Introduction to the noncommutative digital differential geometry

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based on the joint work with Shahn Majid

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- Noncommutative Geometry $\leftrightarrow$ Quantum geometry: Continuum differential geometry cannot be the geometry when both quantum and gravitational effects are present.

  On a curved space one must use the methods of Riemannian geometry but in their quantum version.

- The formalism of noncommutative differential geometry does not require functions and differentials to commute, so is more general even when the algebra is classical.

Plan of the talk:

1. Quantum Riemannian Geometry ingredients
2. What is the digital quantum geometry?
3. Digital quantum geometries in $n \leq 3$
4. Conclusions

Differential Geometry vs NC Differential Geometry

M - manifold and
$C^\infty(M)$ - functions on a manifold

$\rightarrow$ 'coordinate algebra' A

and

Ω^1 space of 1-forms, e.g. differentials:

$$\mathrm{d}f = \sum_i \frac{\partial f}{\partial x^\mu} \mathrm{d}x^\mu$$

$$f\mathrm{d}g = (\mathrm{d}g)f$$

$\rightarrow$ noncommutative differential structure:
differential bimodule (Ω^1, d) of 1-forms with d - obeying the Leibniz rule and
$\rightarrow f\mathrm{d}g \neq (\mathrm{d}g)f$

Bimodule - to associatively multiply such 1-forms by elements of A from the left and the right.

Quantum Riemannian Geometry

Ingredients of noncommutative Riemannian geometry as quantum geometry:

- quantum differentials
- quantum metrics
- quantum-Levi Civita connections
- quantum curvature
- Ricci and Einstein tensors

Quantum differentials

Differential calculus on an algebra A

- A is a 'coordinate' algebra over field k (noncommutative or commutative)

Definition

A first order differential calculus (Ω^1, d) over A means:

1. Ω^1 is an A-bimodule
2. A linear map $\mathrm{d} : A \to \Omega^1$ such that

$$\mathrm{d}(ab) = (\mathrm{d}a)b + a\mathrm{d}b \quad , \forall a, b \in A$$

3. $\Omega^1 = span\{adb\}$
4. (optional) $\ker d = k.1$ - connectedness condition

Differential graded algebra -DGA

Definition

DGA on an algebra A is:

1. A graded algebra $\Omega = \oplus_{n \geq 0} \Omega^n$, $\Omega^0 = A$
2. $\mathrm{d} : \Omega^n \to \Omega^{n+1}$, s.t. $\mathrm{d}^2 = 0$ and
$$\mathrm{d}(\omega\rho) = (\mathrm{d}\omega) \wedge \rho + (-1)^n \omega \wedge \mathrm{d}\rho$$
$\forall \omega, \rho \in \Omega, \quad \omega \in \Omega^n.$
3. $A, \mathrm{d}A$ generate Ω
(optional surjectivity condition - if it holds we say it is an **exterior algebra** on A)

Quantum metrics

When working with algebraic differential forms by **metric** we mean an element

$$g \in \Omega^1 \otimes_A \Omega^1$$

which is:

- 'quantum symmetric': $\wedge(g) = 0$,
- invertible
  in the sense that there exists $(\quad,\quad) : \Omega^1 \otimes_A \Omega^1 \to A$
  $$((\omega, \quad) \otimes id)g = \omega = (id \otimes (\quad, \omega))g \qquad \forall \omega \in \Omega^1$$

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- **central in the 'coordinate algebra'** $A \ni x^\mu$:
  $$[g, x^\mu] = 0$$

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  $$\underline{\dim} = (\ ,\ )(g) \in k.$$

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The general form of the quantum metric:

$$g = g_{\mu\nu} dx^\mu \otimes_A dx^\nu$$

Quantum connections

[Quillen, Karoubi, Michor, Mourad, Dubois-Violette, . . .]

