Divergencies of superfield effective actions in $6D$, $\mathcal{N} = (1, 0)$ and $\mathcal{N} = (1, 1)$ SYM theories

Boris Merzlikin

Tomsk State Pedagogical University

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Aims:

- Construction of the background superfield method for $6D, \mathcal{N} = (1,0)$ non-Abelian vector multiplet coupled to hypermultiplet
- Calculation of the one-loop off-shell divergences in vector multiplet and hypermultiplet sectors for arbitrary $\mathcal{N} = (1,0)$ gauge theory
- Analysis of divergences in the $\mathcal{N} = (1,1)$ SYM theory
- Two-loop divergences

The talk is based on:

I.L.Buchbinder, E.A. Ivanov, K.V. Stepanayantz, B.M., arXiv:1907.12302
*Nucl. Phys. B 936 (**2018**)*, arXiv:1808.08446; *Phys. Lett. B 778 (**2018**)*, arXiv:1711.11514;
*Nucl. Phys. B 921 (**2017**)*, arXiv:1704.02530; *JHEP 1701 (**2017**)*, arXiv:1612.03190;
*Phys. Lett. B 763 (**2016**)*, arXiv:1609.00975;

Plan

- General motivations
- Review part
  1. $6D$ supersymmetry
  2. $6D, \mathcal{N} = (1,0)$ harmonic superspace
  3. $6D, \mathcal{N} = (1,0)$ hypermultiplet
  4. $6D, \mathcal{N} = (1,0)$ vector multiplet
  5. Action for vector multiplet coupled to hypermultiplet
  6. $\mathcal{N} = (1,1)$ SYM theory in terms of $\mathcal{N} = (1,0)$ harmonic supersfields
- Background field method
- Structure of one-loop counterterms
- Divergent part of one-loop effective action
- Two-loop results
- Gauge dependence
- Summary

The modern interest to $6D$ supersymmetric gauge theories is stipulated by the following reasons:

- The problem of describing the quantum structure of six-dimensional supersymmetric gauge theories dimensionally reduced from superstrings and the connection of effective action for the D5-branes at low energies with maximally supersymmetric Yang-Mills theory in six dimensions. [**N.Seiberg (1996), E. Witten (1996); N. Seiberg, (1997)**].
- Lagrangian description of the interacting multiple $M5$-branes is related to $6D$, $\mathcal{N}=(2,0)$ supersymmetric gauge theory. The theory includes self-dual non-Abelian antisymmetric tensor and it is not constructed still (see e.g. review [**J. Bagger, N. Lambert, S. Mikhu, C. Papageorgakis (2013)**]).

Motivations

▶ The problem of miraculous cancellation of on-shell divergences in higher dimensional maximally supersymmetric gauge theories (theories with 16 supercharges). All these theories are non-renormalizable by power counting.

- Field limit of superstring amplitude shows that $6D, \mathcal{N} = (1,1)$ SYM theory is on-shell finite at one-loop [**M.B. Green, J.H. Schwarz, L. Brink, (1982)**].
- Analysis based on on-shell supersymmetries, gauge invariance and field redefinitions [**P.S. Howe, K.S. Stelle, (1984), (2003); G. Bossard, P.S. Howe, K.S. Stelle, (2009)**].
- Direct one-loop and two-loop component calculations (mainly in bosonic sector and mainly on-shell) [**E.S. Fradkin, A.A. Tseytlin, (1983); N. Marcus, A. Sagnotti, (1984), (1985)**].
- Direct calculations of scattering amplitudes in $6D$ theory up to five loops and in $D8, 10$ theories up to four loops [**L.V. Bork, D.I. Kazakov, M.V. Kompaniets, D.M. Tolkachev, D.E. Vlasenko, (2015).**]

Results: On-shell divergences in $6D$ theory start at three loops.

Some properties of $6D, \mathcal{N} = (1,1)$ SYM theory

Purpose

To study one- and two-loop divergences of the superfield effective action in $\mathcal{N} = (1,1)$ SYM theory.

Properties

$6D, \mathcal{N} = (1,1)$ SYM theory possesses some properties close or analogous to $4D, \mathcal{N} = 4$ SYM theory.

- The $6D, \mathcal{N} = (1,1)$ SYM theory can be formulated in harmonic superspace as well as the $4D, \mathcal{N} = 4$ SYM theory.
- The $6D, \mathcal{N} = (1,1)$ SYM theory possesses the manifest $\mathcal{N} = (1,0)$ supersymmetry and additional hidden $\mathcal{N} = (0,1)$ supersymmetry analogous to $4D, \mathcal{N} = 4$ SYM theory where there is the manifest $\mathcal{N} = 2$ supersymmetry and additional hidden $\mathcal{N} = 2$ supersymmetry.
- The $6D, \mathcal{N} = (1,1)$ SYM theory as well as the $4D, \mathcal{N} = 4$ SYM theory is characterized by the non-trivial moduli space.
- The $6D, \mathcal{N} = (1,1)$ SYM theory is anomaly free as well as the $4D, \mathcal{N} = 4$ SYM theory and satisfies some non-renormaization theorems.

P.S. Howe, G. Sierra, P.K. Townsend, (1983).

