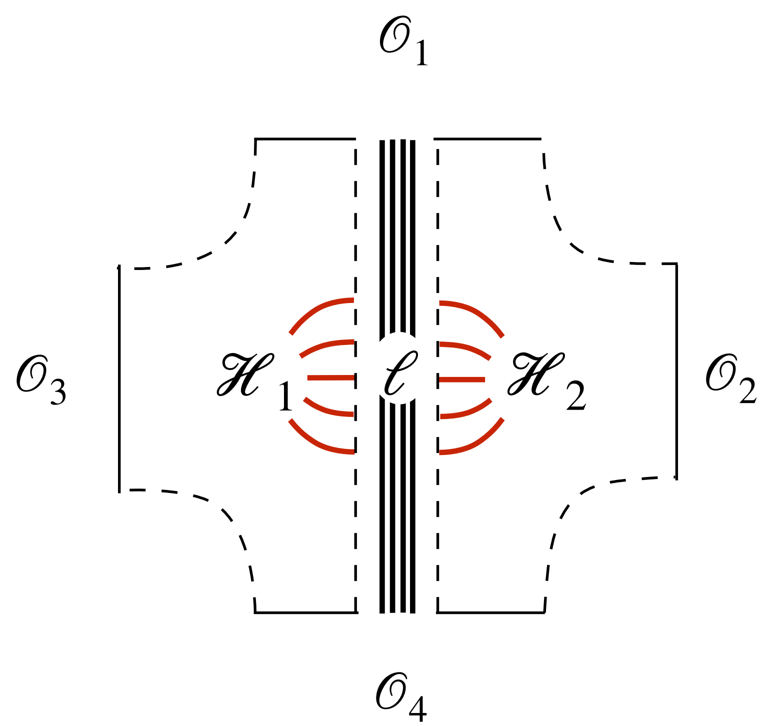
more generally, sum of products

- **Hexagonalization:**
  using integrability inspired technique - geometric decomposition of n-point correlation functions into **hexagon** form factors - used to describe 3-p. functions

Basso, Komatsu, Vieira, 2015

Eden&Sfondrini 2016, Fleury&Komatsu, 2016,2017

$\mathbb{O}_l =$



octagon = two hexagons glued together
contribution of virtual (“mirror”) particles
infinite sum of multiple integrals I_n

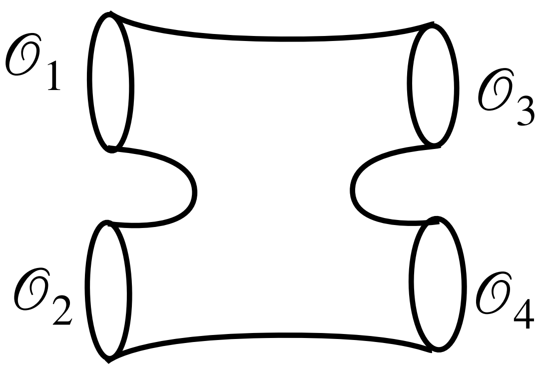
F. Coronado
based on Fleury&Komatsu

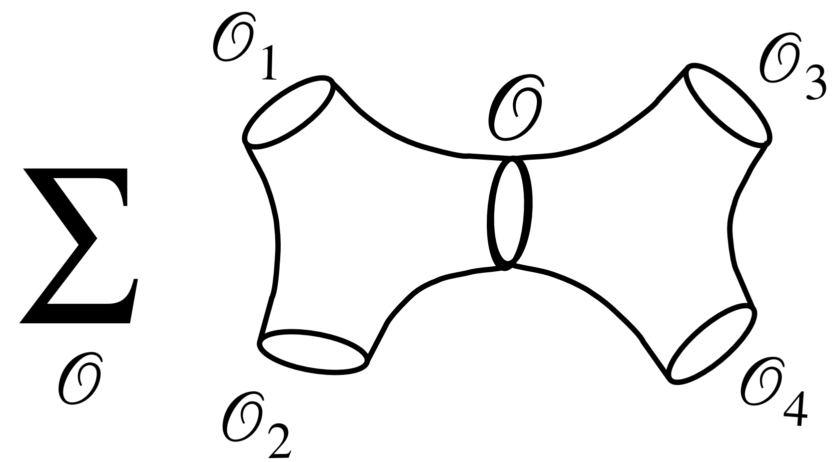
perturbative, weak ’t Hooft coupling $g \to 0$
expansion of the first few ; conjecture:
multilinear combinations of ladder functions

- **Problem :** find a nonperturbative formula for I_n

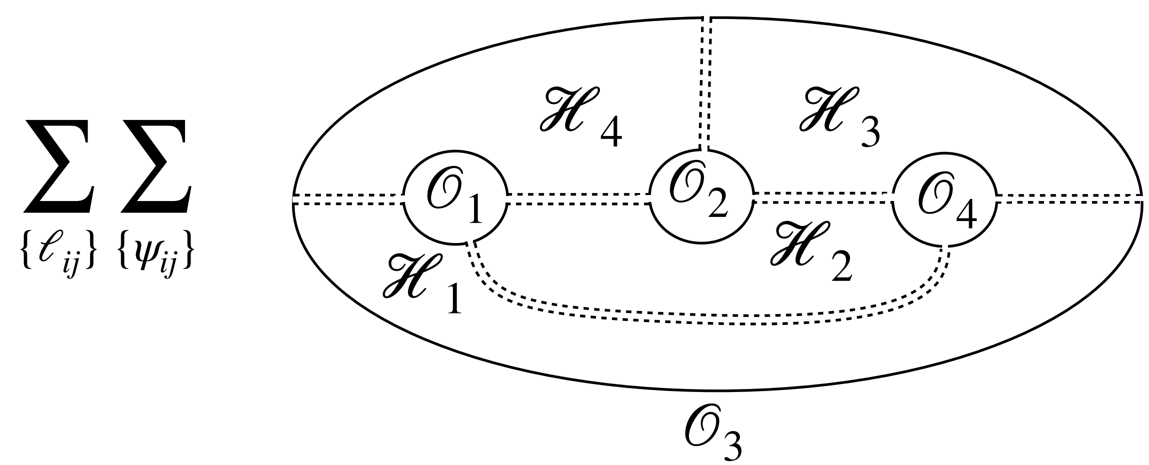
4p function

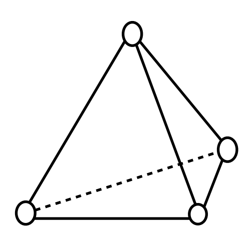
sum over planar graphs





The traditional OPE approach for computing the 4p function:
sum over all intermediate physical states





The new approach: hexagonalization (triangulation); sum over mirror states ψ_{ij}
ℓ_{ij} bridge lengths (Wick contractions)

Eden&Sfondrini 2016, Fleury&Komatsu 2016,2017

precise prescription - F&K

Idea -to simplify - **suppress some of the mirror channels**

The "simplest" four-point function [F. Coronado, arXiv:1811.00467]

