Cubic Interaction for Higher Spins in AdS in the explicit covariant form

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Plan

1. Short Description
2. Setup for Radial Pullback
3. Cubic Self-Interaction and Noncommutative algebra
4. Pullback of the main term of cubic self-interaction
5. Conclusion

Short Description

We present a slightly modified prescription of the radial pullback formalism proposed previously by R. Manvelyan, R. Mkrtchyan and W. Rühl in 2012, where authors investigated possibility to connect the main term of higher spin interaction in flat $d+2$ dimensional space to the main term of interaction in AdS_{d+1} space ignoring all trace and divergent terms but expressed directly through the AdS covariant derivatives and including some curvature corrections. In this paper we succeeded to solve all necessary *recurrence relations* to finalize full radial pullback of the main term of cubic self-interaction for higher spin gauge fields in Fronsdal's formulation from flat to one dimension less AdS_{d+1} space. Nontrivial solutions of recurrence relations lead to the possibility to obtain the full set of AdS_{d+1} dimensional interacting terms with all curvature corrections including trace and divergence terms from any interaction term in $d+2$ dimensional flat space.

Notations and Setup: Coordinate Transformations

$$X^{d+2} = \frac{1}{2}e^{u}[r + \frac{1}{r}(L^2 + x^i x^j \eta_{ij})],$$
$$X^{d+1} = \frac{1}{2}e^{u}[r - \frac{1}{r}(L^2 - x^i x^j \eta_{ij})],$$
$$X^i = e^u L \frac{x^i}{r},$$
$$-e^{2u}L^2 = -(X^{d+2})^2 + (X^{d+1})^2 + X^i X^j \eta_{ij},$$
$$ds^2 = L^2 e^{2u}[-du^2 + \frac{1}{r^2}(dr^2 + dx^i dx^j \eta_{ij})].$$

The restriction $e^u = 1$ leads instead of coordinate transformations to the usual embedding of the Euclidian AdS_{d+1} hypersphere with local coordinates $x^\mu = (x^0, x^i) = (r, x^i)$ into $d+2$ dimensional flat space.

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Jacobian Matrix; Embedding and Frenet Basis

$$E^A_\mu(u, x^\nu) = \frac{\partial X^A}{\partial x^\mu} = e^u e^A_\mu(x^\nu),$$
$$E^A_u(u, x^\nu) = \frac{\partial X^A}{\partial u} = X^A(u, x^\nu) = e^u L n^A(x^\nu),$$
$$E^u_A(u, x) = -\frac{e^{-u}}{L} n_A(x)$$
$$E^\mu_A(u, x) = e^{-u} e^\mu_A(x),$$

where the $d+1$ tangent vectors $\{e^A_\mu(x)\}^d_{\mu=0}$ and one normal vector $n^A(x)$

$$n^A(x) e^B_\mu(x) \eta_{AB} = 0$$
$$n^A(x) n^B(x) \eta_{AB} = -1$$

for embedded AdS_{d+1} space define the standard induced metric $g_{\mu\nu}(x)$ and extrinsic curvature $K_{\mu\nu}(x)$ for our AdS_{d+1} space:

Induced Metric and Extrinsic Curvature

$$g_{\mu\nu}(x) = e^A_\mu(x) e^B_\nu(x) \eta_{AB} = \left(\frac{L}{x^0}\right)^2 \delta_{\mu\nu}$$

and

$$\partial_\mu e^A_\nu(x) = \Gamma^\lambda_{\mu\nu}(g) e^A_\nu(x) + K_{\mu\nu}(x) n^A(x)$$

where

$$\begin{aligned}
\Gamma^\lambda_{\mu\nu}(g) &= \frac{1}{2} g^{\lambda\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\mu g_{\nu\rho} - \partial_\rho g_{\mu\nu}\right), \\
K_{\mu\nu}(x) &= \frac{g_{\mu\nu}(x)}{L}
\end{aligned}$$

Therefor to restrict our flat theory to AdS hypersphere we should first formulate $d+2$ dimensional field theory in the curvilinear coordinates with flat $e^{2u}(AdS_{d+1} \times \mathcal{R}_u)$ metric

$$\begin{aligned}
ds^2 &= e^{2u}[-L^2 du^2 + g_{\mu\nu}(x) dx^\mu dx^\nu] \\
&= G_{uu}(u) du^2 + G_{\mu\nu}(u,x) dx^\mu dx^\nu, \\
G_{uu}(u) &= E^A_u(u,x^\nu) E^B_u(u,x^\nu) \eta_{AB} = X^A X_A = -L^2 e^{2u} \\
G_{\mu\nu} &= E^A_\mu(u,x^\nu) E^B_\nu(u,x^\nu) \eta_{AB} = e^{2u} g_{\mu\nu}(x)
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\end{aligned}$$