$6D$ Minkowski space

- Coordinates x^M, $M = 0, 1, 2, 3, 4, 5$
- Metric $\eta_{MN} = diag(1, -1, -1, -1, -1, -1)$
- Proper Lorentz group $SO(1, 5)$

Two types of $6D$ Spinors

- Left $(1, 0)$ spinors ψ_a, $a = 1, 2, 3, 4$
- Right $(0, 1)$ spinors ϕ^a, $a = 1, 2, 3, 4$

Dirac matrices

- 8×8 Dirac matrices Γ_M,

$$\Gamma_M \Gamma_N + \Gamma_N \Gamma_M = 2\eta_{MN}$$

- Representation of the Dirac matrices

$$\Gamma_M = \begin{pmatrix} 0 & \tilde{\gamma}_M \\ -\gamma_M & 0 \end{pmatrix},$$

- Antisymmetric 4×4 Pauli-type matrices γ_M and $\tilde{\gamma}_M$,

$$\gamma_M \tilde{\gamma}_N + \gamma_N \tilde{\gamma}_M = -2\eta_{MN}$$

$$(\tilde{\gamma}_M)^{ab} = \frac{1}{2}\epsilon^{abcd}(\gamma_M)_{cd}$$

- Spinor representation of the vectors, $V_{ab} = \frac{1}{2}(\gamma^M)_{ab} V_M$.

$6D$ superalgebra

- Two types of independent supercharges
  $Q^I_a, Q^a_J,\ I = 1, ..., 2m; J = 1, ..., 2n$
- $\mathcal{N} = (m, n)$ supersymmetry
- Anticommutational relations for supercharges

$$\{Q^I_a, Q^K_b\} = 2\Omega^{IK} P_{ab}$$

$$\{Q^a_J, Q^b_L\} = 2\Omega_{JL} P^{ab}$$

  Matrix Ω_{IK} belongs to $USp(2n)$ group (R-symmetry group), $\Omega_{IK}\Omega^{KJ} = \delta^J_I$
- $\mathcal{N} = (1, 0)$ superspace, $I = i$, coordinates $z = (x^M, \theta^a_i), i = 1, 2$
- Basic spinor derivatives

$$D^i_a = \frac{\partial}{\partial\theta^a_i} - i\theta^{ib}\partial_{ab}, \quad \{D^i_a, D^j_b\} = -2i\Omega^{ij}\partial_{ab}$$

Harmonic superspace

$4D$

A.Galperin, E. Ivanov, S. Kalitsyn, V. Ogievetsky, E. Sokatchev, (1984).

A.Galperin, E. Ivanov, V. Ogievetsky, E. Sokatchev, *Harmonic Superspace*, **(2001).**

General purpose:
to formulate $\mathcal{N}=2$ models in terms of unconstrained $\mathcal{N}=2$ superfields.
General idea:
to use the parameters $u^{\pm i}(i=1,2)$ (harmonics) related to $SU(2)$ automorphism group of the $\mathcal{N}=2$ superalgebra and parameterizing the 2-sphere, $u^{+i}u_i^{-}=1$.

It allows to introduce the $\mathcal{N}=2$ superfields and formulate the theory with manifest $\mathcal{N}=2$ supersymmetry in harmonic superspace. Price for this is a presence of extra bosonic variables, harmonics $u^{\pm i}$.

$6D$

P.S. Howe, K.S. Stelle, P.C. West, (1985).
B.M. Zupnik, (1986); (1999).
G. Bossard, E. Ivanov, A. Smilga, (2015).

Note! Pure spinor approach to describe $6D$ SYM theories, [**M. Cederwall, (2018)**].

$\mathcal{N} = (1,0)$ harmonic superspace

- $USp(2) \sim SU(2), I = i$ The same harmonics $u^{\pm i}$ as in $4D, \mathcal{N} = 2$ supersymmetry
- Harmonic $6D, (1,0)$ superspace with coordinates $Z = (x^M, \theta^a_i, u^{\pm i})$
- Analytic basis $Z_{(an)} = (x^M_{(an)}, \theta^{\pm a}, u^{\pm}_i)$,
  $x^M_{(an)} = x^M + \frac{i}{2}\theta^{-a}(\gamma^M)_{ab}\theta^{+b}, \quad \theta^{\pm a} = u^{\pm}_i \theta^{ai}$
  The coordinates $\zeta = (x^M_{(an)}, \theta^{+a}, u^{\pm}_i)$ form a subspace closed under $(1,0)$ supersymmetry
- The harmonic derivatives

$$D^{++} = u^{+i}\frac{\partial}{\partial u^{-i}} + i\theta^{+a}\theta^{+b}\partial_{ab} + \theta^{+a}\frac{\partial}{\partial \theta^{-a}},$$

$$D^{0} = u^{+i}\frac{\partial}{\partial u^{+i}} - u^{-i}\frac{\partial}{\partial u^{-i}} + \theta^{+a}\frac{\partial}{\partial \theta^{+a}} - \theta^{-a}\frac{\partial}{\partial \theta^{-a}}$$

- Spinor derivatives in the analytic basis

$$D^{+}_a = \frac{\partial}{\partial \theta^{-a}}, \quad D^{-}_a = -\frac{\partial}{\partial \theta^{+a}} - 2i\partial_{ab}\theta^{-b}, \quad \{D^{+}_a, D^{-}_b\} = 2i\partial_{ab}$$