- 4 heavy half-BPS (protected by supersymmetry) operators:

$$\mathcal{O}_i = \mathrm{tr}[(y_i \cdot \Phi(x_i))^K] \qquad i = 1,2,3,4; \qquad K \to \infty$$

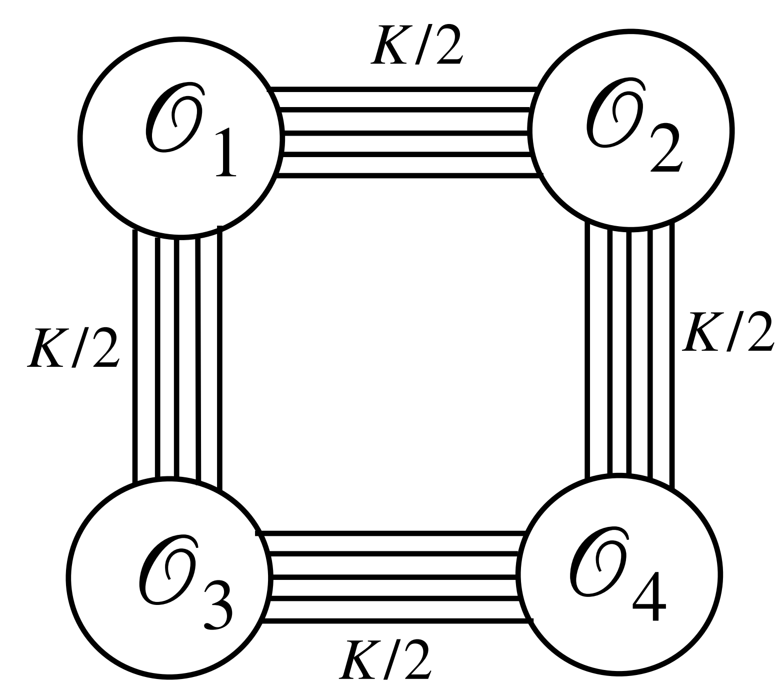
- special choice for the polarisations: $(y_1 \cdot y_4) = (y_2 \cdot y_3) = 0$

For example:

$$\mathcal{O}_1(0) = \mathrm{tr}(Z^{\frac{K}{2}}\bar{X}^{\frac{K}{2}}) + \text{permutations} \qquad \mathcal{O}_2(z,\bar{z}) = \mathrm{tr}(X^K)$$

$$\mathcal{O}_3(1) = \mathrm{tr}(\bar{Z}^K) \qquad \mathcal{O}_4(\infty) = \mathrm{tr}(Z^{\frac{K}{2}}\bar{X}^{\frac{K}{2}}) + \text{permutations}$$

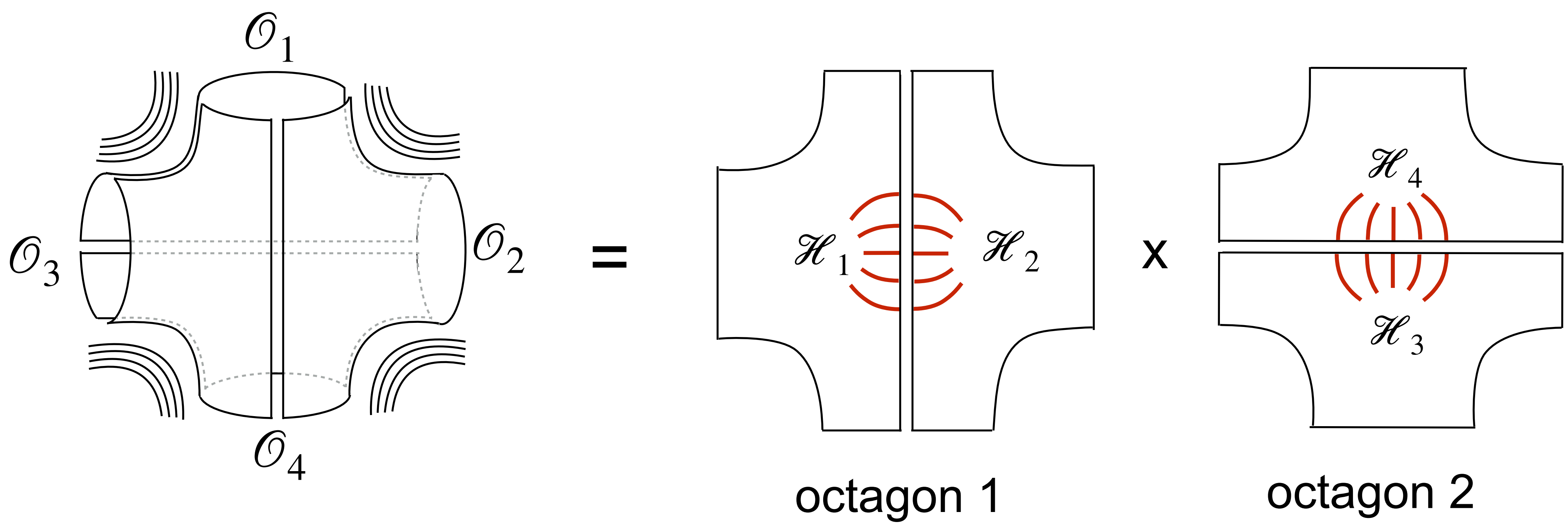
- A single tree-level planar Feynman diagram:
4 large "bridges" of K/2 propagators



$$\overset{1}{}\!\!\text{———}\!\!\overset{2}{} = \frac{(y_1 \cdot y_2)}{(x_1 - x_2)^2} = d_{12}$$

- for large K the bridges act as infinite potential walls for the mirror particles and the 4point functions factorizes into two **octagons**