- Bimodule connection: $\nabla : \Omega^1 \to \Omega^1 \otimes_A \Omega^1$,
  $\sigma \quad : \quad \Omega^1 \otimes_A \Omega^1 \to \Omega^1 \otimes_A \Omega^1$,
  for $a \in A, \omega \in \Omega^1$

$$\nabla(a\omega) = a\nabla\omega + \mathrm{d}a \otimes \omega$$

$$\nabla(\omega a) = (\nabla\omega)a + \sigma(\omega \otimes \mathrm{d}a)$$

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- Such connections extend to tensor products:

$$\nabla(\omega\otimes\eta) = (\nabla\omega)\otimes\eta + (\sigma\otimes id)(\omega\otimes\nabla\eta), \qquad \omega\otimes\eta \in \Omega^1\otimes_A\Omega^1$$

Metric compatibility, torsion and curvature

Metric compatible connection:

$$\nabla(g) = 0$$

Torsion of a connection on Ω^1 is

$$T_\nabla \omega = \wedge \nabla \omega - \mathrm{d}\omega \quad : \qquad T_\nabla : \Omega^1 \to \Omega^2$$

We define a **quantum Levi-Civita connection (QLC connection)** as metric compatible and torsion free connection.

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Curvature:

$$R_\nabla \omega = (\mathrm{d} \otimes id - \wedge(id \otimes \nabla))\nabla \omega \qquad R_\nabla : \quad \Omega^1 \to \Omega^2 \otimes_A \Omega^1$$

Ricci & Einstein tensors

- Ricci tensor:

$$\mathrm{Ricci} = ((\ ,\ )\otimes \mathrm{id})(\mathrm{id}\otimes i \otimes \mathrm{id})R_\nabla$$

  with respect to a 'lifting' bimodule map $i : \Omega^2 \to \Omega^1 \otimes_A \Omega^1$ such that $\wedge \circ i = \mathrm{id}$.
- Then Ricci scalar is $S = (\ ,\ )\mathrm{Ricci}$.

[Beggs,Majid,Class.Quantum.Grav.31(2014)]

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- For Einstein tensor one can consider the usual definition

$$\mathrm{Eins} = \mathrm{Ricci} - \frac{1}{2}Sg$$

  but field independent option would be:

$$\mathrm{Eins} = \mathrm{Ricci} - \alpha Sg, \qquad \alpha \in k$$

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- one could take $\mathrm{Eins} = \mathrm{Ricci} - \frac{1}{\underline{\mathrm{dim}}} S g$

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Aim

- to study bimodule quantum Riemannian geometries over the field $\mathbb{F}_2 = \{0, 1\}$ of two elements (**'digital'** (quantum) geometries)
- to classify all (parallelisable) such geometries for coordinate algebras up to dimension $n \leq 3$

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- to study bimodule quantum Riemannian geometries over the field $\mathbb{F}_2 = \{0, 1\}$ of two elements (**'digital'** (quantum) geometries)
- to classify all (parallelisable) such geometries for coordinate algebras up to dimension $n \leq 3$

Preview of results:
A rich moduli of examples for $n = 3$, including **9 that are Ricci flat but not flat**
(with commutative coordinate algebras but with noncommuting differentials $x^\mu x^\nu = x^\nu x^\mu, \quad x^\mu dx^\rho \neq dx^\rho x^\mu$, $x^\mu, x^\nu \in A, dx^\rho \in \Omega^1$).

Why digital?

Finite field $\mathbb{F}_2 = \{0, 1\}$

- The choice of the finite field leads to a new kind of 'discretisation scheme', which adds **'digital'** to quantum geometry.
- A standard technique in physics and engineering is to replace geometric backgrounds by discrete approximations such as a lattice or graph, thereby rendering systems more calculable.
- A repertoire of digital quantum geometries $\Rightarrow$ to test ideas and conjectures in the general theory if we expect them to hold for any field, even if we are mainly interested in the theory over $\mathbb{C}$.

Digital Geometry set up

'Coordinate algebra' A (unital associative algebra) over $\mathbb{F}_2$ - the field of two elements $0, 1$.