- Analytic superfields ϕ do not depend on $\theta^{-\alpha}$, $D^{+}_{\alpha}\phi = 0$

Hypermultiplet in conventional $6D$ superspace

- The $\mathcal{N}=(1,0)$ hypermultiplet is described in conventional $6D,\mathcal{N}=(1,0)$ superspace by the superfields $q^i(x,\theta)$ and their conjugate $\bar{q}_i(x,\theta)$, where $\bar{q}_i=(q^i)^+$ under the constraint

$$D_a^{(i}q^{j)}(x,\theta)=0$$

- On-shell component form of the hypermultiplet

$$q^i(z)=f^i(x)+\theta^{ai}\psi_a(x)$$

where the scalar field $f^i(x)$ and the spinor field $\psi_a(x)$ satisfy the equations $\Box f^i=0,\partial^{ab}\psi_b=0$

Hypermultiplet in harmonic superspace: off-shell Lagrangian formulation

- Off-shell hypermultiplet is described by the analytic superfield $q^+_A(\zeta, u)$, $D^+_a q^+_A(\zeta, u) = 0$, satisfying the reality condition $\widetilde{(q^{+A})} \equiv q^+_A = \varepsilon_{AB} q^{+B} = (q^+ \; - \tilde{q}^+)$. Pauli-Gürsey indices $A, B = 1, 2$
- Off-shell hypermultiplet harmonic superfield contains infinite set of auxiliary fields which vanish on-shell due to the equations of motion

$$D^{++} q^+(\zeta, u) = 0$$

- The equations of motion follow from the action

$$S_{HYPER} = -\frac{1}{2} \int d\zeta^{(-4)} du \; q^{+A} D^{++} q^+_A$$

Here $d\zeta^{(-4)} = d^6x d^4\theta^+$.

The $\mathcal{N}=(1,0)$ non-Abelian vector multiplet in $6D$ conventional superspace

- Gauge covariant derivatives

$$\nabla_{\mathcal{M}} = D_{\mathcal{M}} + i\mathcal{A}_{\mathcal{M}}, \quad [\nabla_{\mathcal{M}}, \nabla_{\mathcal{N}}] = iT_{\mathcal{M}\mathcal{N}}{}^{\mathcal{L}}\nabla_{\mathcal{L}} + F_{\mathcal{M}\mathcal{N}}$$

  with $D_{\mathcal{M}} = (\partial_M, D^i_a)$ being the flat covariant derivatives and $\mathcal{A}_M$ being the gauge connection taking the values in the Lie algebra of the gauge group.

- The constraints

$$F^{ij}_{ab} = 0, \quad \{\nabla^i_a, \nabla^j_b\} = 2i\varepsilon^{ij}\nabla_{ab}, \quad [\nabla^i_c, \nabla_{ab}] = \frac{i}{2}\varepsilon_{abcd}W^{id}$$

  Here W^{ia} is the superfield strength obeying the Bianchi identities.

The constraints are solved in the framework of the harmonic superspace.

The $\mathcal{N}=(1,0)$ non-Abelian vector multiplet in $6D$, $\mathcal{N}=(1,0)$ harmonic superspace

- Harmonic covariant derivative

$$\nabla^{++} = D^{++} + iV^{++}$$

  Connection V^{++}, takes the values in the Lie algebra of the gauge group, this is an unconstrained analytic potential of the $6D, \mathcal{N}=(1,0)$ SYM theory.

- On-shell contents: $V^{++} = \theta^{+a}\theta^{+b}A_{ab} + 2(\theta^{+})_a\lambda^{-a}$, A_{ab} is a vector field, $\lambda^{-a} = \lambda^{ai}u_i^{-}, \lambda^{ai}$ is a spinor field.
- The superfield action of $6D, \mathcal{N}=(1,0)$ SYM theory is written in the form

$$S_{SYM} = \frac{1}{f_0^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\mathrm{tr}\int d^{14}z du_1 \dots du_n \frac{V^{++}(z,u_1)\dots V^{++}(z,u_n)}{(u_1^+u_2^+)\dots(u_n^+u_1^+)}$$

  Here f_0 is the dimensional coupling constant ($[f_0]=-1$)

- Gauge transformations

$$V^{++\prime} = -ie^{i\lambda}D^{++}e^{-i\lambda} + e^{i\lambda}V^{++}e^{-i\lambda}, \qquad q^{+\prime} = e^{i\lambda}q^{+}$$

Theory of $\mathcal{N}=(1,0)$ non-Abelian vector multiplet coupled to hypermultiplet
(chiral anomaly in harmonic superspace formulation calculated by **S. M. Kuzenko, J. Novak and I. B. Samsonov (2017).**]

