On unfolded off-shell formulation for higher-spin theory
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Higher-Spin Theory

Higher-Spin (HS) theory

- is an interacting theory of massless fields of all spins (including gravity)
- possesses an infinite-dimensional HS gauge symmetry (String Theory as broken phase?)
- lives on Anti-de Sitter background (has no sensible flat limit with HS symmetry unbroken)
- is dual to different boundary vectorial models (Klebanov–Polyakov conjecture)
- is formulated in the form of Vasiliev equations (action is unknown $\Rightarrow$ direct check of AdS/CFT, standard quantization procedure are unavailable)

Unfolded equations

- Unfolded equations [Vasiliev'80]

$$\mathrm{d}\,W^A(x) = G^A(W)\,, \tag{1}$$

where $\mathrm{d} = \mathrm{d}x^{\underline{m}}\partial_{\underline{m}}$ – space-time de Rham differential, $W^A(x)$ – differential forms of unfolded fields.

- From $\mathrm{d}^2 \equiv 0$ a consistency condition follows

$$Q^2 \equiv 0, \qquad Q := G^A \frac{\delta}{\delta W^A}. \tag{2}$$

- Unfolded system is manifestly invariant under gauge transformations

$$\delta W^A = \mathrm{d}\varepsilon^A(x) - \varepsilon^B \frac{\delta G^A(W)}{\delta W^B}. \tag{3}$$

- An example of unfolded system – Minkowski space: 1-forms of vielbein $e^a = e^a{}_{\underline{m}}\mathrm{d}x^{\underline{m}}$ and Lorentz spin-connection $\omega_L^{a,b} = \omega_L^{a,b}{}_{\underline{m}}\mathrm{d}x^{\underline{m}} = -\omega_L^{b,a}$ obeying

$$\mathrm{d}e^a + \omega_L^{a,b} e_b = 0, \tag{4}$$

$$\mathrm{d}\omega_L^{a,b} + \omega_L^{a,}{}_c\,\omega_L^{c,b} e_b = 0. \tag{5}$$

Unfolded scalar field

- Unfolded free massless scalar field:

$$D^L C_{a(n)} = e^b C_{ba(n)}, \tag{6}$$

  where $D^L = \mathrm{d} + \omega_L$, and $C_{a(n)}(x)$ are symmetric rank-n Lorentz tensors.
- In Cartesian coordinates $e^a{}_{\underline{m}} = \delta^a{}_{\underline{m}}$, $\omega_L^{a,b} = 0$

$$C_{a(n)}(x) = \partial_{a_1} ... \partial_{a_n} C(x), \tag{7}$$

  i.e. $C_{a(n)}(x)$, $n > 0$ are descendants, forming the tower of all derivatives of the primary scalar $C(x)$.
- For traceful $C_{a(n)}(x)$ the primary is off mass shell: no constraints on $C(x)$.
- For traceless $C_{a(n)}(x)$ the primary is on mass shell:

$$C_{aa}(x) = \partial_a \partial_a C(x) \Longrightarrow \Box C(x) = 0. \tag{8}$$

- Unfolded gauge-invariant actions are determined by Q-cohomology of the off-shell unfolded system [Vasiliev'06].

Free $d = 4$ spinorial HS equations

- The most elaborated is $4d$ Vasiliev system due to $so(3,1) \approx sl(2,\mathbb{C})$.
- A spectrum of $4d$ HS unfolded fields includes master 1-form ω and master 0-form C depending on spinors $Y^A = (y^\alpha, \bar{y}^{\dot\alpha})$, $\alpha, \dot\alpha = \overline{1,2}$. The tracelessness of HS fields is equivalent to commutativity of Y^A

$$\omega = \sum_{n,m} \frac{1}{n!m!} \omega_{\alpha(n),\dot\beta(m)} y^{\alpha_1} \ldots y^{\alpha_n} \bar{y}^{\dot\beta_1} \ldots \bar{y}^{\dot\beta_m}, \; C = \sum_{n,m} \frac{1}{n!m!} C_{\alpha(n),\dot\beta(m)} y^{\alpha_1} \ldots y^{\alpha_n} \bar{y}^{\dot\beta_1} \ldots \bar{y}^{\dot\beta_m}, \tag{9}$$

spin-s subspace corresponds to $(N+\bar{N})\,\omega = (2s-2)\,\omega$, $|N-\bar{N}|C = 2sC$