$\{x^\mu\}$ - basis of A where $x^0 = 1$ the unit and $\mu = 0, \cdots, n-1$.

Structure constants $V^{\mu\nu}{}_\rho \in \mathbb{F}_2$

$$x^\mu x^\nu = V^{\mu\nu}{}_\rho x^\rho.$$

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We have classified all possible such algebras over $\mathbb{F}_2$ up to $n \leq 4$.
[S.Majid,A.P.,J.Math.Phys.59 (2018)]

'Coordinate algebras' over $\mathbb{F}_2$ in low dim

$\{x^\mu\}$ is a basis of A where $x^0 = 1$ the unit and $\mu = 0, \cdots, n-1$
For $n = 1$ There is only **one** unital algebra of dimension 1 $(x^0 x^0 = x^0)$

For $n = 2$ There are **3*** inequivalent (commutative) algebras A, B, C:
A: $x^1 x^1 = 0$
B: $x^1 x^1 = x^1$
C: $x^1 x^1 = x^0 + x^1 = 1 + x^1$.

For $n = 3$ There are **6*** inequivalent (commutative) algebras: A, B, C, D, E, F and **one** noncommutative G.

For $n = 4$ There are **16*** inequivalent (commutative) algebras: A - P and 9 noncommutative ones.

* up to isomorphisms

Differential calculus for coordinate algebra A over $\mathbb{F}_2$

- Ω^1 - space of 1-forms with a basis $\{\omega^i\}$, $i = 1, \cdots, m$ over A ($m \leq n-1$ is the dimension of the calculus over A).
- The case $m = n-1$ is the **'universal calculus'**.

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- The case $m = n-1$ is the **'universal calculus'**.
- $\Omega^1 = A.\{\omega^i\}$ (free left module by the product in A)
  We require a right action of A specified by structure constants

$$\omega^i . x^\mu = a^{i\mu}{}_{\nu j} x^\nu . \omega^j, \quad a^{i\mu}{}_{\nu j} \in \mathbb{F}_2.$$

- the structure constants for the exterior differential $\mathrm{d} : A \to \Omega^1$

$$\mathrm{d}x^\mu = d^\mu{}_{\nu i} x^\nu . \omega^i, \quad d^\mu{}_{\nu i} \in \mathbb{F}_2.$$

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  Such calculus is called **'left parallelisable'**.
- Ω^1 also needs to satisfy the **surjectivity condition** and optionally to be connected.

Classification of quantum digital geometries for $n = 3$

- We have considered each of the 6 commutative (A-F) and one noncommutative (G) algebras with two dimensional Ω^1 (the universal calculus) and with 1 dimensional Ω^1.
- To keep things simple, for the universal calculus, we considered geometries with basis $\omega^1 = \mathrm{d}x^1, \omega^2 = \mathrm{d}x^2$ for Ω^1 and we take 1 dimensional Ω^2

Digital quantum geometries - one algebra example

- From the 6 algebras (A - F) let's choose algebra D (an example of 3-dimensional unital commutative algebra with the basis $1, x^1, x^2$).
- Relations: $x^1x^1 = x^2, \quad x^2x^2 = x^1, \quad x^1x^2 = x^1 + x^2 = x^2x^1$
- Universal differential calculus with relations:

$$dx^1.x^2 = x^1dx^2 + dx^1 + dx^2, \qquad dx^2.x^1 = x^2dx^1 + dx^1 + dx^2$$

$$[dx^1, x^1] = dx^2, \qquad [dx^2, x^2] = dx^1$$

- This algebra (D) is isomorphic to $\mathbb{F}_2\mathbb{Z}_3$ the group algebra on the group $\mathbb{Z}_3$ since $z = 1 + x^1$ obeys $(z)^2 = 1 + x^2$ and $(z)^3 = 1$.