Frenet Basis and AdS_{d+1} Riemann Curvature

Differentiation rules for Frenet basis:

$$\begin{aligned}
\nabla_\mu e^A_\nu(x) &= \frac{g_{\mu\nu}(x)}{L} n^A(x) \\
\partial_\mu n^A(x) &= \frac{1}{L} e^A_\mu(x),
\end{aligned}$$

and then taking commutator :

$$[\nabla_\mu, \nabla_\nu] e^A_\lambda = R_{\mu\nu,\lambda}{}^{\rho} e^A_\rho = K_{\lambda[\nu} K^\rho_{\mu]} e^A_\rho$$

we get the standard expression for AdS_{d+1} Riemann curvature and Ricci tensors

$$\begin{aligned}
R_{\mu\nu,\lambda}{}^{\rho} &= -\frac{1}{L^2}(g_{\mu\lambda}\delta^\rho_\nu - g_{\nu\lambda}\delta^\rho_\mu) \\
R_{\mu,\lambda} &= -\frac{d}{L^2} g_{\mu\nu}, \quad R = g^{\mu\lambda} R_{\mu\lambda} = -\frac{d(d+1)}{L^2}
\end{aligned}$$

Turning to higher spins in flat ambient space we should introduce first the following conventions. As usual, we utilize instead of symmetric tensors such as $h^{(s)}_{A_1 A_2 \ldots A_s}(X)$ polynomials homogeneous in the vector a^A of degree s at the base point X

$$h^{(s)}(X;a) = \sum_{A_i} (\prod_{i=1}^{s} a^{A_i}) h^{(s)}_{A_1 A_2 \ldots A_s}(X).$$

Then we can write the symmetrized gradient, trace, and divergence

$$\begin{aligned}
&Grad : h^{(s)}(X;a) \Rightarrow Gradh^{(s+1)}(X;a) = a^A \partial_A h^{(s)}(X;a), \\
&Tr : h^{(s)}(X;a) \Rightarrow Trh^{(s-2)}(X;a) = \frac{1}{s(s-1)} \Box_a h^{(s)}(X;a), \\
&Div : h^{(s)}(X;a) \Rightarrow Divh^{(s-1)}(X;a) = \frac{1}{s} \eta^{AB} \partial_A \partial_{a^B} h^{(s)}(X;a).
\end{aligned}$$

Moreover, we introduce the notation $*_a, *_b, \ldots$ for a contraction in the symmetric spaces of indices a or b

$$*^s_{a^A} \quad = \quad \frac{1}{(s!)^2} \prod_{i=1}^{s} \overleftarrow{\partial}_{a^{A_i}} \eta^{A_i B_i} \overrightarrow{\partial}_{a^{B_i}}$$

So we should fix two important points to perform correct pullback of higher spin theory from flat ambient to one dimensional less AdS space:

- We should fix the ansatz for $d+2$ dimensional HS field in a way to get from one spin s field exactly one spin s field in AdS_{d+1}. The natural condition here send to zero all components normal to the embedded hypersphere

$$n^A h^{(s)}_{AA_2...A_s}(u,x^\nu) \sim X^A(u,x^\nu) h^{(s)}_{AA_2...A_s}(u,x^\nu) = 0 \tag{1}$$

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- Our auxiliary vector a^A is constant in flat space

$$\begin{aligned} a^A =& E^A_u(u, x) a^u(u, x^\nu) + E^A_\mu(u, x) a^\mu(u, x^\nu) \\ =& e^u \left(L n^A(x) a^u(u, x) + e^A_\mu(x) a^\mu(u, x) \right) \end{aligned} \tag{2}$$

$$\partial_B a^A = 0, \tag{3}$$

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- from (3) we obtain the following four relations for derivatives of components $a^u(u,x), a^\mu(u,x)$:

$$\partial_u a^u(u,x) + a^u(u,x) = 0 \tag{4}$$

$$\partial_u a^\mu(u,x) + a^\mu(u,x) = 0 \tag{5}$$

$$\partial_\mu a^u(u,x) + \frac{1}{L^2} a_\mu(u,x) = 0 \tag{6}$$

$$\nabla_\mu a^\nu(u,x) + \delta^\nu_\mu a^u(u,x) = 0 \tag{7}$$

- First two equations we can solve directly:

