# Supersymmetry and deterministic chaos

Stam Nicolis

CNRS–Institut Denis–Poisson(UMR7013)
Université de Tours, Université d'Orléans
Parc Grandmont, 37200 Tours, France

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# Outline

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# Another approach towards describing supersymmetry

Supersymmetry appears in many guises. While most of the effort has been devoted towards trying to understand how it *doesn't* appear within the Standard Model, and to describe its effects in terms of particles, that are distinct from the "known" particles, it has escaped notice that it has much broader implications for physical systems at "mesoscopic" scales.

The starting point was the work of Parisi and Sourlas (1982) and of Nicolai (1980).

The bottom line: A non–relativistic particle, in equilibrium with a bath, realizes worldline supersymmetry. This means that the correlation functions of the position of the particle satisfy the identities that describe the fact that the system particle+bath is closed–and, moreover, that how the particle is "picked out" from within the bath, doesn't matter.

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# The superpartners hide among the partners

The superpartners of $x_I(t)$, $(\psi_I(t), \chi_I(t))$, come along for the ride. The trajectory becomes a component of a superfield, but the superpartners aren't distinct particles. Revealing their existence is done indirectly, through the identities, that certain composite fields, of the position, $x_I(t)$, satisfy.

This is true, also, for the relativistic particle, where the superpartners of the position are identified with the spin degrees of freedom (classically; fluctuations implicate *three* supersymmetries!)

So it is these identities that reveal the relevance of supersymmetry.

Indeed, a subtle point is that it is not possible to write an interaction between a fixed number of particles, in a way consistent with (global) target space Poincaré invariance.(Leutwyler 1965). Fields can be used, however, as the study of the 2-dimensional WZ model shows, where global SO(2) symmetry can be recovered.

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# The two cases of interest

Now there are two distinct possibilities:

- The (classical) dynamics of the particle is described by a scalar potential.
- The (classical) dynamics of the particle isn't described, solely, by a scalar potential.

The second case, in particular, includes that, where the classical motion is deterministically chaotic. So it's useful to understand what insights supersymmetry can provide, regarding the description of the fluctuations.

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# A particle in a bath, subject to a scalar potential

Everything about the properties of a particle, in equilibrium with a bath, are described by the "canonical" partition function

$$Z = \int [\mathscr{D}x_I]\, e^{-\int dt\,\left(\frac{1}{2}\dot{\boldsymbol{x}}^2 + V(\boldsymbol{x})\right)}$$

where $I = 1, 2, \ldots, d$.

For this expression to make any sense, $V(\boldsymbol{x}) \geq V_{\mathrm{min}} > -\infty$ and $\lim_{|\boldsymbol{x}|\to\infty} V(\boldsymbol{x}) = +\infty$. Otherwise the description is incomplete, anyway.

One class of potentials, that do satisfy these conditions, can be written as

$$V(\boldsymbol{x}) = \frac{1}{2}\sum_{I=1}^{d}\left(\frac{\partial W}{\partial x_I}\right)^2$$

The function $W(\boldsymbol{x})$ is known as the "superpotential".

If it is globally defined, $V(\boldsymbol{x})$ can be found. The converse entails solving the above differential equation–known as the "eikonal equation", for $W(\boldsymbol{x})$–which isn't that straightforward.


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Upon imposing periodic boundary conditions, the (Euclidian) action takes the form

$$S[\boldsymbol{x}] = \int dt \left( \frac{1}{2}\dot{\boldsymbol{x}}^2 + \left( \frac{\partial W(\boldsymbol{x})}{\partial x_I} \right)^2 \right) = \int dt \left( \frac{1}{2}\dot{\boldsymbol{x}} + \frac{\partial W(\boldsymbol{x})}{\partial x_I} \right)^2 + \text{boundary terms}$$


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# The consistent closure

This implies that the fluctuations, described by the canonical partition function can be repackaged as follows:

$$Z = \int [\mathscr{D}x_I]\, e^{-S[\boldsymbol{x}]} \left|\det\frac{\delta\eta_I}{\delta x_J}\right| = 1$$

where

$$\eta_I \equiv \frac{\partial x_I}{\partial t} + \frac{\partial W}{\partial x_I}$$

are the "noise fields". This relation defines the "Nicolai map". It's not a differential equation to be solved–rather it defines a change of variables in the functional integral.