Quantum metric on $\mathbb{F}_2\mathbb{Z}_3$

We define a metric as an invertible element of $g \in \Omega^1 \otimes_D \Omega^1$.

$$g = g_{ij}\omega^i \otimes \omega^j = g_{\mu ij}x^\mu \omega^i \otimes \omega^j, \quad g_{ij} \in D, \quad g_{\mu ij} \in \mathbb{F}_2$$

- Quantum metric (central and quantum symm.) on $D = \mathbb{F}_2\mathbb{Z}_3$:

  $$g_D = \beta z^2 \omega^1 \otimes \omega^1 + \beta z(\omega^1 \otimes \omega^2 + \omega^2 \otimes \omega^1) + \beta \omega^2 \otimes \omega^2$$

  with β - a functional parameter.

- We take special cases for $\beta = 1, z, z^2$
- For these there are 12 QLC connections (11 of them not flat! $R_\nabla \neq 0$ - purely 'quantum' phenomenon.)

Digital quantum connection and curvature

must also have the structure constants in $\mathbb{F}_2$:

$$\nabla\omega^i = \Gamma^i{}_{\nu km} x^\nu \omega^k \otimes \omega^m, \quad \sigma\left(\omega^i \otimes \omega^j\right) = \sigma^{ij}{}_{\mu km} x^\mu \omega^k \otimes \omega^m.$$

For curvature $R_\nabla : \Omega^1 \to \Omega^2 \otimes_D \Omega^1$ we require the same:

$$R_\nabla = (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla)\nabla$$

$$R_\nabla \omega^i = \rho^i{}_{j\mu} x^\mu \mathrm{Vol} \otimes \omega^j = \rho^i{}_j \mathrm{Vol} \otimes \omega^j$$

Ω^2

Differential graded or 'exterior algebra' $\Omega = \oplus_i \Omega^i$ (with $D = \Omega^0$ and Ω^1)

- For $\Omega^2 = D.\mathrm{Vol}$ we take 1-dimensional free module over D, with basis Vol.
- If Vol exists we define

  $$\omega^i \wedge \omega^j = \epsilon^{ij}{}_{\mu} x^{\mu} \mathrm{Vol} = \epsilon^{ij} \mathrm{Vol}, \quad \epsilon^{ij}{}_{\mu} \in \mathbb{F}_2,\ \epsilon^{ij} \in D.$$

  and require it to be central in D.
- **wedge product** $\wedge$ - associative
  (including the action by elements of D, centrality of the volume form $\longrightarrow$ certain commutation relations between $\omega^i \wedge \omega^j$)
- we extend d to general 1-forms by the Leibniz rule

Once we have specified at least Ω^2, we can:

- ask for our metric to be 'quantum symmetric' in the sense

$$\wedge(g) = 0$$

- Look for a *quantum Levi-Civita connection* (QLC):
  $\nabla g = T_\nabla = 0$

QLC connections and curvature on $\mathbb{F}_2\mathbb{Z}_3$

Recall: $g_D = \beta z^2\omega^1 \otimes \omega^1 + \beta z(\omega^1 \otimes \omega^2 + \omega^2 \otimes \omega^1) + \beta\omega^2 \otimes \omega^2$.

For $\beta = 1$ one of QLC's looks like this:

$$\nabla_{D.1.1}\omega^1 = z^2\omega^1 \otimes \omega^1 + (1+z)(\omega^1 \otimes \omega^2 + \omega^2 \otimes \omega^1) + \omega^2 \otimes \omega^2$$

$$\nabla_{D.1.1}\omega^2 = z^2\omega^1 \otimes \omega^1 + z\omega^1 \otimes \omega^2 + z^2\omega^2 \otimes \omega^1 + \omega^2 \otimes \omega^2$$

$$R_{\nabla_{D.1.1}}\omega^1 = \mathrm{Vol} \otimes \omega^1 + z^2\mathrm{Vol} \otimes \omega^2, \quad R_{\nabla_{D.1.1}}\omega^2 = z^2\mathrm{Vol} \otimes \omega^1;$$