- Unfolded system describing free HS fields on Minkowski (Central on-mass-shell theorem, COMST) [Vasiliev'92]:

$$\begin{aligned} & D^L\omega(Y|x) + e^{\alpha\dot\beta} y_\alpha \bar{\partial}_{\dot\beta} \Pi^- \omega(Y|x) + e^{\alpha\dot\beta} \partial_\alpha \bar{y}_{\dot\beta} \Pi^+ \omega(Y|x) = \\ & = \frac{i}{4} \bar{H}^{\dot\alpha\dot\beta} \bar{\partial}_{\dot\alpha} \bar{\partial}_{\dot\beta} C(0,\bar{y}|x) + \frac{i}{4} H^{\alpha\beta} \partial_\alpha \partial_\beta C(y,0|x), \\ & D^L C(Y|x) + i e^{\alpha\dot\beta} \partial_\alpha \bar{\partial}_{\dot\beta} C(Y|x) = 0. \end{aligned} \tag{10}$$

Here $H^{\alpha\beta} = e^\alpha{}_{\dot\gamma} e^{\beta\dot\gamma}$, $\bar{H}^{\dot\alpha\dot\beta} = e_\gamma{}^{\dot\alpha} e^{\gamma\dot\beta}$, Π^+ (Π^-) project onto components with $N > \bar{N}$ ($N < \bar{N}$, respectively), where $N = y^\alpha \partial_\alpha$ and $\bar{N} = \bar{y}^{\dot\alpha} \bar{\partial}_{\dot\alpha}$.

- **Our goal is to build an off-shell extension of this system.**

Sources and off-shell extension

- HS theory formulated in terms of Lorentz tensors [Vasiliev'03] admits an off-shell extension via relaxing tracelessness condition [Sagnotti, Sezgin, Sundell'05].
- In $4d$ spinorial framework traces are absent by construction, so one should look for another solution.
- Switch on external source:

$$\Box\phi(x) = 0 \longrightarrow \Box\phi(x) = J(x)\,. \tag{11}$$

  Treating this as definition of $J(x)$ we get off-shell theory.
- Off-shell extension of COMST = coupling it to HS sources.
- Fronsdal equation for double-traceless spin-s field $\phi_{a(s)}(x)$ coupled to double-traceless external current $J_{a(s)}(x)$ [Fronsdal'78]

$$\Box\phi_{a(s)} - s\partial_a\partial^b\phi_{ba(s-1)} + \frac{s(s-1)}{2}\partial_a\partial_a\phi^b{}_{ba(s-2)} = J_{a(s)}, \tag{12}$$

  with generalized conservation condition

$$\partial^b J_{ba(s-1)} = \frac{(s-1)}{2}\partial_a J^b{}_{ba(s-2)}. \tag{13}$$

Off-shell scalar

- On-shell equations for unfolded scalar $C(Y|x) = \sum_n \frac{1}{(n!)^2} C_{\alpha(n),\dot\beta(n)} \left(y^\alpha \bar{y}^{\dot\beta}\right)^n$

$$D^L C + ie^{\alpha\dot\beta}\partial_\alpha\bar\partial_{\dot\beta}C = 0. \tag{14}$$