damping factors $e^{-\tilde{E}\,\frac{K}{2}}$

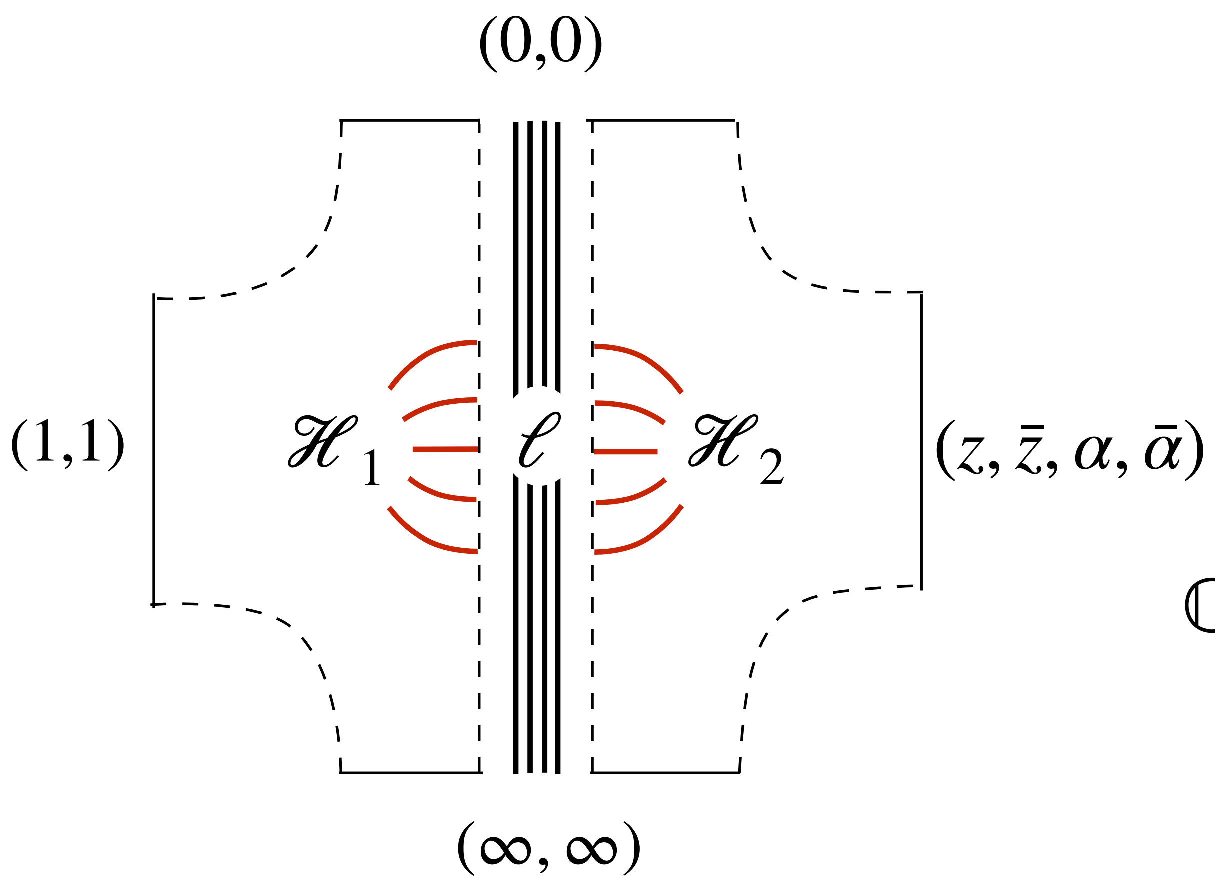


$psu(2|2)^{\text{mirror}} \times psu(2|2)^{\text{mirror}}$

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle \underset{K\to\infty}{=} \frac{\mathbb{O}_0(z,\bar{z},1,1)^2}{(x_{12}^2 x_{34}^2 x_{13}^2 x_{24}^2)^{K/2}}$$

Correlators - with small ℓ
bridges on two of the mirrors seams
- sums of products of octagons

The octagon form factor



More general correlators - small ℓ bridges on two of the mirrors seams - sums of products of octagons

$$\mathbb{O}_\ell(z,\bar{z},\alpha,\bar{\alpha}) = \sum_\psi \langle \mathcal{H}_2 | \psi \rangle \, e^{-\tilde{E}_\psi \ell} \, \langle \psi | \mathcal{H}_1 \rangle$$

- ℓ = the length of the 'bridge' between the two hexagons
- $z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = u \qquad (1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = v$
- $\alpha\bar{\alpha} = \frac{(y_1 \cdot y_2)(y_3 \cdot y_4)}{(y_1 \cdot y_3)(y_2 \cdot y_4)}, \qquad (1-\alpha)(1-\bar{\alpha}) = \frac{(y_1 \cdot y_4)(y_2 \cdot y_3)}{(y_1 \cdot y_3)(y_2 \cdot y_4)}$

$$z = e^{-\xi + i\phi}$$
$$\bar{z} = e^{-\xi - i\phi}$$
$$\alpha = e^{\varphi - \xi + i\theta}$$
$$\bar{\alpha} = e^{\varphi - \xi - i\theta}$$

The series expansion for the octagon

$$\mathbb{O}_\ell = = \sum_{n=0}^{\infty} \frac{(\lambda^+)^n + (\lambda^-)^n}{2} e^{-n\xi}\, \mathrm{I}_{n,\ell}(z,\bar{z})$$

$$\lambda^\pm = 2\cos\phi - 2\cosh(\varphi \pm i\theta)$$

character of fund. rep. su(2|2)

Finite coupling g representation as an infinite sum of multiple integrals

Fleury&Komatsu 2016, 2017
Coronado, arXiv:1811.00467]

Notation

energy and momentum $\tilde{E}_a(u),\, \tilde{p}_a(u)$ of mirror particles and their bound states are parametrised by the integers a=1,2,… and the Zhukovsky variable

$$\frac{u}{g} = x + \frac{1}{x}\,, \qquad x(u) = \frac{u + \sqrt{u^2 - 4g^2}}{2g}$$

$$\tilde{E}_a(u) = \left(\mathbb{D}^a + \mathbb{D}^{-a}\right)\log x\,, \quad i\,\tilde{p}_a(u) = \frac{i}{2} g\left(\mathbb{D}^a + \mathbb{D}^{-a}\right)(x - \frac{1}{x})$$

$\mathbb{D} = e^{\frac{i}{2}\partial_u}$ shift operator

$\mathbb{D}^{\pm a} f(u) = f(u \pm ia/2) =: f^{[\pm a]}(u)$

$$\mathbb{O}_\ell = = \sum_{n=0}^{\infty} \frac{(\lambda^+)^n + (\lambda^-)^n}{2} e^{-n\xi}\, \mathrm{I}_{n,\ell}(z,\bar{z})$$

$$\boxed{\mathrm{I}_{n,\ell}(z,\bar{z}) = \frac{1}{n!} \sum_{a_1,\ldots,a_n \geq 1} \prod_{j=1}^{n} \frac{\sin a_j \phi}{\sin \phi} \int_{\mathbb{R}} \prod_{j=1}^{n} \frac{du_j}{2\pi i} \quad \hat{\mu}_{a_j}(u_j, \ell, z, \bar{z}) \quad \prod_{j<k}^{n} \tilde{H}_{a_j,a_k}(u_j, u_k)}$$

$$\hat{\mu}_a(u) = \frac{x^{[+a]} - x^{[-a]}}{x^{[+a]}x^{[-a]} - 1}\, \Omega_\ell(u + ia/2)\ \Omega_\ell(u - ia/2)$$

$$\tilde{H}_{ab}(u,v) = \prod_{\varepsilon,\delta=\pm} \frac{x^{[\varepsilon a]} - y^{[\delta b]}}{x^{[\varepsilon a]}\ y^{[\delta b]} - 1}$$

$$x^{[\epsilon a]} = x(u + i\epsilon a/2)$$

$$K(x,y) = \frac{x - y}{x\,y - 1}$$

$$\Omega_\ell(u) \equiv \frac{e^{ig\xi\,[x(u) - 1/x(u)]}}{x(u) - 1/x(u)}\ \frac{x(u)^{-\ell-1}}{\sqrt{g}}$$

How to compute the multiple integrals?