- Action

$$S[V^{++},q^{+}] = \frac{1}{f_0^2}\sum_{n=2}^{\infty}\frac{(-i)^n}{n}\mathrm{tr}\int d^{14}z\,du_1\dots du_n\,\frac{V^{++}(z,u_1)\dots V^{++}(z,u_n)}{(u_1^+u_2^+)\dots(u_n^+u_1^+)}$$

$$-\int d\zeta^{-4}du\tilde{q}^{+}\nabla^{++}q^{+}$$

- Harmonic covariant derivative

$$\nabla^{++} = D^{++} + iV^{++}$$

- Equations of motion

$$F^{++} - 2if_0^2\tilde{q}^{+}q^{+} = 0\,, \qquad \nabla^{++}q^{+} = 0\,.$$

$$F^{++} = (D^{+})^4V^{--}, \qquad D^{++}V^{--} - D^{--}V^{++} + i[V^{++},V^{--}] = 0$$

$\mathcal{N} = (1,1)$ SYM theory

$\mathcal{N} = (1,1)$ SYM theory can be formulated in terms of $\mathcal{N} = (1,0)$ harmonic superfields as the $\mathcal{N} = (1,0)$ vector multiplet coupled to hypermultiplet in adjoint representation. The theory is manifestly $\mathcal{N} = (1,0)$ supersymmetric and possesses the extra hidden $\mathcal{N} = (0,1)$ supersymmetry.

- Action

$$S[V^{++}, q^+] = S_{SYM}[V^{++}] + S_{HYPER}[q^+, V^{++}]$$

- The action is manifestly $\mathcal{N} = (1,0)$ supersymmetric.
- The action is invariant under the transformations of extra hidden $\mathcal{N} = (0,1)$ supersymmetry

$$\delta V^{++} = \epsilon^+ q^+, \qquad \delta q^+ = -(D^+)^4(\epsilon^- V^{--})$$

where the transformation parameter $\epsilon^{\pm}_A = \epsilon_{aA}\theta^{\pm A}$.

General scheme of calculations

- We start with harmonic superfield formulations of vector multiplet coupled to hypermultiplet.
- Effective action is formulated in the framework of the background field method in harmonic superspace. It provides manifest $\mathcal{N}=(1,0)$ supersymmetry and gauge invariance of effective action under the classical gauge transformations.
- Effective action can be calculated on the base of superfield proper-time technique. It provides preservation of manifest $\mathcal{N}=(1,0)$ supersymmetry and manifest gauge invariance at all steps of calculations.
- The effective action can also be calculated perturbatively on the base of Feynman diagrams in superspace (supergraph technique).
- We study the model where the $\mathcal{N}=(1,0)$ vector multiplet interacts with hypermultiplet in the arbitrary representation of the gauge group. Then, we assume in the final result for one-loop divergences, that this representation is adjoint what corresponds to $\mathcal{N}=(1,1)$ SYM theory.

Aim: construction of gauge invariant effective action, (see, e.g., [**B.DeWitt (1965)**]).
Realization:

- The superfields V^{++}, q^{+} are splitting into the sum of the background V^{++}, Q^{+} and the quantum v^{++}, q^{+} superfields

$$V^{++} \to V^{++} + f_0 v^{++}, \qquad q^{+} \to Q^{+} + q^{+}$$

- The action is expending in a power series in quantum fields.
- The gauge-fixing function are imposed only on quantum superfiled

$$\mathcal{F}_{\tau}^{(+4)} = D^{++} v_{\tau}^{++} = e^{-ib}(\nabla^{++} v^{++}) e^{ib} = e^{-ib} \mathcal{F}^{(+4)} e^{ib} \ ,$$

where $b(z)$ is a background-dependent gauge bridge superfield and τ means τ-frame.

- Faddev-Popov procedure is used. The effective action $\Gamma[V^{++}, Q^{+}]$ gauge invariant under the classical gauge transformations. Background field construction in the case under consideration is analogous to one in $4D, \mathcal{N}=2$ SYM theory [**I.L.Buchbinder, E.I. Buchbinder, S.M. Kuzenko, B.A. Ovrut, (1998)**].

Background field method

- The effective action $\Gamma[V^{++}, Q^{+}]$ is written in terms of path integral

$$e^{i\Gamma[V^{++},Q^{+}]} = \mathrm{Det}^{1/2}\, \widehat{\Box} \int \mathcal{D}v^{++}\, \mathcal{D}q^{+}\, \mathcal{D}\mathbf{b}\, \mathcal{D}\mathbf{c}\, \mathcal{D}\varphi\; e^{iS_{quant}}$$

- The quantum action S_{quant} has the structure

$$S_{quant} = S + S_{GF}[v^{++}, V^{++}] + S_{FP}[\mathbf{b}, \mathbf{c}, v^{++}, V^{++}] + S_{NK}[\varphi, V^{++}].$$

- Gauge fixing action $S_{GF}[v^{++}, V^{++}]$, Faddeed-Popov ghost action $S_{FP}[\mathbf{b}, \mathbf{c}, v^{++}, V^{++}]$, Nelson-Kalosh ghost action $S_{NK}[\varphi, V^{++}]$
- Operator $\widehat{\Box} = \frac{1}{2}(D^{+})^{4}(\nabla^{--})^{2}$

$$\widehat{\Box} = \eta^{MN}\nabla_{M}\nabla_{N} + W^{+a}\nabla_{a}^{-} + F^{++}\nabla^{--} - \frac{1}{2}(\nabla^{--}F^{++})$$