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There are 3 more for this choice of β (none flat):

$$\nabla_{D.1.2}\omega^1 = z^2\omega^1 \otimes \omega^1 + z(\omega^1 \otimes \omega^2 + \omega^2 \otimes \omega^1) + \omega^2 \otimes \omega^2$$

$$\nabla_{D.1.2}\omega^2 = z^2\omega^2 \otimes \omega^1$$

$$R_{\nabla_{D.1.2}}\omega^1 = R_{\nabla_{D.1.2}}\omega^2 = \left(1+z^2\right)\mathrm{Vol} \otimes (\omega^1 + \omega^2);$$

$$\nabla_{D.1.3}\omega^1 = (z+z^2)\omega^1 \otimes \omega^1 + (1+z)\omega^1 \otimes \omega^2 + z\omega^2 \otimes \omega^1 + \left(1+z^2\right)\omega^2 \otimes \omega^2$$

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$$R_{\nabla_{D.1.3}}\omega^1 = \mathrm{Vol} \otimes \omega^1 + z^2\mathrm{Vol} \otimes \omega^2, \quad R_{\nabla_{D.1.3}}\omega^2 = z^2\mathrm{Vol} \otimes \omega^1;$$

$$\nabla_{D.1.4}\omega^1 = (z+z^2)\omega^1 \otimes \omega^1 + z\omega^1 \otimes \omega^2 + (1+z)\omega^2 \otimes \omega^1 + \left(1+z^2\right)\omega^2 \otimes \omega^2$$

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There are further 8 QLCs for $\beta = z$, $\beta = z^2$ (only 1 flat).

The Ricci tensor

$$\mathrm{Ricci} = ((\ ,\ ) \otimes \mathrm{id})(\mathrm{id} \otimes i \otimes \mathrm{id})R_\nabla$$

- 'lifting' bimodule map $i : \Omega^2 \to \Omega^1 \otimes_A \Omega^1$ such that $\wedge \circ i = \mathrm{id}$.
- When Ω^2 is 1-dim (with central basis Vol) then:

  $$i(\mathrm{Vol}) = I_{ij}\omega^i \otimes \omega^j, \qquad I_{ij} \in A$$

  for some central element of $\Omega^1 \otimes_A \Omega^1$ such that $\wedge(I) = \mathrm{Vol}$.
- I - not unique (we can add any functional multiple γg for $\gamma \in A$ if g is central and quantum symmetric)

Then
$\mathrm{Ricci} = g_{ij}((\omega^i,\ ) \otimes \mathrm{id})(i \otimes \mathrm{id})R_\nabla\omega^j = g_{ij}(\omega^i, \rho^j{}_k I_{mn}\omega^m)\omega^n \otimes \omega^k.$

- We are interested in the choices for γ when

$$\nabla \cdot \mathrm{Ricci} = 0$$

where $\nabla\cdot$ means to apply ∇ in the element of $\Omega^1 \otimes_D \Omega^1$ (same as for the metric) and then contract the first two factors with $(\ ,\ )$.

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For $D = \mathbb{F}_2\mathbb{Z}_3$ we take

$$i(Vol) = z^2\omega^2 \otimes \omega^1 + z\omega^2 \otimes \omega^2 + \gamma g$$

where $\gamma \in D, \quad \gamma = \gamma_1 + \gamma_2 z + \gamma_3 z^2$.