- **Weak coupling** expansion of the multiple integrals : disentangles the n integrations and each of the integrals can be computed by residues;
  - the first few up to n=4 up to ? loops

Coronado

- \- if one restricts in $\mathrm{I}_{n,\ell}$ to the leading contribution of the expansion - it is of order $g^{2n(n+l)}$ - coincides with the integral encountered and computed in the study of the ***fishnet graphs***

Basso&Dixon 2017

- The full expansion - too complicated - we need to insert the weak coupling expansion of the Zhukovsky variables $x^{[\pm]}(u)$ in each factor in the integrand, even the parts in the measure which are factorized into “chiral” factors

What is not factorized are only the anti-symmetric kernels
\- both as parts of the measure and in $\tilde{H}_{ab}$

$$K(x,y) = \frac{x-y}{x\,y-1} = < 0\,|\,\psi(x)\psi(y)\,|\,0 >$$

fermion 2-point function

Aim - reduce all order computation to a **single integral** for **finite coupling g**

The **strategy** : two steps

1. compute the logarithms of the two pieces of the octagon

2. use a different expansion - affects only the propagators K(x,y)

2. could be applied directly to each of the Coronado integrals - but messy, hence step 1.

— compare with the data from weak coupling expansion

Derivation: - generalization of the "Pfaffian integration formula"

Borodin&Kanzieper arXiv:0707.2784

- CFT interpretation , Coulomb gas on the Zhukovsky plane

Pfaffians

take $a_j = 1$

Basso&Coronado&Komatsu
&Lam&Vieira&Zhong 2017

n=1

from the measure

$$\frac{x^+(u) - x^-(u)}{x^+(u)x^-(u) - 1} = K(x^+, x^-) = <0|\psi(x^+)\psi(x^-)|0> = \mathrm{Pf}(\boldsymbol{K}_1(u))$$

$$\boldsymbol{K}_1(u) = \begin{pmatrix} 0 & \frac{x^+ - x^-}{x^+x^- - 1} \\ \frac{x^- - x^+}{x^+x^- - 1} & 0 \end{pmatrix}$$

pfaffian of anti-symmetric 2 x 2 matrix

n=2

$$K(x^+, x^-)\, K(y^+, y^-)\, \tilde{H}_{11}(u_1, u_2) = <0|\psi(x^+)\,\psi(x^-)\,\psi(y^+)\,\psi(y^-)|0>$$

by Wick - 3 terms

$$= \mathrm{Pf}\left[\boldsymbol{K}_2(u_1, u_2)\right]$$

4 x 4 anti-symmetric matrix

2 x 2 matrix blocks

$$\boxed{K(u_1; u_2) = \begin{pmatrix} K(x^+, y^+) & K(x^+, y^-) \\ K(x^-, y^+) & K(x^-, y^-) \end{pmatrix}}$$

$$K(u; u) = K_1(u)$$

2n-point free fermion correlator

$$< 0\,|\,\psi(x^+(u_1))\,\psi(x^-(u_1))\ldots,\psi(x^+(u_n))\,\psi(x^-(u_n))\,|\,0 > = \prod_{j=1}^{n} K(x^+(u_j), x^-(u_j)) \prod_{j<k}^{n} \tilde{H}_{11}(u_j, u_k)$$

"bosonization"

$$\prod_{j<k}^{2n} \frac{x_j - x_k}{x_j x_k - 1} = \mathrm{Pf}\Big(\Big[\frac{x_j - x_k}{x_j x_k - 1} \Big]_{i,j=1}^{2n} \Big)$$

$$= \mathrm{Pf}\,\big[\boldsymbol{K}_n(u_1,\ldots,u_n)\big] = \mathrm{Pf}\Big(\big[\boldsymbol{K}(u_j;u_k)\big]_{j,k=1}^{n}\Big)$$

$$x_{2j+1} = x^+(u_j)\,,\, x_{2j+2} = x^-(u_j)\,,\, j = 0,\ldots,n-1$$

Restoring the dependence on the label a of the bound states we have a **2 x 2 matrix kernel** $\boldsymbol{K}(u_1, a_1; u_2, a_2)$ defined on $(\mathbb{R} \times \mathbb{N})^{\times 2}$ with matrix elements

$$K^{\varepsilon_1,\varepsilon_2}(u_1, a_1; u_2, a_2) = K(u_1^{[\varepsilon_1 a_1]}, u_2^{[\varepsilon_2 a_2]}), \quad \varepsilon_{1,2} = \pm$$

The octagon as a Fredholm pfaffian

$$\mathbb{O}_\ell = \frac{1}{2}\sum_{\pm}\sum_{n=0}^{\infty}\frac{(\lambda^{\pm})^n}{n!}\sum_{a_1,\ldots,a_n\geq 1}\int_{\mathbb{R}}\prod_{j=1}^{n} d\mu(u_j,a_j)\ \mathrm{Pf}\left[\boldsymbol{K}_n(u_1,a_1;\ldots;u_n,a_n)\right]$$

$$\boldsymbol{K}_n(u_1,a_1;\ldots;u_n,a_n) = \Big[\boldsymbol{K}(u_j,a_j;u_k,a_k\Big]_{1\geq j,k\geq n}$$

$$d\mu(u,a) = \frac{\sin a\phi}{\sin\phi}\frac{du}{2\pi i}\ \Omega_\ell(u+ia/2)\ \Omega_\ell(u-ia/2)$$

2 x 2 matrix kernel

$$\mathbb{O}_\ell(z,\bar{z},\alpha,\bar{\alpha}) = \frac{1}{2}\sum_{\pm}\ \mathrm{Pf}\Big(\boldsymbol{J}+\lambda^{\pm}\ \boldsymbol{K}\Big) = \tfrac{1}{2}\textstyle\sum_{\pm}\ \sqrt{\mathrm{Det}(\,I-\lambda^{\pm}\boldsymbol{J}\boldsymbol{K})}$$

noted by
Basso&Coronado&Komatsu
&Lam&Vieira&Zhong 2017

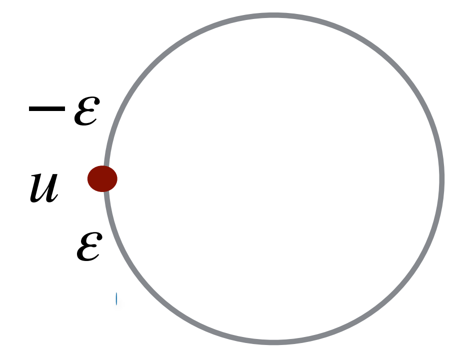
$$\boldsymbol{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\ \delta(u,v)$$

the expansion of the octagon is a sum of
two **Fredholm pfaffians,**
square roots of a **Fredholm determinant**

Why is this representation of the octagon useful?

$$\mathbb{O}_\ell(z,\bar{z},\alpha,\bar{\alpha}) = \frac{1}{2}\sum_{\pm} e^{\frac{1}{2}\mathrm{Tr}\log(I-\lambda_\pm JK)} = \frac{1}{2}\sum_{\pm} e^{\frac{1}{2}\sum_{n=1}^{\infty}\cdots}$$

n=1 $$-\frac{1}{2}\lambda_\pm \sum_{a\geq 1}\sum_{\varepsilon=\pm}\frac{\sin a\phi}{\sin\phi}\int_{\mathbb{R}}\frac{du}{2\pi i}\;\varepsilon\,\hat{K}(u^{[-\varepsilon a]},u^{[\varepsilon a]})$$