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# To trust is good; to check is better

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The only reason the partition function wouldn't be equal to 1, would be that the zero modes of the Jacobian would be of non–zero measure. If tunneling can occur, however, it's not necessary to worry about them. And this can be checked by computing the identities, the noise fields are expected to satisfy:

$$\langle \eta_I(t) \rangle = 0$$
$$\left\langle \eta_I(t)\eta_J(t') \right\rangle = \delta_{IJ}\delta(t-t')$$
$$\langle \eta_{I_1}(t_1)\eta_{I_2}(t_2)\cdots\eta_{I_{2n}}(t_{2n}) \rangle =$$
$$\sum_\pi \left\langle \eta_{I_{\pi(1)}}(t_{\pi(1)})\eta_{\pi(2)}(t_{\pi(2)}) \right\rangle \cdots \langle \eta_{\pi(2n-1)}(t_{\pi(2n-1)})\eta_{\pi(2n)}(t_{\pi(2n)}) \rangle$$

where the averages are taken with the weight $e^{-S[x]}/Z$.

# The linear term of the superpotential

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We note that it isn't possible to distinguish, in an invariant way, whether $\langle\eta\rangle\neq 0$ from $\langle\partial W\rangle\neq 0$, assuming periodic boundary conditions, so that $\langle\dot{x}_I\rangle=0$. So it is possible to absorb the value of $\langle\eta\rangle$ into a contribution $c_I x_I$ to the superpotential–that's a boundary term.

# Where SUSY is hidden

We may write $Z$ as follows:

$$
\begin{array}{l}
Z = \int [\mathscr{D}x_I]\, e^{-S[\boldsymbol{x}]} \left|\det\frac{\delta\eta_I}{\delta x_J}\right| = \\
\int [\mathscr{D}x_I]\, e^{-S[\boldsymbol{x}]}\, e^{\mathrm{i}\theta_{\mathrm{det}}} \det\left(\delta_{IJ}\frac{d}{dt} + \frac{\partial W}{\partial x_I \partial x_J}\right) = \\
\int [\mathscr{D}x_I][\mathscr{D}\psi_I][\mathscr{D}\chi_I]\, e^{\mathrm{i}\theta_{\mathrm{det}}}\, e^{-S[\boldsymbol{x}] + \int\, dt\, \psi_I\left(\delta_{IJ}\frac{d}{dt} + \frac{\partial W}{\partial x_I \partial x_J}\right)\chi_J} \overset{?}{=} \\
\left\langle e^{\mathrm{i}\theta_{\mathrm{det}}}\right\rangle_{\mathrm{SUSY}} Z_{\mathrm{SUSY}} = \left\langle e^{\mathrm{i}\theta_{\mathrm{det}}}\right\rangle_{\mathrm{SUSY}}
\end{array}
$$

since $Z_{\mathrm{SUSY}} = 1$ (?) It's this factorization that breaks down when the classical potential has multiple minima. But we don't need this factorization–we, just, need to check that the noise fields satisfy Wick's theorem.

This can be done using Monte Carlo simulations of the measure $e^{-S[\boldsymbol{x}]}/Z$, that's perfectly well behaved.

Things get more subtle, however, when the equations $\partial_I W = 0$ don't have real solutions. This leads to SUSY breaking à la Fayet–O'Raifeartaigh. Could it be restored?

# From point attractors to extended structures

One way to evade this conclusion is to realize that the action expresses the "BPS bound" $S[\boldsymbol{x}] \geq 0$, that's saturated when the system of differential equations

$$\frac{dx_I}{dt} + \frac{\partial W}{\partial x_I} = 0$$

is satisfied–and this need not have, only, constant solutions! Non–constant solutions describe extended objects, not points! Another context when this can occur is for deterministically chaotic motion.