$$a^u(u,x) = e^{-u} a^u(x) \tag{8}$$
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- Substituting these solutions in $a^A(a^u, a^\mu)$ and using restriction $n^A h^{(s)}_{A\dots} = 0$ we see that in curvilinear coordinates our ansatz leads to the following relation:

$$\begin{aligned} h^{(s)}(X, a^B) &= h^{(s)}_{A_1A_2\dots A_s}(X)a^{A_1}a^{A_2}\dots a^{A_s}|_{X^A=(u,x^\mu),n^A h^{(s)}_{A\dots}=0} \\ &= h^{(s)}_{\mu_1\mu_2\dots\mu_s}(u,x)a^{\mu_1}(x)a^{\mu_2}(x)\dots a^{\mu_s}(x) = h^{(s)}(u,x,a^\mu(x)) \end{aligned} \tag{10}$$

where:

$$h^{(s)}_{\mu_1\mu_2\dots\mu_s}(u,x) = h^{(s)}_{A_1A_2\dots A_s}(u,x)e^{A_1}_{\mu_1}(x)e^{A_2}_{\mu_1}(x)\dots e^{A_s}_{\mu_s}(x) \tag{11}$$

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- Expand auxiliary vectors a^A using Frenet basis for embedded AdS space we have finally, the following embedding rule

$$a^A => Ln^A(x)a^u + e^A_\mu(x)a^\mu \tag{12}$$

The initial gauge variation of order zero for the spin s field is

$$\delta_{(0)} h^{(s)}(X^A; a^A) = s(a^A \partial_A)\epsilon^{(s-1)}(X^A; a^A), \tag{13}$$

with the traceless gauge parameter for the double traceless gauge field

$$\Box_{a^A} \epsilon^{(s-1)}(X^A; a^A) = 0, \tag{14}$$

$$\Box^2_{a^A} h^{(s)}(X^A; a^A) = 0 \tag{15}$$

Then we obtain

$$a^A \partial_A \epsilon^{(s-1)}(X^A; a^A) = e^{-u} \left(a^u(x)\partial_u + a^\mu(x)\partial_{x^\mu} \right) \epsilon^{(s-1)}(u, x; a^\mu(x)) \tag{16}$$

where parameter $\epsilon^{(s-1)}(X^A; a^A)$ obeys to the same type ansatz rule as the $h^{(s)}(X^A; a^A)$

$$\epsilon^{(s-1)}(X^A; a^A) = \epsilon^{(s-1)}(u, x; a^\mu(x)) \tag{17}$$

The next important observation is about derivatives $\partial_{x^\mu} \equiv \partial_\mu$ in respect to AdS_{d+1} coordinates x^μ:

- First note that we mapped scalar object in flat space constructed from X - dependent tensor contracted with constant vectors a^A to the scalar object in curve space constructed from x-dependent tensor contracted with x-dependent vectors $a^\mu(x)$. So as a result we obtain for scalars ordinary derivative ∂_{x^μ}
- To see appearance of the AdS_{d+1} covariant derivatives we should use Leibnitz rule in curve space:

$$\partial_{x^\mu}(T_\nu(x)a^\nu(x)) = \nabla_\mu T_\nu(x)a^\nu(x) + T_\nu(x)\nabla_\mu a^\nu(x)$$
$$= (\nabla_\mu T_\nu(x))a^\nu(x) - T_\mu(x)a^u(x) = (\nabla_\mu T_\nu(x))a^\nu(x) - a^u(x)\frac{\partial}{\partial a^\mu}(T_\nu(x)a^\nu) \quad (18)$$

From this example we see that we should replace the usual derivative with the following operators in Frenet basis:

$$\partial_A \quad => \quad (e^{-u}\partial_u, e^{-u}\partial_\mu), \quad (19)$$
$$\partial_\mu \quad => \quad D_\mu = \nabla_\mu - a^u\partial_{a^\mu} - \frac{a_\mu}{L^2}\partial_{a^u}, \quad (20)$$

where ∇_μ is AdS covariant derivative constructed from the AdS_{d+1} Christoffel symbols with the following action rule:

$$\nabla_\mu h^{(s)}(u,x;a) = \nabla_\mu h_{\mu_1\mu_2\ldots\mu_s}(u,x)a^{\mu_1}a^{\mu_2}\ldots a^{\mu_s}. \quad (21)$$

So from now on we have instead of usual differential operator and coordinate dependent auxiliary vector components "constant" objects a^u and a^μ and covariant derivative operator working on rank s symmetric tensors as operators working in both x and a spaces.