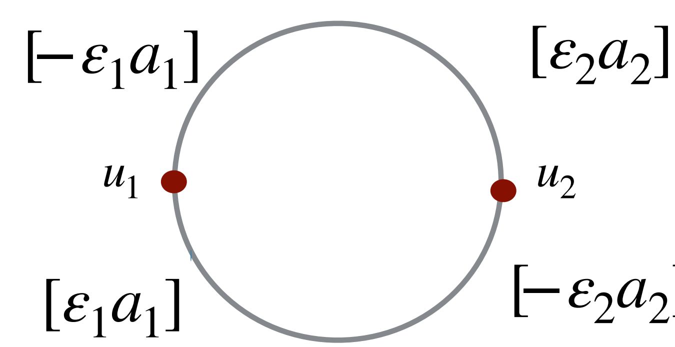


$$= I_{1,\ell}$$

$$\hat{K}(u_j,u_k) := \Omega_\ell(u_j)\,K(u_j,u_k)\,\Omega_\ell(u_k)$$

n=2 $$-\frac{1}{2}\frac{\lambda_\pm^2}{2}\sum_{a_1,a_2\geq 1}\sum_{\varepsilon_1,\varepsilon_2=\pm}\prod_{j=1}^{2}\frac{\sin a_j\phi}{\sin\phi}\int_{\mathbb{R}}\frac{du_j}{2\pi i}\;\varepsilon_1\,\hat{K}(u_1^{[-\varepsilon_1 a_1]},u_2^{[\varepsilon_2 a_2]})\,\varepsilon_2\,\hat{K}(u_2^{[-\varepsilon_2 a_2]},u_1^{[\varepsilon_1 a_1]})$$

compare with $I_{2,\ell}$



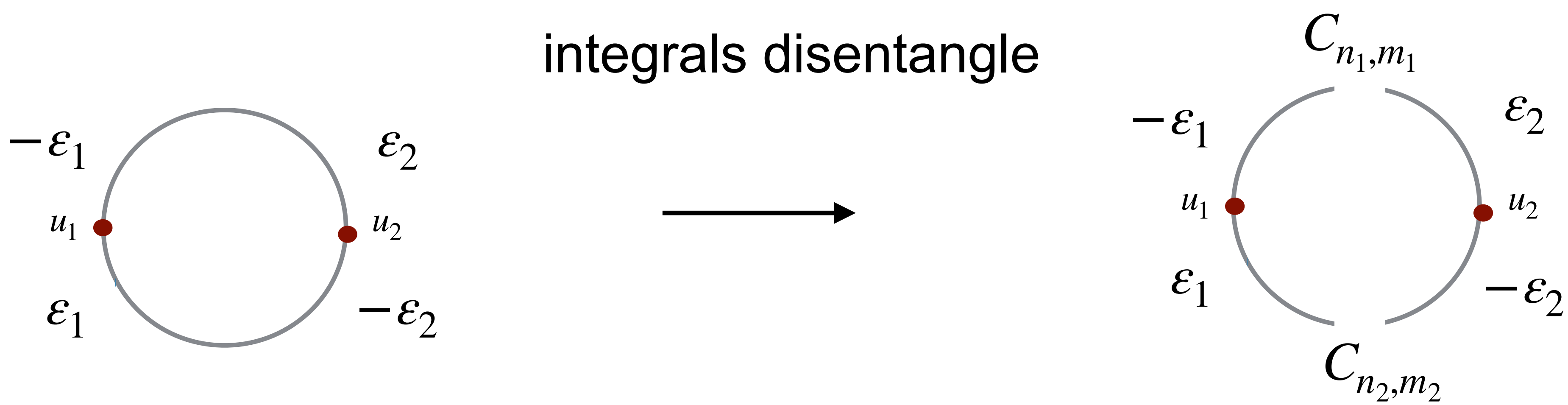
From Fredholm kernel to semi-infinite matrix

1. discrete basis $$K(u,v) = \frac{x-y}{xy-1} = \sum_{m,n=0}^{\infty} x^{-n}\, \mathrm{C}_{nm}\, y^{-m} \qquad \text{for } |x|>1, |y|>1$$

$$\mathrm{C}_{nm} = \delta_{n+1,m} - \delta_{n,m+1}, \qquad m,n \geq 0.$$

$$\Omega_{\ell}(u) \;\to\; \Omega_{\ell+n}(u) \equiv \sqrt{g}\,\frac{dx}{du}\,\frac{e^{ig\xi\,[x(u)-1/x(u)]}}{x(u)^{\ell+n+1}}$$

$$\hat{K}(u_j,u_k) := \Omega_{\ell+n}(u_j)\, C_{nm}\, \Omega_{\ell+m}(u_k)$$



2. Fourier transform - integral computed by residues, $\mathbb{D}^{2a} \to e^{-at}$

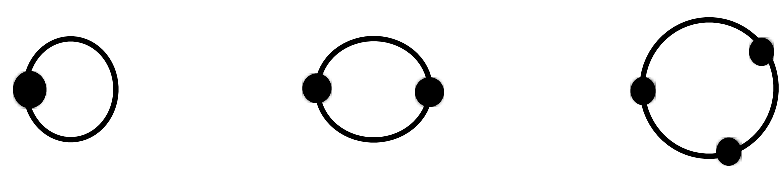
Trace of a matrix product $[\mathbf{CK}]^2$

$$\mathrm{K}_{mn} = \frac{g}{2i}\int_{|\xi|}^{\infty} dt \frac{\left(i\sqrt{\frac{t+\xi}{t-\xi}}\right)^{m-n} - \left(i\sqrt{\frac{t+\xi}{t-\xi}}\right)^{n-m}}{\cos\phi - \cosh t} J_{m+\ell}(2g\sqrt{t^2-\xi^2})\, J_{n+\ell}(2g\sqrt{t^2-\xi^2})$$

$$2\,\xi = -\log z\,\bar{z}$$

$\mathrm{K}_{mn}\,,\; m,n = 0,1,2\ldots$

$$\mathbb{O}_{\ell} = \tfrac{1}{2}\textstyle\sum_{\pm} \exp\tfrac{1}{2}\left(-\lambda_{\pm}\mathrm{tr}[\mathbf{CK}] - \tfrac{\lambda_{\pm}^2}{2}\mathrm{tr}[(\mathbf{CK})^2] - \tfrac{\lambda_{\pm}^3}{3}\mathrm{tr}[(\mathbf{CK})^3] - \ldots\right) = \tfrac{1}{2}\textstyle\sum_{\pm}\sqrt{\mathrm{Det}\left[\mathbf{I} - \lambda_{\pm}\mathbf{CK}\right]}$$



$$= \frac{1}{2}\sum_{\pm}\frac{\mathrm{Pf}\left[\mathbf{C}^{-1} - \lambda_{\pm}\mathbf{K}\right]}{\mathrm{Pf}\left[\mathbf{C}^{-1}\right]}$$