Ricci tensor and scalar for $\mathbb{F}_2\mathbb{Z}_3$

Metric	QLC	Ricci (central for all γ_i)	$S = (,)(\mathrm{Ricci})$	q. symmetric
$g_{D.1}$ $(\beta = 1)$	$\nabla_{D.1.2}$	$\mathrm{Ricci} = 0$	$S = 0$	—
	$\nabla_{D.1.1}$, $\nabla_{D.1.3}$, $\nabla_{D.1.4}$	$\mathrm{Ricci} = \left(\gamma_3 + \gamma_2 z^2\right)\omega^1 \otimes \omega^1 + \left(\gamma_2 z + \gamma_3 z^2\right)\omega^1 \otimes \omega^2 + \left(\gamma_1 + z + \gamma_3 z^2\right)\omega^2 \otimes \omega^1 + \left(1 + \gamma_3 z + \gamma_1 z^2\right)\omega^2 \otimes \omega^2$	$\gamma_2 + \gamma_3 z$	$\gamma_1 = 0, \gamma_2 = 1:$ $\mathrm{Ricci} = (1+\gamma_3 z)z^2\omega^1 \otimes \omega^1 + (1+\gamma_3 z)z\omega^1 \otimes \omega^2 + (1+\gamma_3 z)z\omega^2 \otimes \omega^1 + (1+\gamma_3 z)\omega^2 \otimes \omega^2$
$g_{D.2}$ $(\beta = z)$	$\nabla_{D.2.4}$	$\mathrm{Ricci} = 0$	$S = 0$	—
	$\nabla_{D.2.1}$, $\nabla_{D.2.2}$, $\nabla_{D.2.3}$	$\mathrm{Ricci} = \left(1 + \gamma_3 z + \gamma_1 z^2\right)\omega^1 \otimes \omega^1 + \left(\gamma_3 + \gamma_1 z + z^2\right)\omega^1 \otimes \omega^2 + \left(\gamma_1 z + (1+\gamma_2)z^2\right)\omega^2 \otimes \omega^1 + (\gamma_1 + (1+\gamma_2)z)\,\omega^2 \otimes \omega^2$	$1 + \gamma_2 + \gamma_1 z^2$	$\gamma_2 = 0 = \gamma_3:$ $\mathrm{Ricci} = (\gamma_1 + z)z^2\omega^1 \otimes \omega^1 + (\gamma_1 + z)z\omega^1 \otimes \omega^2 + (\gamma_1 + z)z\omega^2 \otimes \omega^1 + (\gamma_1 + z)\omega^2 \otimes \omega^2$
$g_{D.3}$ $(\beta = z^2)$	$\nabla_{D.3.1}$	$\mathrm{Ricci} = 0$ (flat connection)	$S = 0$	—
	$\nabla_{D.3.2}$, $\nabla_{D.3.3}$, $\nabla_{D.3.4}$	$\mathrm{Ricci} = (\gamma_1 + (1+\gamma_2)z)\,\omega^1 \otimes \omega^1 + \left(1 + \gamma_2 + \gamma_1 z^2\right)\omega^1 \otimes \omega^2 + (\gamma_2 + \gamma_3 z)\,\omega^2 \otimes \omega^1 + \left(\gamma_3 + \gamma_2 z^2\right)\omega^2 \otimes \omega^2$	$1 + \gamma_3 z + \gamma_1 z^2$	never qsymm

For each metric one connection is Ricci flat.

$\underline{\dim}_{D.1} = \underline{\dim}_{D.2} = 1, \underline{\dim}_{D.3} = 0.$

The Einstein tensor

$$\begin{aligned}\mathrm{Eins} &= \mathrm{Ricci} + Sg \\ &= (\mathrm{Ricci}_{\mu ij} + S_\nu g_{\rho ij} V^{\nu\rho}{}_{\mu}) x^\mu \omega^i \otimes \omega^j\end{aligned}$$

Note: the usual definition $\mathrm{Eins} = \mathrm{Ricci} - \frac{1}{2} Sg$ makes no sense over $\mathbb{F}_2$.
In $\mathbb{F}_2$ we actually have only two choices, $0, 1$, for the coefficient of Sg.

We are interested in the values of Eins and if this is not zero (as it would be classically for a 2D manifold) then we look when

$$\nabla \cdot \mathrm{Eins} = 0$$

$$\nabla \cdot \mathrm{Eins} = \nabla \cdot \mathrm{Ricci} + ((\ ,\ ) \otimes \mathrm{id})(\mathrm{d}S \otimes g) = \nabla \cdot \mathrm{Ricci} + \mathrm{d}S$$

given the properties of a connection, the inverse metric and $\nabla g = 0$ for a QLC.