- All ghosts are the analytic superfields

- The gauge fixing action

$$S_{\mathsf{gf}}[v^{++}, V^{++}] = -\frac{1}{2\xi_0}\mathrm{tr}\int d^{14}z du_1 du_2 \frac{v_\tau^{++}(1)v_\tau^{++}(2)}{(u_1^+ u_2^+)^2} + \frac{1}{4\xi_0}\mathrm{tr}\int d^{14}z du\, v_\tau^{++}(D^{--})^2 v_\tau^{++}. \quad (1)$$

- Faddeed-Popov ghosts and Nelson-Kalosh ghost actions

$$S_{FP} = \mathrm{tr}\int d\zeta^{(-4)} du\, b\nabla^{++}(\nabla^{++} + iv^{++})c\,, \quad (2)$$

$$S_{NK} = \frac{1}{2}\mathrm{tr}\int d\zeta^{(-4)} du\, \varphi(\nabla^{++})^2\varphi\,. \quad (3)$$

In what follows we assume gauge fixing parameter $\xi_0 = 1$. The case $\xi_0 \neq 1$ will be considered latter.

Supergraphs

- Perturbation theory can be given in terms of Feynman diagrams formulated in superspace
- Vector multiplet propagator

$$G^{(2,2)}(1|2) = -2\frac{(D_1^+)^4}{\widehat{\Box}_1}\delta^{14}(z_1 - z_2)\delta^{(-2,2)}(u_1, u_2)$$

- Hypermultiplet propagator

$$G^{(1,1)}(1|2) = \frac{(D_1^+)^4(D_2^+)^4}{\widehat{\Box}_1}\frac{\delta^{14}(z_1 - z_2)}{(u_1^+ u_2^+)^3}$$

- Ghost propagators have the analogous structure
- Superspace delta-function

$$\delta^{14}(z_1 - z_2) = \delta^6(x_1 - x_2)\delta^8(\theta_1 - \theta_2)$$

- The vertices are taken from the superfield action as usual

Superficial degree of divergence ω is a total degree of momenta in loop integral.

- One can prove that due to structure of the propagators and the Grassmann delta-functions in the propagators, any supergraph for effective action can be written through the integrals over full $\mathcal{N} = (1, 0)$ superspace and contains only a single integral over $d^8\theta$ (non-renormalization theorem).
- Mass dimensions: $[x] = -1$, $[p] = 1$, $[\int d^6p] = 6$, $[\theta] = -\frac{1}{2}$, $[\int d^8\theta] = 4$, $[q^+] = 1$, $[V^{++}] = 0$.
- After summing all dimensions and using some identities, power counting gives $\omega(G) = 2L - N_Q - \frac{1}{2}N_D$
- N_D is a number of spinor derivarives acting on external lines
- A number of space-time derivatives in the counterterms increases with L. The theory is multiplicatively non-renormalizable.
- One loop approximation $\omega_{1-loop}(G) = 2 - N_Q$
- The possible divergences correspond to $\omega_{1-loop} = 2$ and $\omega_{1-loop} = 0$

Calculations of ω are analogous to ones in $4D, \mathcal{N} = 2$ gauge theory [**I.L. Buchbinder, S.M. Kuzenko, B.A. Ovrut, (1998)**].

Structure of one-loop counterterm

According to the general analysis performed in [**G. Bossard, E. Ivanov, A. Smilga,**(2015)] the logarithmic divergences in the one-loop approximation can be written as

$$\Gamma^{(1)}_{\ln} = \int d\zeta^{(-4)}\,du\left[c_1(F^{++A})^2 + ic_2F^{++A}(\widetilde{q}^{+})^m(T^A)_m{}^n(q^+)_n + c_3\Big((\widetilde{q}^{+})^m(q^+)_m\Big)^2\right], \quad (4)$$

where c_1, c_2, and c_3 are numerical real coefficients.
We consider a possible form of one-loop counterterms on the base of superficial degree of divergences.

- Let $N_Q = 0$, $N_D = 0$, so that ω=2, and we use the dimensional regularization. The corresponding counterterm has to be quadratic in momenta and given by the full $\mathcal{N} = (1,0)$ superspace integral. The only admissible possibility is

$$\Gamma^{(1)}_1 \sim \int d^{14}z du V^{--}\Box V^{++}$$

  After some transformation it coincides with first term in $\Gamma^{(1)}_{div}$, with dimensionless divergent coefficient c_1. Being dimensionless, this coefficient must be proportional to $1/\varepsilon$, where $\varepsilon = d-6$ is a regularization parameter.

- Let $N_Q = 2$, $N_D = 0$ so that $\omega = 0$ and we use the dimensional regularization. The corresponding counterterm has to be momentum independent and given by the full $\mathcal{N} = (1,0)$ superspace integral. The only admissible possibility is

$$\Gamma_2^{(1)} \sim \int d^{14}z du \widetilde{q}^{+} V^{--} q^{+}$$

After some transformation it coincides with second term in $\Gamma_{div}^{(1)}$, with dimensionless divergent coefficient c_2. Being dimensionless, this coefficient must be proportional to $1/\varepsilon$, where $\varepsilon = d - 6$ is a regularization parameter.

- Let $N_Q = 4$, $N_D = 0$ so that $\omega = -2$ and the corresponding supergraph is convergent. It means that $c_3 = 0$ in $\Gamma_{div}^{(1)}$. As a result, all one-loop $(q^{+})^4$ possible contributions to effective action are finite.