Then we can write:

$$a^A\partial_A\epsilon^{(s-1)}(X^A;a^A) = e^{-u}\left(a^u\partial_u + a^\mu D_\mu\right)\epsilon^{(s-1)}(u,x;a^\mu)$$
$$= e^{-u}\left[a^u(\partial_u - s + 1) + a^\mu\nabla_\mu\right]\epsilon^{(s-1)}(u,x;a^\mu) \quad (22)$$

Using this and restricting the dependence on additional "u" coordinates for all fields and gauge parameters in the following (exponential) way

$$h^{(s)}(u,x^\mu;a^\mu) = e^{\Delta_h u}h^{(s)}(x^\mu;a^\mu), \quad (23)$$
$$\epsilon^{(s-1)}(u,x^\mu;a^\mu) = e^{\Delta_\epsilon u}\epsilon^{(s-1)}(x^\mu;a^\mu), \quad (24)$$

we obtain for $\Delta_h, \Delta_\epsilon$ the following relation:

$$e^{\Delta_h u}\delta h^{(s)}(x^\mu;a^\mu) = e^{(\Delta_\epsilon-1)u}s\left[a^u(\Delta_\epsilon - s + 1) + a^\mu\nabla_\mu\right]\epsilon^{(s-1)}(x;a^\mu). \quad (25)$$

So we see that for getting from gauge transformation in $d+2$ dimensional flat space (13) the correct AdS_{d+1} gauge transformation

$$\delta h^{(s)}(x^\mu;a^\mu) = sa^\mu\nabla_\mu\epsilon^{(s-1)}(x;a^\mu) \quad (26)$$

we should fix the last freedom in our ansatz in unique form

$$\Delta_\epsilon = s-1 \quad (27)$$
$$\Delta_h = \Delta_\epsilon - 1 = s-2 \quad (28)$$

which is in agreement with other consideration.

After all, we can formulate our final prescription for radial pullback in the massless AdS case slightly differs from our reduction formulated in arXiv:1210.7227 and can be summarized by the following three points.

1. Expand auxiliary vectors a^A using Frenet basis for embedded AdS space

$$a^A => Ln^A(x)a^u + e^A_\mu(x)a^\mu \tag{29}$$

2. Replace all derivatives in the following way:

$$\partial_A => e^{-u}\left(-\frac{n_A(x)}{L}\partial_u + e^\mu_A(x)D_\mu\right) \tag{30}$$

where D_μ is

$$D_\mu = \nabla_\mu - a^u\partial_{a^\mu} - \frac{a_\mu}{L^2}\partial_{a^u} \tag{31}$$

3. Restrict the dependence on additional "u" coordinates for all fields to preserve gauge invariants during pullback.

$$h^{(s)}(u, x^\mu; a^\mu) = e^{(s-2)u}h^{(s)}(x^\mu; a^\mu), \tag{32}$$

$$\tag{33}$$

4. Write very simple form of the pullback of star contractions:

$$*^s_{a^A} = \frac{1}{(s!)^2}\prod_{i=1}^{s}\left(-\overleftarrow{\partial}_{a^{u_i}}\overrightarrow{\partial}_{a^{u_i}} + \overleftarrow{\partial}_{a_{\mu_i}}\overrightarrow{\partial}_{a^{\mu_i}}\right) = \sum_{n=0}^{s}\frac{(-1)^n}{\binom{s}{n}} *^n_{a^u} *^{s-n}_{a^\mu}. \tag{34}$$

After some straightforward calculation using our reduction rules we can prove that $d+2$ dimensional gauge invariant Fronsdal tensor

$$\begin{aligned}\mathcal{F}^{(s)}(X^A;a^A) \quad &= \Box_{d+2}h^{(s)}(X^A;a^A) - a^A\partial_A\Big(\partial^B\partial_{a^B}h^{(s)}(X^A;a^A) \\ &-\frac{1}{2}(a^B\partial_B)\Box_{a^A}h^{(s)}(X^A;a^A)\Big),\end{aligned} \tag{35}$$

reduces to the AdS_{d+1} gauge invariant Fronsdal tensor

$$\begin{aligned}&\mathcal{F}^{(s)}(x;a^\mu) = \Box_{d+1}h^{(s)}(x^\mu;a^\mu) \\ &-(a^\mu\nabla_\mu)\Big[(\nabla^\nu\partial_{a^\nu})h^{(s)}(x;a^\mu) - \frac{1}{2}(a^\nu\nabla_\nu)\Box_{a^\mu}h^{(s)}(x;a^\mu)\Big] \\ &-\frac{1}{L^2}[s^2+s(d-5)-2(d-2)]h^{(s)}(x^\mu;a^\mu)) - \frac{1}{L^2}a^\mu a_\mu\Box_{a^\mu}h^{(s)}(x^\mu;a^\mu).\end{aligned} \tag{36}$$

in the following way

$$\mathcal{F}^{(s)}(X^A;a^A) \quad = \quad e^{(s-4)u}\mathcal{F}^{(s)}(x;a^\mu), \tag{37}$$