Non-perturbative formula, determines Coronado integrals as polynomials of traces of $\mathbf{CK}$

$$\Rightarrow\quad n\mathrm{I}_{n,\ell} = -\frac{1}{2}\sum_{k=0}^{n-1}\mathrm{I}_{k,\ell}\,\mathrm{tr}\left[(e^{\xi}\mathbf{CK})^{n-k}\right]$$

Remark: compare with the generating function for the ladder integrals

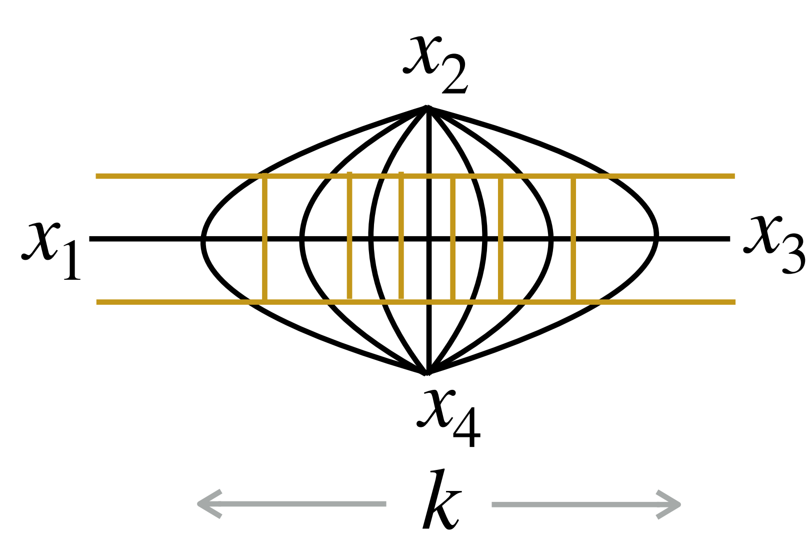
$$\sum_{n=1}^{\infty}(-g^2)^n\frac{f_n(z,\bar z)}{n!(n-1)!}=\frac{g\,e^{\xi}}{2}\int_{\xi}^{\infty}\frac{\left(\frac{\sqrt{t-\xi}}{\sqrt{t+\xi}}+\frac{\sqrt{t+\xi}}{\sqrt{t-\xi}}\right)J_1\left(2g\sqrt{t^2-\xi^2}\right)}{\cosh t-\cos\phi}\,dt$$

Broadhurst&Davydychev, arXiv:1007.0237

$$f_k(z,\bar z)=\sum_{j=k}^{2k}\frac{(k-1)!\,j!}{(j-k)!(2k-j)!}\,(-\log z\bar z)^{2k-j}\,\frac{\mathrm{Li}_j(z)-\mathrm{Li}_j(\bar z)}{z-\bar z}$$

Ladder Feynman integrals

$$f_k(z,\bar z)=$$



Usyukina&Davydychev 1993

Weak coupling expansion of the matrix $\mathbf{K}$

$$\mathrm{I}_{n,\ell} = \sum_{J=n(n+\ell)}^{\infty} \sum_{j_1+\dots j_n=J} c^{(\ell)}_{j_1,\cdots j_n} f_{j_1} \cdots f_{j_n} \, g^{2J}$$

Conjecture Coronado
established by Math.
to high orders in g

$$\mathrm{K}_{m+r,m} = e^{-\xi} \sum_{p=0}^{\infty} C_p^{m,r} \sum_{k=1}^{[\frac{r+1}{2}]} \binom{r-k}{k-1} |2g\,\xi|^{r-2k+1} g^{2\ell+2m+2p+2k} f_{\ell+m+p+k}(z,\bar{z}) \qquad (r \geq 1)\,.$$

- but besides ladders, also powers of $2\xi = -\log|z|^2$ for $r > 1$??

Empirical observation: only odd r = 2s-1
and 2k=r+1=2s contribute to the traces of $(\mathbf{CK})^n$

? change of basis $O\mathbf{CK}O^{-1}$

truncated series

$$\mathrm{K}^{\circ}_{m+2s,m} = 0$$

$$\mathrm{K}^{\circ}_{m+2s-1,m} = e^{-\xi} \sum_{j=l+m+s}^{\infty} \binom{2j-1}{j-l-m-s} \frac{(-1)^{j-l-m}}{(j-s)!(j+s-1)!} g^{2j} f_j(z,\bar{z})$$

reproduces the weak g expansions with coeffs computed from K°

$$\mathrm{I}_{n,\ell} = \sum_{J=n(n+\ell)}^{\infty} \sum_{j_1+\ldots j_n=J} c^{(\ell)}_{j_1,\cdots j_n} f_{j_1} \cdots f_{j_n} g^{2J}$$

resumming - by Mathematica; difficult to reproduce it analytically, ${}_{m+1}F_m$

The perturbative octagon as a determinant

To compute the octagon up to 2*N* loops, one can replace the semi-infinite matrices by 2N x 2N matrices $\{\mathrm{C}_{m,n}\}_{0\le m,n\le 2N-1}\,,\ \{\mathrm{K}^{\circ}_{m,n}\}_{0\le m,n\le 2N-1}\,.$

Using that their matrix elements **vanish if m=n mod 2**, we can introduce an N x N matrix

$$\mathbf{R}_{N\times N} = \{\mathrm{R}_{k,j}\}_{0\le k,j\le N-1}\,, \qquad R_{k,j} := -\,e^{\xi}\sum_p C_{2k,p}\,K^{\circ}_{p,2j} \quad \Rightarrow \quad \mathrm{tr}\,[\mathbf{R}^m] = -\,2\,e^{\xi}\ \mathrm{tr}\,[(\mathbf{C}\,\mathbf{K})^m]$$

$$\mathbb{O}_{\ell} = \frac{1}{2}\sum\nolimits_{\pm}\det(1+\lambda_{\pm}e^{-\xi}\,\mathbf{R})$$

For example the 3x3 matrix gives the expansion up to 6 loops:

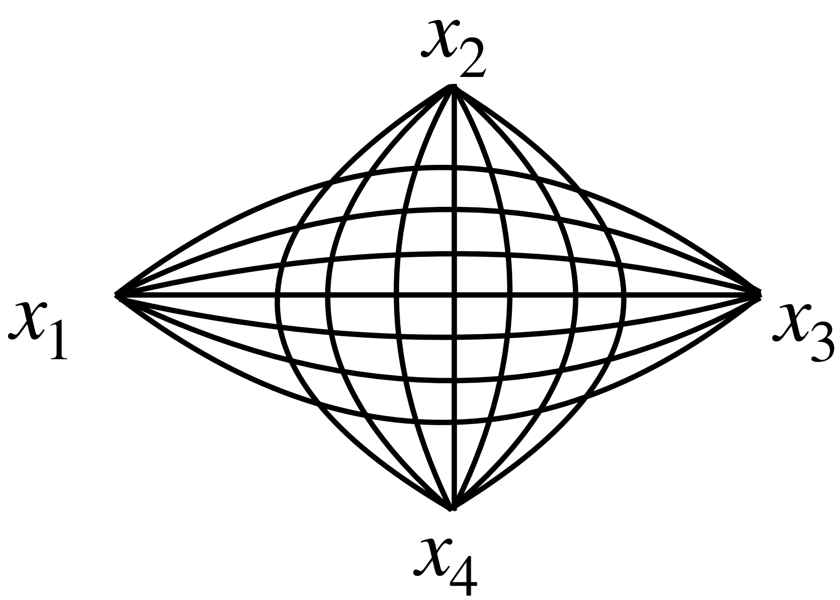
$$\mathbb{O}_{\ell=0} = \frac{1}{2}\sum\nolimits_{\pm}\det(1+\lambda_{\pm}e^{-\xi}\,\mathbf{R})_{3\times 3} + o(g^{12})$$