Calculating the one-loop divergences of superfield functional determinants is carried out in the framework of proper-time technique (superfield version of Schwinger-De Witt technique). Such technique allows us to preserve the manifest gauge invariance and manifest $\mathcal{N} = (1,0)$ supersymmetry at all steps of calculations.

General scheme of calculations

- Proper-time representation

$$\mathrm{Tr}\ln O \sim \mathrm{Tr}\int_0^\infty \frac{d(is)}{(is)^{1+\varepsilon}} e^{isO_1}\delta(1,2)|_{2=1}$$

- Here s is the proper-time parameter and ε is a parameter of dimensional regularization.
- Typically the $\delta(1,2)$ contains $\delta^8(\theta_1 - \theta_2)$, which vanishes at $\theta_1 = \theta_2$
- Typically the operator O contains some number of spinor derivatives D^+_a, D^-_a which act on the Grassmann delta-functions $\delta^8(\theta_1 - \theta_2)$ and can kill them. Non-zero result will be only if all these δ-functions are killed.
- Only these terms are taking into account which have the pole $\frac{1}{\varepsilon}$ after integration over proper-time.

In the one-loop approximation, the first quantum correction to the classical action, $\Gamma^{(1)}[V^{++}, Q^{+}]$, is given by the quadratic action S_2:

$$
\begin{aligned}
S_2 \;=\; & \frac{1}{2}\int d\zeta^{(-4)}du\, v^{++A}\, \widehat{\square}^{AB}\, v^{++B} + \int d\zeta^{(-4)}du\, \mathbf{b}^{A}(\nabla^{++})^{2\,AB}\mathbf{c}^{B} \\
& +\frac{1}{2}\int d\zeta^{(-4)}du\, \varphi^{A}(\nabla^{++})^{2\,AB}\varphi^{B} - \int d\zeta^{(-4)}du\, \tilde{q}^{+m}(\nabla^{++})_{m}{}^{n}q^{+}_{n} \\
& -\int d\zeta^{(-4)}du\Big\{\widetilde{Q}^{+m} i f_0 (v^{++})^{C}(T^{C})_{m}{}^{n}q^{+}_{n} + \tilde{q}^{+m} i f_0 (v^{++})^{C}(T^{C})_{m}{}^{n}Q^{+}_{n}\Big\},
\end{aligned}
\qquad (5)
$$

We consider the special change of hypermultiplet variables [**I.L. Buchbinder, N.G. Pletnev,** *JHEP* 0704; **S. M. Kuzenko, S. J. Tyler,** *JHEP* 0705] in the one-loop effective action

$$
q^{+}_{n}(1) = h^{+}_{n}(1) - f_0 \int d\zeta^{(-4)}_{2} du_2\, G^{(1,1)}(1|2)_{n}{}^{p}\, i v^{++C}(2)\, (T^{C})_{p}{}^{l}\, Q^{+}_{l}(2)\,, \qquad (6)
$$

with h^{+}_{n} are new independent variable in the path integral describing the hypermultiplet.

The one-loop quantum correction $\Gamma^{(1)}[V^{++}, Q^+]$ to the classical action, which has the following formal expression

$$\begin{aligned}\Gamma^{(1)}[V^{++}, Q] &= \frac{i}{2}\mathsf{Tr}\ln\Big\{ \widehat{\Box}^{AB} - 4f_0^2 \widetilde{Q}^{+\,m} \big(T^A G T^B\big)_m{}^n Q_n^+ \Big\} - \frac{i}{2}\mathsf{Tr}\ln \widehat{\Box}_{\rm Adj} \\ &\quad - i\mathsf{Tr}\ln(\nabla^{++})^2_{\rm Adj} + \frac{i}{2}\mathsf{Tr}\ln(\nabla^{++})^2_{\rm Adj} + i\mathsf{Tr}\ln\nabla^{++}_{\rm R}\,. \end{aligned} \quad (7)$$

The $(F^{++})^2$ is defined by the last three terms in Eq. (16).

$$\Gamma^{(1)}_{F^2}[V^{++}] = -i\mathsf{Tr}\ln\nabla^{++}_{\rm Adj} + i\mathsf{Tr}\ln\nabla^{++}_{\rm R}\,. \quad (8)$$

The divergent contribution reads

$$\Gamma^{(1)}_{F^2} = \frac{C_2 - T(R)}{6(4\pi)^3\varepsilon} \int d\zeta^{(-4)} du\, (F^{++A})^2\,. \quad (9)$$

The hypermultiplet-dependent part $\widetilde{Q}^{+}F^{++}Q^{+}$ of the one-loop counterterm comes out from the first term in (16).

$$\frac{i}{2}\mathsf{Tr}\ln\Big\{\,\widehat{\Box}^{AB}-4f^2\widetilde{Q}^{+\,m}\big(T^AGT^B\big)_m{}^nQ^+_n\Big\}=\frac{i}{2}\mathsf{Tr}\ln\widehat{\Box}+\frac{i}{2}\mathsf{Tr}\ln\Big\{\delta^{AB}-4f^2(\widehat{\Box}^{-1})^{AC}\widetilde{Q}^{+\,m}\big(T^CGT^B\big)_m{}^nQ^+_n\Big\}. \quad (10)$$