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# No superpotential–no problem

So let us consider the general situation where

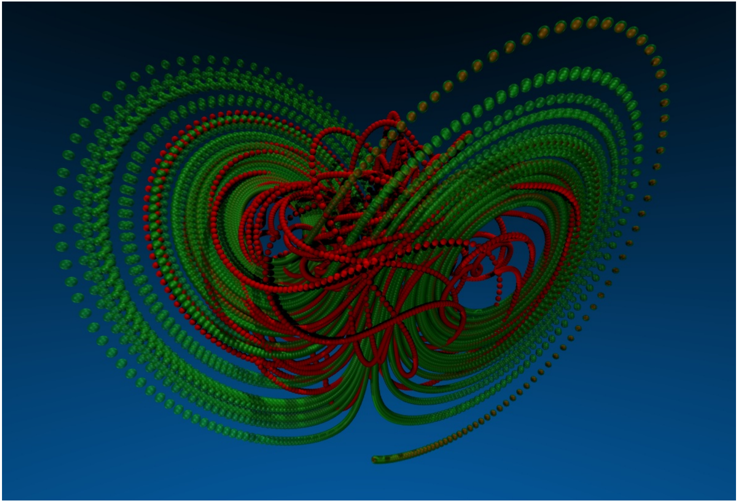
$$\frac{dx_I}{dt} = F_I(\boldsymbol{x})$$

where $I = 1, 2, \ldots, d$.
These equations can be identified with the equations of motion of a particle, whose (Euclidian) action is given by the expression

$$S[\boldsymbol{x}] = \int dt\, \frac{1}{2}\sum_{I=1}^{d} \left(\dot{x}_I - F_I\right)^2 = \int dt\, \left(\frac{1}{2}\left(\dot{\boldsymbol{x}}^2 + F_I^2\right) - \sum_{I=1}^{d} \dot{x}_I F_I(\boldsymbol{x})\right)$$

Since $F_I \neq \partial_I W$, the last term isn't a total derivative!
If $\nabla \cdot \boldsymbol{F} < 0$, the motion is "dissipative" and, if $\nabla \times \boldsymbol{F} \neq 0$, "mixing" may occur and the *stable* solution can be a strange attractor:

# Classical and “quantum” attractors: The Lorenz butterflies

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# How supersymmetry can provide insight into the fluctuations

Even though $F_I \neq \partial_I W$, the Euclidian action is perfectly suited for sampling the configurations of the particle's motion. It's, just, that the attractor isn't a point, but an extended structure. If we do impose periodic boundary conditions, we can check, whether the system is consistently closed by monitoring

$$
\begin{array}{l}
\langle \eta_I(t) \rangle = 0 \\
\left\langle \eta_I(t)\eta_J(t') \right\rangle = \delta_{IJ}\delta(t-t') \\
\langle \eta_{I_1}(t_1)\eta_{I_2}(t_2)\cdots\eta_{I_{2n}}(t_{2n}) \rangle = \\
\sum_\pi \left\langle \eta_{I_{\pi(1)}}(t_{\pi(1)})\eta_{\pi(2)}(t_{\pi(2)}) \right\rangle \cdots \langle \eta_{\pi(2n-1)}(t_{\pi(2n-1)})\eta_{\pi(2n)}(t_{\pi(2n)}) \rangle
\end{array}
$$

where $\eta_I(t) = \dot{x}_I + \partial_I W$. Even when $\langle \eta_I(t) \rangle \neq 0$, it is useful to check that $\eta_I(t) - \langle \eta_I(t) \rangle$ does satisfy the other identities. In particular, if the 2–point function isn't ultra–local, this means that the system isn't consistently closed; there exist additional degrees of freedom.


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# Bloch–Bloembergen

The reason it's of "practical" interest to study the Lorenz equations is that they can describe the motion of a magnetic moment, in a medium, described by Bloch and Bloembergen:

$$\begin{aligned}
\frac{dm_1}{dt} &= (\eta_2 - \eta_3)m_2 m_3 - \beta_2 m_3 + \beta_3 m_2 - \frac{m_1}{\tau_1} \\
\frac{dm_2}{dt} &= (\eta_3 - \eta_1)m_3 m_1 - \beta_3 m_1 + \beta_1 m_3 - \frac{m_2}{\tau_2} \\
\frac{dm_3}{dt} &= (\eta_1 - \eta_2)m_1 m_2 - \beta_1 m_2 + \beta_2 m_1 - \frac{m_3}{\tau_3}
\end{aligned}$$