The Einstein tensor on $\mathbb{F}_2\mathbb{Z}_3$

Metric	QLC	$\mathrm{Eins} = \mathrm{Ricci} + Sg$	Ricci qsymm	$\nabla \cdot \mathrm{Eins} = 0$
$g_{D.1}$	$\nabla_{D.1.2}$	$\mathrm{Eins} = 0$	—	—
	$\left.\begin{matrix}\nabla_{D.1.1}\\ \nabla_{D.1.3}\\ \nabla_{D.1.4}\end{matrix}\right\}$	$\mathrm{Eins} = (\gamma_1 + z(1+\gamma_2))\,\omega^2\otimes\omega^1 + \left(1+\gamma_2+\gamma_1 z^2\right)\omega^2\otimes\omega^2$	$\mathrm{Eins} = 0$	$\gamma_1 = 0$: $\mathrm{Eins} = (1+\gamma_2)\,z\omega^2\otimes\omega^1 + (1+\gamma_2)\,\omega^2\otimes\omega^2$
$g_{D.2}$	$\nabla_{D.2.4}$	$\mathrm{Eins} = 0$	—	—
	$\left.\begin{matrix}\nabla_{D.2.1}\\ \nabla_{D.2.2}\\ \nabla_{D.2.3}\end{matrix}\right\}$	$\mathrm{Eins} = (\gamma_2 + \gamma_3 z))\,\omega^1\otimes\omega^1 + \left(\gamma_3+\gamma_2 z^2\right)\omega^1\otimes\omega^2$	$\mathrm{Eins} = 0$	$\gamma_3 = 0$: $\mathrm{Eins} = \gamma_2\omega^1\otimes\omega^1 + \gamma_2 z^2\omega^1\otimes\omega^2$
$g_{D.3}$	$\nabla_{D.3.1}$	$\mathrm{Eins} = 0$ (flat connection)	—	—
	$\left.\begin{matrix}\nabla_{D.3.2}\\ \nabla_{D.3.3}\\ \nabla_{D.3.4}\end{matrix}\right\}$	$\mathrm{Eins} = \left(\gamma_2 z + \gamma_3 z^2\right)\omega^1\otimes\omega^1 + (\gamma_2+\gamma_3 z)\,\omega^1\otimes\omega^2 + \left(1+\gamma_2+\gamma_1 z^2\right)\omega^2\otimes\omega^1 + \left(\gamma_1 z + (1+\gamma_2)z^2\right)\omega^2\otimes\omega^2$	never qsymm	$\gamma_1 = 0 = \gamma_3$: $\mathrm{Eins} = \gamma_2 z\omega^1\otimes\omega^1 + \gamma_2\omega^1\otimes\omega^2 + (1+\gamma_2)\,\omega^2\otimes\omega^1 + (1+\gamma_2)z^2\omega^2\otimes\omega^2$

Metrics where $\underline{\dim} = 1$ have zero Einstein tensor when Ricci is lifted to be quantum symmetric.
The metric $g_{D.3}$ where $\underline{\dim} = 0$ has two lifts for the non-flat connections with $\nabla \cdot \mathrm{Eins} = 0$ and $S = 1$.

Digital Quantum Geometries on $D = \mathbb{F}_2\mathbb{Z}_3$:

- for each metric one connection is Ricci flat for all lifts (and only actually flat for $g_{D.3}$)
- and the other three connections all have the same Ricci curvature
- when Ricci is quantum symmetric (choice of γ_i) then $\mathrm{Eins} = 0$
- we can chose the lift so that $\nabla \cdot \mathrm{Eins} = 0$

$$g_{D.1}: \quad \gamma_1 = \gamma_3 = 0, \gamma_2 = 1, \quad \mathrm{Ricci} = g_{D.1}, \quad S = 1, \quad \nabla{\cdot}\mathrm{Ricci} = 0, \quad \mathrm{Eins} = 0$$