- To go off-shell we add source which is another (unconstrained) scalar $D^L C + ie^{\alpha\dot\beta}\partial_\alpha\bar\partial_{\dot\beta}C \sim eJ$.
- To avoid imposing $\Box J = 0$ we need to introduce "source for source" $J^{(1)}$: $D^L J + ie^{\alpha\dot\beta}\partial_\alpha\bar\partial_{\dot\beta}J \sim eJ^{(1)}$, then source $J^{(2)}$ for $J^{(1)}$and so on.
- It is convenient to introduce

$$J(Y|b|x) = \sum_{k=0}^{\infty} \frac{b^k}{k!} J^{(k)}(Y|x). \tag{15}$$

- Then unfolded system for off-shell scalar ($NC = \bar{N}C$, $NJ = \bar{N}J$)

$$D^L C + ie^{\alpha\dot\beta}\partial_\alpha\bar\partial_{\dot\beta}C = ie^{\alpha\dot\beta}y_\alpha\bar{y}_{\dot\beta}\frac{1}{(N+1)(N+2)}J(b=0), \tag{16}$$

$$D^L J + ie^{\alpha\dot\beta}\partial_\alpha\bar\partial_{\dot\beta}J = ie^{\alpha\dot\beta}y_\alpha\bar{y}_{\dot\beta}\frac{1}{(N+1)(N+2)}\frac{\partial}{\partial b}J, \tag{17}$$

- A simple way of imposing higher-order equations:

$$\Box^n C(x) = 0 \Longleftrightarrow J(Y|b|x) = \sum_{k=0}^{n-1} \frac{b^k}{k!} J^{(k)}(Y|x). \tag{18}$$

Unfolding unconstrained traceless tensor field $T_{a(n)}(x)$

- The space of descendants: one- and two-row traceless Young diagrams with the second row with no more than n cells.
- In the language of multispinors this is equivalent to the set of $T_{\alpha(p),\dot\beta(q)}(x)$ with $|p-q|\leq 2n$.
- What operators can appear in unfolded equations?
  - symmetrized derivatives, adding cells: $ie^{\alpha\dot\beta}\partial_\alpha\bar\partial_{\dot\beta}$, $e^{\alpha\dot\beta}y_\alpha\bar\partial_{\dot\beta}\Pi^-$ and $e^{\alpha\dot\beta}\partial_\alpha\bar y_{\dot\beta}\Pi^+$
  - boxes, removing cells: $ie^{\alpha\dot\beta}y_\alpha\bar y_{\dot\beta}$, $e^{\alpha\dot\beta}y_\alpha\bar\partial_{\dot\beta}\Pi^{+0}$ and $e^{\alpha\dot\beta}\partial_\alpha\bar y_{\dot\beta}\Pi^{-0}$; needs b-expansion
  - divergencies, also removing cells; needs an introduction of one more parameter f
- Unconstrained rank-n tensor field corresponds to the following unfolded module

$$T\left(Y|b,f|x\right)=\sum_{p=0}^{\infty}\sum_{q=0}^{n}\frac{f^q}{q!}\frac{b^p}{p!}T^{(p,q)}\left(Y|x\right), \tag{19}$$

$$\left(N+\bar N\right)T\geq 2\left(n-f\frac{\partial}{\partial f}\right)T,\ |N-\bar N|T\leq 2\left(n-f\frac{\partial}{\partial f}\right)T. \tag{20}$$

- Unfolded system describing unconstrained traceless rank-n tensor field

$$D^L T+ie^{\alpha\dot\beta}\partial_\alpha\bar\partial_{\dot\beta}T+e^{\alpha\dot\beta}y_\alpha\bar\partial_{\dot\beta}\tfrac{1}{(N+1)(N+2)}\Pi^-T+e^{\alpha\dot\beta}\partial_\alpha\bar y_{\dot\beta}\tfrac{1}{(\bar N+1)(\bar N+2)}\Pi^+T=$$
$$=-e^{\alpha\dot\beta}y_\alpha\bar\partial_{\dot\beta}\tfrac{1}{(N+1)(N+2)}\Pi^{+0}\left(\tfrac{\partial}{\partial b}+\tfrac{\partial}{\partial f}\right)T-e^{\alpha\dot\beta}\partial_\alpha\bar y_{\dot\beta}\tfrac{1}{(\bar N+1)(\bar N+2)}\Pi^{-0}\left(\tfrac{\partial}{\partial b}+\tfrac{\partial}{\partial f}\right)T+$$
$$+ie^{\alpha\dot\beta}y_\alpha\bar y_{\dot\beta}\tfrac{1}{(N+1)(N+2)(\bar N+1)(\bar N+2)}\left(\tfrac{\partial}{\partial b}+\tfrac{\partial}{\partial f}\right)T. \tag{21}$$