$$= 1 + \mathcal{X}_1\ \left(f_1 g^2 - f_2 g^4 + \frac{1}{2} f_3 g^6 - \frac{5}{36} f_4 g^8 + \frac{7}{288} f_5 g^{10}\right)$$

$$+\mathcal{X}_2\ \left(\frac{1}{12}(f_1 f_3 - f_2^2)g^8 - \frac{1}{24}(f_1 f_4 - f_2 f_3)g^{10}\right)\ + o(g^{12})$$

$\mathcal{X}_n = \frac{(\lambda^+)^n + (\lambda^-)^n}{2}\, e^{-n\,\xi}$

Relation to fishnets [Zamolodchikov, Gurdogan-Kazakov, Gromov, Korchemsky,…]

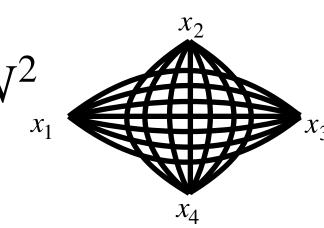


$$C_N = \frac{\det\left[f_{i+j+1}\right]_{i,j=0,\dots,N-1}}{\prod_{i=0}^{N-1}(2i)!(2i+1)!}$$

Fishnet is a determinant of ladders

Basso&Dixon 2017

Fishnets appear as the lowest order coefficients in the N-parricle contribution

$$\mathbb{O}_{\ell=0} = \sum_{N=0}^{\infty} \mathcal{X}_N \, g^{2N^2}$$  $+\dots$

Coronado 2018

The determinant representation gives

$$\mathbb{O}_{\ell=0} = \sum_{N=0}^{\infty} \mathcal{X}_N \sum_{\substack{0\le i_1<\dots<i_N \\ 0\le j_1<\dots<j_N}} \det\left(\left[\mathbf{R}_{i_\alpha j_\beta}\right]_{\alpha,\beta=1,\dots,N}\right)$$

$$\det \mathbf{R}_{N\times N} = C_N \, g^{2N^2} + o(g^{2N^2+2})$$

The octagon in a **Coulomb gas representation**

- Start with a gaussian field in the plane:

$$\varphi(x) = \hat{q} + \hat{p}\log x + \sum_{n\neq 0} \frac{J_n}{n} x^{-n}$$

$$[J_n, J_m] = n\delta_{m+n,0}; \quad [\hat{p}, \hat{q}] = 1$$

$$J_n|0\rangle = 0, \quad (n > 0); \quad \hat{p}|0\rangle = 0$$

$$\langle 0|J_n = 0, \quad (n < 0); \quad \langle 0|\hat{q} = 0$$

$$\langle \varphi(x)\varphi(y)\rangle = \log(x - y)$$

- Go to the Zhukovsky plane $x + \frac{1}{x} = \frac{u}{g}, \quad x(u) = \frac{u \pm \sqrt{u^2 - 4g^2}}{2g}$

$\varphi^{(\pm)}(u)$:= the value of the gaussian field in the upper/lower sheet $\qquad x \to 1/x$

$$\varphi^{(+)}(u)\varphi^{(+)}(v)\rangle = \log(x(u) - x(v)) \qquad \varphi^{(+)}(u)\varphi^{(-)}(v)\rangle = \log(x(u) - \frac{1}{x(v)})$$

$$\varphi^{(-)}(u)\varphi^{(+)}(v)\rangle = \log(\frac{1}{x(u)} - x(v)) \qquad \varphi^{(-)}(u)\varphi^{(-)}(v)\rangle = \log(\frac{1}{x(u)} - \frac{1}{x(v)})$$

The monodromy around the branch points at $u = \pm 2g$ is diagonalised by the combinations

$$\Phi(u) = \frac{\varphi^{(+)}(u) - \varphi^{(-)}(u)}{\sqrt{2}}, \quad \tilde{\Phi}(u) = \frac{\varphi^{(+)}(u) + \varphi^{(-)}(u)}{\sqrt{2}}$$

Correlator of the twisted component:

$$\langle 0 | \Phi(u)\Phi(v) | 0 \rangle = \frac{1}{2} \log \frac{(x-y)(\frac{1}{x} - \frac{1}{y})}{(x - \frac{1}{y})(\frac{1}{x} - y)} = \log \frac{x-y}{xy-1}$$

Correlator of the untwisted component:

$$\langle 0 | \tilde{\Phi}(u)\tilde{\Phi}(v) | 0 \rangle = \frac{1}{2} \log(x-y)(\frac{1}{x} - \frac{1}{y})(x - \frac{1}{y})(\frac{1}{x} - y) = \log(\frac{u-v}{g})$$

$$\langle 0 | \Phi(u) \ \tilde{\Phi}(v) | 0 \rangle = 0$$

- The real fermions are bosonized: $\Psi(u) = :e^{\Phi(u)}:$ vertex operator

$$:e^{\Phi(u)}::e^{\Phi(v)}: = K(u,v) :e^{\Phi(u)+\Phi(v)}:$$

- The bi-local factors $\tilde{H}_{ab}(u,v)$ are generated by the correlations of the vertex operator $\mathscr{V}_a(u) = :e^{\Phi(u+ia/2)+\Phi(u-ia/2)}:$

$$\tilde{H}_{ab}(u,v) = \langle 0 | \mathscr{V}_a(u) \, \mathscr{V}_b(v) | 0 \rangle$$

- Part of the measure is generated by modifying the expectation value

$$\langle \mathscr{O} \rangle_{\xi,\ell} := \langle 0 | \mathscr{O} \, e^{i\sqrt{2}g\xi J_{-1} - \frac{\ell+1}{\sqrt{2}}\hat{q}} | 0 \rangle$$

$$\langle \Phi(u) \rangle_{\xi,\ell} = -ig\xi(x - 1/x) - (\ell+1)\log x$$

Operator representation of the octagon

The rest of the measure originates from the (regularized) expectation value of the exponential field:

$$e^{\hat{\Phi}(u)} = \sqrt{g}\frac{dx}{du}\, e^{\Phi(u)} \qquad\qquad <: e^{\hat{\Phi}(u)} :>_{\xi,\ell} = \Omega_\ell(u) = e^{\Phi_c(u)}$$

$$<: e^{\hat{\Phi}(u)} :: e^{\hat{\Phi}(v)} :>_{\xi,\ell} = \Omega_\ell(u)\, K(u,v)\, \Omega_\ell(v) = \hat{K}(u,v)$$