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# It looks like Bloch–Bloombergen; it's Lorenz

$$\boldsymbol{\eta} = (2, 1, 1) \qquad \boldsymbol{\beta} = (0, 0, \sigma) \qquad \boldsymbol{\tau} = (1/\sigma, 1, 1/b)$$
$$d = -b(r + \sigma)$$

change variables:

$$\boldsymbol{m} = (x, y, z - r - \sigma)$$

The Bloch–Bloembergen equations take the form:

$$\frac{dx}{dt} = \sigma(y - x)$$
$$\frac{dy}{dt} = x(r - z) - y$$
$$\frac{dz}{dt} = xy - bz$$

Attention: There are constants that have been set to 1–so the lattice action requires some care (work in progress).


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- Supersymmetry is relevant for describing the consistent closure (or not!) of systems that display deterministic chaos. The reason is that such systems are, typically, defined by ordinary differential equations, $\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x})$ and these can be repackaged as the equations of motion deduced from the classical action

$$S[\boldsymbol{x}] = \int dt \, \frac{1}{2} \sum_{I=1}^{d} \left(\dot{x}_I - F_I(\boldsymbol{x})\right)^2$$

The fluctuations are, in turn described by the partition function

$$Z = \int [\mathscr{D}x_I] \, e^{-S[\boldsymbol{x}]}$$

whose complete description is provided by the insertion of

$$\left|\det\left(\delta_{IJ}\frac{d}{dt} - \frac{\partial F_I}{\partial x_J}\right)\right|$$

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- and whose presence can be revealed by computing the correlation functions of

$$\eta_I(t) = \dot{x}_I + F_I(\boldsymbol{x})$$

- By Helmholtz/Clebsch-Monge/Hodge decomposition, it's possible to identify a part of the $F_I$ as $\partial_I W$.
- In the non–chaotic phase, the Lorenz equations are known to describe knots (cf. Birman and Williams).
- Imposing *open* boundary conditions means exploring the manifold the "end" of the Lorenz attractor can explore. Could be a toy model for D-branes.
- When deterministic chaos is relevant, the attractors are extended objects, not points.


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Another way of describing the fluctuations of the magnetic moment of nanomagnets is through the Landau–Lifshitz–Gilbert equation

$$\frac{d\boldsymbol{m}}{dt} = \boldsymbol{\omega} \times \boldsymbol{m} + \lambda(\boldsymbol{m} \times \boldsymbol{\Omega}(t)) \times \boldsymbol{m}$$

where $\boldsymbol{\Omega}(t) \equiv \boldsymbol{\omega} + \boldsymbol{\varphi}(t)$, with $\boldsymbol{\varphi}(t)$ drawn from a "colored" noise, viz.

$$\langle \boldsymbol{\varphi}(t) \rangle = \mathbf{0}$$
$$\langle \varphi_I(t) \varphi_J(t') \rangle = \frac{\delta_{IJ} D}{\tau} e^{-\frac{|t-t'|}{\tau}}$$

which leads to an action of a particle on a quite intricate target space geometry–in particular, the dreibeine aren't invertible. (cf. arXiv:1610.01622 for a discussion of this point).
Cf. also, 1404.7774, 1504.06161. For an overview, cf. 1405.0820v2.

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# How about gauge theories?

The gauge group defines a *curved* manifold, so the appropriate description requires multiplicative, not additive, noise. Perturbative approaches probe the tangent space, not the full group manifold.

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Supersymmetry is hiding in plain sight–however its probes aren't easy to find; once found, however, they're easy to compute (at least, numerically) and, hopefully, will be measured in real experiments with quantum matter, that can measure non–linear susceptibilities and the identities that relate them, by expressing consistent closure of the physical system.

For target space supersymmetry, for instance, in four dimensions, the corresponding Nicolai map reads

$$\eta_I = \gamma^{\mu}_{IJ}\frac{\partial\phi_J}{\partial x_\mu} + \frac{\partial W}{\partial\phi_I}$$

where $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ (In Euclidian signature). So it's necessary to have, at the very least, four scalars–which are present in the Standard Model, incidentally–they're, just, repackaged differently. Some of the consequences were worked out, for the 2d WZ model, in arXiv:1712.07045

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