We decompose the logarithm up to the first order and compute the functional trace

$$\begin{aligned}\Gamma^{(1)}_{QFQ} &= -2if^2\int d\zeta^{(-4)}du\,\widetilde{Q}^{+\,m}Q^+_n\,(\widehat{\Box}^{-1})^{AB}\big(T^BGT^A\big)_m{}^n\Big|^{2=1}_{\text{div}}\\ &= -2if^2\int d\zeta^{(-4)}du\,\widetilde{Q}^{+\,m}Q^+_n\\ &\quad\times(\widehat{\Box}^{-1})^{AB}\big(T^B\,\widehat{\Box}^{-1}\,T^A\big)_m{}^n(u^+_1u^+_2)\,\delta^6(x_1-x_2)\Big|_{2=1}. \end{aligned} \quad (11)$$

Passing to momentum representation we finally obtain

$$\begin{aligned}\Gamma^{(1)}_{QFQ}[V^{++},Q^{+}] \quad &= \quad -\frac{2if^2}{(4\pi)^3\varepsilon}\int d\zeta^{(-4)}du \\ &\widetilde{Q}^{+\,m}(C_2\delta^l_m - C(R)_m{}^l)(F^{++})^A\ (T^A)_l{}^n\,Q^+_n. \qquad (12)\end{aligned}$$

Summing up the contributions (9) and (12), we finally obtain the total divergent contribution

$$\begin{aligned}\Gamma^{(1)}_{div}[V^{++},Q^{+}] = \frac{C_2 - T(R)}{3(4\pi)^3\varepsilon}\,\mathrm{tr}\int d\zeta^{(-4)}du\,(F^{++})^2 \\ -\frac{2if_0^2}{(4\pi)^3\varepsilon}\int d\zeta^{(-4)}du\,\widetilde{Q}^{+}(C_2 - C(R))F^{++}Q^{+}. \qquad (13)\end{aligned}$$

Results of calculations

$$\Gamma^{(1)}_{div}[V^{++},Q^+] = \frac{C_2 - T(R)}{3(4\pi)^3\varepsilon}\mathrm{tr}\int d\zeta^{(-4)}du(F^{++})^2 - \frac{2if_0^2}{(4\pi)^3\varepsilon}\int d\zeta^{(-4)}du\widetilde{Q}^{+m}(C_2\delta_m{}^n - C(R)_m{}^n)F^{++}Q^+{}_n. \quad (14)$$

- The quantities $C_2, T(R), C(R)$ are defined as follows

$$\mathrm{tr}\,(T^AT^B) = T(R)\delta^{AB}\,, \quad T(Adj) = C_2\,, \quad (T^AT^A)_m{}^n = C(R)\delta_m{}^n.$$

- Results of calculations correspond to analysis done on the base of power counting. The coefficients c_1, c_2 are found. The coefficient $c_3 = 0$ as we expected.
- In $\mathcal{N} = (1,1)$ SYM theory, the hypermultiplet is in the same representation as the vector multiplet. Then $C_2 = T(R) = C(R)$. Then $\Gamma^{(1)}_{div}[V^{++},Q^+] = 0$.
- The result depends on choice of gauge fixing function.

Supergraph calculations

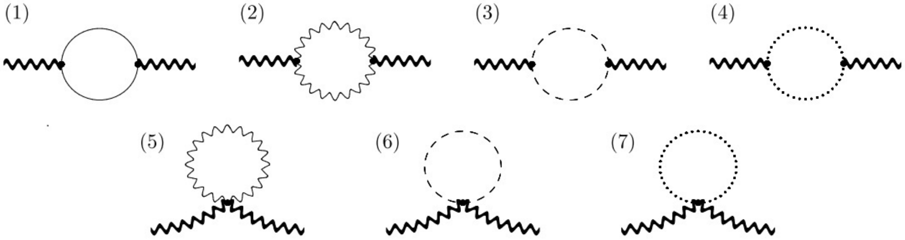

Figure: Supergraphs for two-point function of vector multiplet

Supergraph calculations

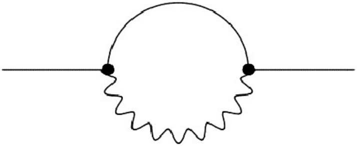

Figure: Supergraph for two-point function of hypermultiplet

Supergraph calculations

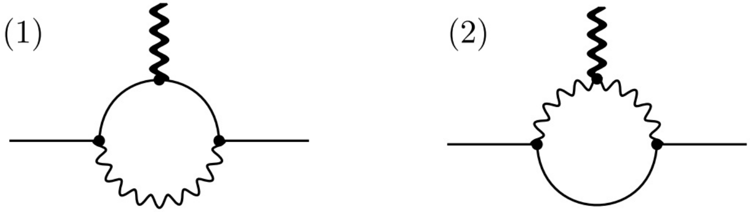

Figure: Supergraphs for three-point vector multiplet - hypermultiplet function

Let we discuss the structure of the two-loop divergences in the hypermultiplet sector.