Using this operator representation the sum over the bound state labels *a* can be performed explicitly:

- $$\mathbb{O}_\ell = \frac{1}{2}\sum_\pm \left\langle \exp\left[\frac{\lambda^\pm}{2}\int\frac{du}{2\pi i}\ : e^{\hat{\Phi}(u-i0)} :\ \frac{1}{\cos\phi - 2\cos\partial_u}\ : e^{\hat{\Phi}(u+i0)} :\right]\right\rangle_{\xi,\ell}$$

Alternatively from mode expansion of the free fermion on the Zhukovsky plane

$$\Psi(u) = \sum_{m\geq 0} \Psi_m \; x(u)^{-m}, \quad \langle 0 | \Psi_m \Psi_n | 0 \rangle := \mathrm{C}_{mn} = \delta_{n+1,m} - \delta_{n,m+1}, \qquad m, n \geq 0.$$

$$\langle 0 | \Psi(u_1)\Psi(u_2) | 0 \rangle = \frac{x(u_1) - x(u_2)}{x(u_1)x(u_2) - 1} = \sum_{n,m=0}^{\infty} x(u_1)^{-n} \; C_{nm} \; x(u_2)^{-m}$$

Inserting this mode expansion :

- $$\mathbb{O}_\ell = \frac{1}{2} \sum_\pm \langle 0 | e^{-\frac{1}{2}\lambda_\pm \sum_{m,n\geq 0} \Psi_m \mathrm{K}_{nm} \Psi_n} | 0 \rangle$$

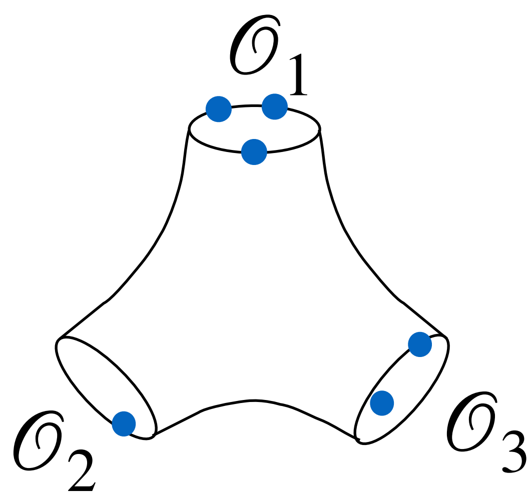
Thanks

The Hexagon proposal:

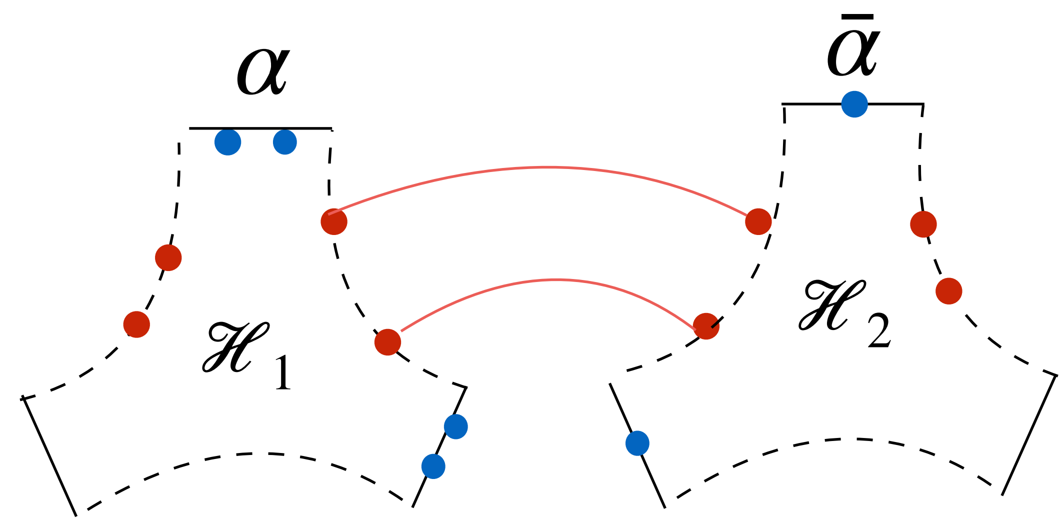
Correlation functions - described by a “path integral” on a sphere with ***n*** boundaries associated with one-trace operators

1) Cut the world sheet into hexagons (assign special form factors $\mathscr{H}_i$)

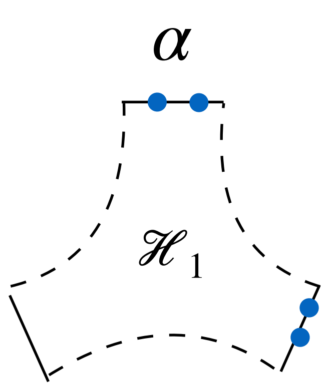
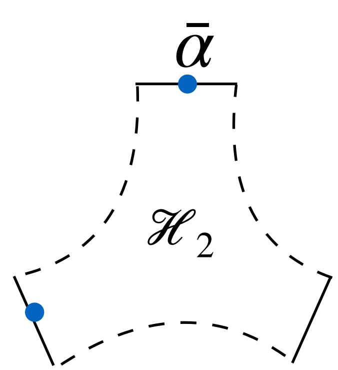
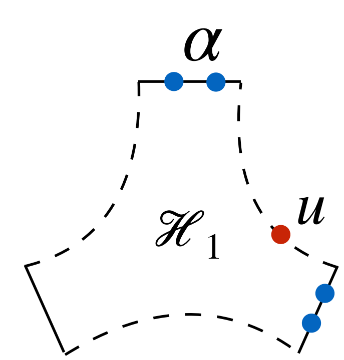
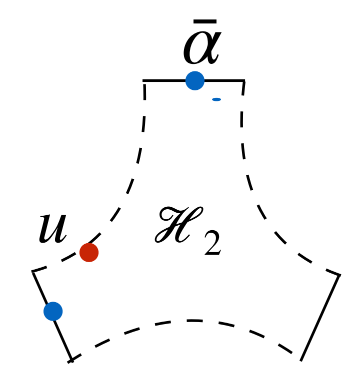
Basso, Komatsu, Vieira, 2015



$\sum$
partitions of
phys. rapidities
mirror states



2) Glue back by inserting a complete set of virtual (**mirror)** states at each cut.

$= \sum_{\alpha\cup\bar\alpha=\mathbf{v}}$   $+ \int_{-\infty}^{\infty}\frac{du}{2\pi}\, e^{-\tilde{E}\,\ell_{13}}$   $+\ldots$

$$\ell_{13} = \frac{L_1 + L_3 - L_2}{2}$$

“analogs” of the ‘screening charges’ in Liouville theory

$$\sum_a \int d\mu_a(u)\ V(u,a)$$

$$V(u,a) =: e^{\Phi(u+ia/2)} : : e^{\Phi(u-ia/2)} :$$

bulk operator in boundary CFT, with “defects”