HS sources

- To describe a space of all HS sources we need to introduce last additional parameter m encoding spin to avoid degeneracy

$$T\left(Y|b,f,m|x\right)=\sum_{2s-4=0}^{\infty}\frac{m^{2s}}{(2s)!}\sum_{p=0}^{\infty}\frac{b^p}{p!}\sum_{q=0}^{s-2}\frac{f^q}{q!}T^{(p,q,2s-4)}\left(Y|x\right). \tag{22}$$

- To describe Fronsdal currents we treat T as a space of totally unconstrained traces of currents and complement it with analogous system for traceless components $J\left(Y|b,f,m|x\right)$

$$J\left(Y|b,f,m|x\right)=\sum_{s=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{s}\frac{m^{2s}}{(2s)!}\frac{f^q}{q!}\frac{b^p}{p!}J^{(p,q,2s)}\left(Y|x\right), \tag{23}$$

which have divergences proportional to first symmetrized derivatives of T:

$$\begin{aligned} &D^L J+ie^{\alpha\dot\beta}\partial_\alpha\bar\partial_{\dot\beta}J+e^{\alpha\dot\beta}y_\alpha\bar\partial_{\dot\beta}\tfrac{1}{(N+1)(N+2)}\Pi^-J+e^{\alpha\dot\beta}\partial_\alpha\bar y_{\dot\beta}\tfrac{1}{\left(\bar N+1\right)\left(\bar N+2\right)}\Pi^+J=\\ &=-e^{\alpha\dot\beta}y_\alpha\bar\partial_{\dot\beta}\tfrac{1}{(N+1)(N+2)}\Pi^{+0}\left(\tfrac{\partial}{\partial b}J+T\right)-e^{\alpha\dot\beta}\partial_\alpha\bar y_{\dot\beta}\tfrac{1}{\left(\bar N+1\right)\left(\bar N+2\right)}\Pi^{-0}\left(\tfrac{\partial}{\partial b}J+T\right)+\\ &+ie^{\alpha\dot\beta}y_\alpha\bar y_{\dot\beta}\tfrac{1}{(N+1)(N+2)\left(\bar N+1\right)\left(\bar N+2\right)}\left(\tfrac{\partial}{\partial b}J+T\right). \end{aligned} \tag{24}$$

Off-shell HS equations

- To get an off-shell completion of COMST we couple it to HS currents:

$$D^L\omega(Y|x)+e^{\alpha\dot\beta}y_\alpha\bar\partial_{\dot\beta}\Pi^-\omega(Y|x)+e^{\alpha\dot\beta}\partial_\alpha\bar y_{\dot\beta}\Pi^+\omega(Y|x)=\frac{i}{4}H^{\alpha\beta}\partial_\alpha\partial_\beta C(y,0|x)+$$
$$+\frac{i}{4}\bar H^{\dot\alpha\dot\beta}\bar\partial_{\dot\alpha}\bar\partial_{\dot\beta}\Pi^{+0}\frac{\bar N!\,(\bar N-2)!}{N+\bar N}\oint_{m=0}\frac{dm}{2\pi i m}J\left(\frac{1}{m}y,\frac{1}{m}\bar y\right)|_{b=f=0}+$$
$$+\frac{i}{4}H^{\alpha\beta}y_\alpha y_\beta\frac{\bar N!\,(\bar N+1)!N!}{(N+3)!\,(N+\bar N+4)}\Pi^{+0}\oint_{m=0}\frac{dm}{2\pi i m^5}T\left(\frac{1}{m}y,\frac{1}{m}\bar y\right)|_{b=f=0}+h.c.,\quad(25)$$