$$g_{D.2}: \quad \gamma_1 = \gamma_2 = \gamma_3 = 0, \quad \mathrm{Ricci} = g_{D.2}, \quad S = 1, \quad \nabla{\cdot}\mathrm{Ricci} = 0, \quad \mathrm{Eins} = 0$$

$$g_{D.3}: \quad \gamma_1 = \gamma_3 = 0, \qquad S = 1, \quad \nabla \cdot \mathrm{Ricci} = \nabla \cdot \mathrm{Eins} = 0, \quad \mathrm{Eins} \neq 0$$

where the last case is unusual in that classically the Einstein tensor in 2D would vanish , but this is also the 'unphysical' case where $\underline{\dim} = 0$.

- Similar results were obtained for two other (commutative) algebras $B = \mathbb{F}_2(\mathbb{Z}_3)$ and $F = \mathbb{F}_8$.

- We have also investigated the properties of the geometric Laplacians:
$$\Delta = (\ ,\ )\nabla \mathrm{d} : \qquad A \to A$$

- For algebras A, C, E, G there are no invertible central metrics for the universal calculus.

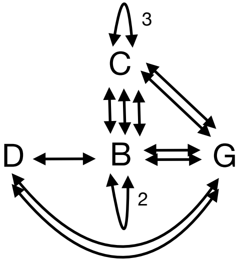
- All results - see S.Majid, A.P., J.Phys. A 2019 (in press) [arXiv:1807.08492].

Quantum symmetries

[work in progress]

- In 3 dimensions we are currently classifying all the Hopf algebras on associative algebras (A-G) over $\mathbb{F}_2$.
- Only $B = \mathbb{F}_2(\mathbb{Z}_3)$ and $D = \mathbb{F}_2\mathbb{Z}_3$ admit a Hopf algebra structure (namely the unique one indicated by the notation as group algebra or function algebra on a group).
- The algebras B,C,D,G admit many bialgebras (but no further Hopf algebras) and the algebras A,E,F admit no bialgebra structures.

If we make a graph where A$\rightarrow$B means that algebra A admits bialgebra structure with coalgebra isomorphic to the dual of B then we can graph our results as



In the Hopf algebra (as opposed to bialgebra) version we have only

$$\mathrm{B} \leftrightarrow \mathrm{D}$$

Summary

- we have mapped out the landscape of all reasonable up to 2D quantum geometries over the field $\mathbb{F}_2$ on unital algebras of dimension $n \leq 3$
- the interesting ones up to this dimension have **commutative coordinate algebras**
- even under this restricted set of assumptions there are a **lot of such 'digital' finite quantum geometries**
- In $n = 3$ with 2-dim Ω^1 we find that only **three** of the six algebras, namely B$= \mathbb{F}_2(\mathbb{Z}_3)$, D$= \mathbb{F}_2\mathbb{Z}_3$, F$= \mathbb{F}_8$, meet our full requirements on the calculus including Ω^2 as top form and existence of a quantum symmetric metric.

Conclusions

- For each of them we find an essentially **unique calculus and a unique quantum metric** up to an invertible functional factor
- When these quantum metrics admit QLC, each pair produces 'digital quantum Riemannian geometry' of which some are flat in the sense of zero Riemann curvature R_∇
- For the Ricci tensor and scalar S: we have found 2, 2, 5 (for alg. B, D, F resp.) - a **total of 9 interesting Ricci flat but not flat quantum geometries** over $\mathbb{F}_2$.
- These deserve more study in view of the important role of Ricci flat metrics in classical GR (as vacuum solutions of Einstein's equations).

Perspectives

- Finite field setting allows one to test definitions and conjectures - full classification possible.
- Quantum gravity is normally seen as a weighted 'sum' over all possible metrics
- once we have a good handle on the moduli of classes of small $\mathbb{F}_{p^d}$ quantum Riemannian geometries, we could consider quantum gravity, for example as a weighted sum over the moduli space of them much as in lattice approximations, but now finite.

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Thank you for your attention!