The total contribution to the divergent part of effective action from two-point supergraphs with the hypermultiplet legs vanishes off shell in case $\mathcal{N}=(1,1)$ SYM theory.

$$4f_0^4\int\frac{d^6pd^8\theta}{(2\pi)^6}\int\frac{du_1\,du_2}{(u_1^+u_2^+)}\Big[q_2^+(-p)_j\Big(C_2-T(R)\Big)C(R)_i{}^j\widetilde{q}_1^+(p)^i\times I_1(p)$$
$$+\widetilde{q}_1^+(p)^i\Big(-C(R)^2+C_2C(R)\Big)_i{}^j q_2^+(-p)_j\times I_2(p)\Big]. \tag{15}$$

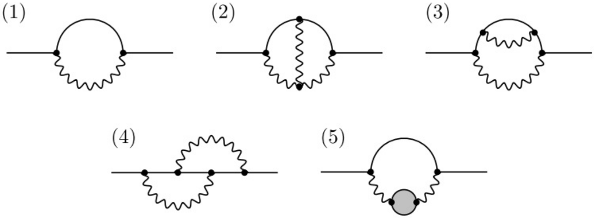

Fig. 1. One- and two-loop diagrams contributing to the two-point Green function of the hypermultiplet.

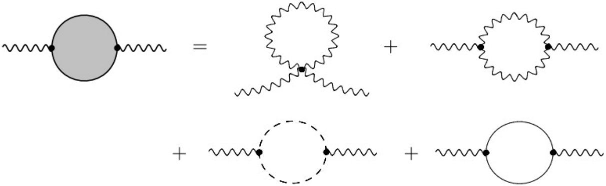

Fig. 2. One-loop subdiagrams entering the two-loop diagram (5) in Fig. 1.

The absence of one-loop divergencies in $\mathcal{N}(1,1)$ SYM theory is known to be a gauge dependent (see [**D. I. Kazakov, JHEP (2003)**]).
The one-loop contribution $\Gamma^{(1)}$ to the superfield effective action in case $\xi_0 \neq 1$ reads

$$\begin{aligned}\Gamma^{(1)}[V^{++}, Q; \xi_0] \quad = \quad & \frac{i}{2} \mathsf{Tr} \ln \Big\{ \frac{1}{\xi_0} \widehat{\Box}^{AB} + \Big(1 - \frac{1}{\xi_0}\Big) \delta^{AB} \frac{(D_1^+)^4}{(u_1^+ u_2^+)^2} \\ & \qquad -4 f_0^2 \widetilde{Q}_1^+ (T^A G^{(1,1)} T^B)(1|2) Q_2^+ \Big\} \\ & - \frac{i}{2} \mathsf{Tr} \ln \widehat{\Box} - \frac{i}{2} \mathsf{Tr} \ln (\nabla^{++}_{\mathrm{Adj}})^2 + i \mathsf{Tr} \ln \nabla^{++}_R \,. \end{aligned} \tag{16}$$

The divergent part of the one-loop effective action can be written as

$$\Gamma^{(1)}_{\infty}[V^{++},Q^+;\xi_0] = \frac{1}{(4\pi)^3\varepsilon}\Big(\frac{1}{3}(C_2 - T(R)) + 2(\xi_0-1)C_2\Big)\,\mathrm{tr}\int d\zeta^{(-4)}du\,(F^{++})^2 - \frac{2i\xi_0\, f_0^2(C_2 - C(R))}{(4\pi)^3\varepsilon}\int d\zeta^{(-4)}du\,\widetilde{Q}^+F^{++}Q^+ - \frac{f_0^2(\xi_0-1)C(R)}{(4\pi)^3\varepsilon}\int d^{14}z\,du\,\Big(\widetilde{Q}^+Q^- - \widetilde{Q}^-Q^+\Big). \qquad (17)$$

In case of on-shell background multiplet we have $Q^- = \nabla^{--}Q^+$ and

$$\int d^{14}z\,du\,\widetilde{Q}^+\,Q^- = i\int d\zeta^{(-4)}\,du\,\widetilde{q}^+\,F^{++}q^+. \qquad (18)$$

The gauge dependence vanishes! (see,e.g., [**A.O. Barvinsky, G.A. Vilkovisky, Phys.Rep. (1985)**]).

Summury

- Background field method in $\mathcal{N} = (1, 0)$ harmonic superspace was developed.
- Superficial degree of divergence was evaluated and structure of one-loop counterterms was studied.
- The one-loop divergences in the theories under consideration were calculated for arbitrary gauge fixing parameter ξ_0.
- The one-loop divergences have been calculated independently with help of $\mathcal{N} = (1, 0)$ supergraphs technique in case $\xi_0 = 1$.
- It is proved that $\mathcal{N} = (1, 1)$ SYM theory is one-loop off-shell finite in case $\xi_0 = 1$.
- Two-loop divergences of the 2-point Green function is calculated.

Outlook

- Superfield Euler-Heisenberg effective action in $6D$, $\mathcal{N} = (1, 1)$ SYM theory.
- Background field method and study of one-loop divergences in $\mathcal{N} = (1, 0)$ SYM theories with high derivatives (component analysis [**E.A. Ivanov, A.V.Smilga, (2006)**] and [**L. Casarina, A. Tseytlin, (2019)**]).