$$D^LC(Y|x)+ie^{\alpha\dot\beta}\partial_\alpha\bar\partial_{\dot\beta}C(Y|x)=-ie^{\alpha\dot\beta}y_\alpha\bar y_{\dot\beta}\frac{1}{(N+1)(N+2)}J-$$
$$-e^{\alpha\dot\beta}y_\alpha\bar\partial_{\dot\beta}\frac{1}{(N+1)(N+2)(N-\bar N+2)}\Pi^{+0}\oint_{m=0}\frac{dm}{2\pi i m^3}J\left(\frac{1}{m}y,m\bar y\right)|_{b=f=0}+$$
$$+ie^{\alpha\dot\beta}y_\alpha\bar y_{\dot\beta}\frac{1}{(N+1)(N+2)(N-\bar N)}\Pi^{+}\oint_{m=0}\frac{dm}{2\pi i m}J\left(\frac{1}{m}y,m\bar y\right)|_{b=f=0}+h.c.\quad(26)$$

- All coefficients before HS sources get fixed by consistency.
- Contour integrals ensure that only spin-s current sources spin-s field.
- J and T are 0-forms $\Longrightarrow$ the gauge symmetries did not changed.
- On-shell reduction: $J=T=0$.

Two-point functions

- In standard QFT from classical e.o.m. one finds Schwinger–Dyson equations for the partition

$$\frac{\delta S}{\delta\varphi_n}\left[\varphi_k(x)\right]=0 \implies \frac{\delta S}{\delta\varphi_n}\left[-i\frac{\delta}{\delta J_k(x)}\right]Z=-J_n(x)\,Z. \tag{27}$$

- In Cartesian coordinates unfolded equation for 0-form C

$$\left(D^L+ie^{\alpha\dot\alpha}\partial_\alpha\bar\partial_{\dot\alpha}\right)C(Y|x)=e^{\alpha\dot\alpha}F_{\alpha\dot\alpha}(Y|x)\,, \tag{28}$$

can be solved as

$$C(Y|x)=-\frac{i}{2(2\pi)^8}\int d^4p\int d^4z\,\frac{e^{ip(x-z)}}{p^2}\left(p_{\alpha\dot\alpha}-\partial_\alpha\bar\partial_{\dot\alpha}\right)F^{\alpha\dot\alpha}(Y|z)\,, \tag{29}$$

(analogous formulas exist for 1-forms).

- Then traceless components of Fronsdal fields in the Feynman gauge

$$\phi_{\alpha(s),\dot\alpha(s)}(x)=\frac{i\left((s-1)!\right)^2}{4(2\pi)^8(2s)!}\left(1+s+s^2\right)\int d^4p\int d^4z\,\frac{e^{ip(x-z)}}{p^2}J_{\alpha(s),\dot\alpha(s)}(z)\,. \tag{30}$$

- Treating (30) as the J-derivative of $W=\log Z$ one finds 2-pt. functions for traceless HS fields (k_s is s-dependent constant)

$$\left\langle\phi_{\alpha(s),\dot\alpha(s)}(x_1)\,\phi_{\beta(s),\dot\beta(s)}(x_2)\right\rangle=k_s\int d^4p\,\frac{e^{ip(x_1-x_2)}}{p^2}\left(\epsilon_{\alpha\beta}\right)^s\left(\epsilon_{\dot\alpha\dot\beta}\right)^s. \tag{31}$$

Conclusions

- We found an off-shell completion for unfolded system of free HS fields in $4d$ flat spacetime in spinorial formalism.
- We derived an unfolded description of HS currents.
- Proposed off-shell unfolded system can be reinterpreted as Schwinger–Dyson system allowing one to evaluate two-point functions.
- To do: AdS space, nonlinear theory.