Supplementing this with the reductions for field and for integration volume:

$$\int d^{d+2}X = \int dud^{d+1}x\sqrt{-G} = L\int dud^{d+1}x\sqrt{g}e^{(d+2)u} \tag{38}$$

we obtain the following reduction rule for Fronsdal actions :

$$S_0[h^{(s)}(X^A; a^A)] \;\; = \;\; \left[L\int due^{(d+2s-4)u}\right] \times S_0[h^{(s)}(x^\mu; a^\mu)], \tag{39}$$

The overall infinite factor

$$\left[L\int due^{(d+2s-4)u}\right], \tag{40}$$

here the same as in arXiv:1210.7227 , where we described prescription to get correct additional AdS correction terms from the full "u" derivative part of interaction terms. This additional terms can be found with insertion of the dimensionless delta function in measure (38)

$$\int d^{d+2}X\delta\left(\frac{\sqrt{-X^2}}{L}-1\right) \tag{41}$$

then full derivative terms will survive only for normal u derivatives:

$$\int d^{d+2}X\delta\left(\frac{\sqrt{-X^2}}{L}-1\right)\partial^A\mathfrak{L}_A = \int d^{d+2}X\delta^{(1)}\left(\frac{\sqrt{-X^2}}{L}-1\right)\frac{X^A}{L^2}E^u_A\mathfrak{L}_{\mathfrak{u}}$$
$$= \int dud^{d+1}x\sqrt{g}e^{(d+2)u}\frac{\delta^{(1)}(e^u-1)}{L}\mathfrak{L}_u \tag{42}$$

Cubic Self-Interaction

We look at the main term in the case of a cubic self-interaction in flat space

$$
\begin{aligned}
\mathcal{L}_I^{main} \;=\; & \sum_{\substack{\alpha,\beta,\gamma \\ \alpha+\beta+\gamma=s}} \binom{s}{\alpha,\beta,\gamma} \int d^{d+2}X \\
& *_a^{\gamma+\alpha} (a^A\partial_{b^A})^{\gamma} (a^B\partial_B)^{\alpha} h^{(s)}(X;b^C) \\
& *_b^{\alpha+\beta} (b^D\partial_{c^D})^{\alpha} (b^E\partial_E)^{\beta} h^{(s)}(X;c^F) \\
& *_c^{\beta+\gamma} (c^G\partial_{a^G})^{\beta} (c^H\partial_H)^{\gamma} h^{(s)}(X;a^K),
\end{aligned} \tag{43}
$$

and see that the main object of cubic interaction above is the bitensorial function

$$
K^{(s)}(Q,n;a^A,b^A;X) = (a^A\partial_{b^A})^Q (a^B\partial_B)^n h^{(s)}(X;b^C). \tag{44}
$$

This term should generate all AdS curvature corrections coming from main term. For that we study these operators in a representation that act on pullback HS field

$$
h^{(s)}(X;b^A)|_{X=X(u,x)} = h^{(s)}(u,x^\mu;b^\mu) = e^{(s-2)u} h^{(s)}(x^\mu;b^\mu). \tag{45}
$$

Then we can obtain these AdS corrections expanding all flat $d+2$ dimensional objects in Frenet basis or in other words in term of $d+1$ dimensional AdS space derivatives and vectors and normal components surviving after applying our ansatz rules:

$$
(a^B\partial_B)^n|_{X=X(u,x)} = \left[e^{-u}(a^u\partial_u + a^\mu D_\mu)\right]^n \tag{46}
$$

$$
a^\mu D_\mu = (a,D) = (a,\nabla) - a^u(a,\partial_a) - b^u(a,\partial_b) - a^2\partial_{a^u} - (a,b)\partial_{b^u} \tag{47}
$$

$$
\text{where} \qquad a^2 = (a,a) = a^\mu a^\nu g_{\mu\nu}(x)
$$

and contracting over all a^u, b^u, c^u.

Noncommutative algebra and a^u stripping

So we must deal with the $d+1$ dimensional expansion for the n'th power of $d+2$ dimensional derivatives (46), where the operator

$$
\begin{aligned}
a^u\partial_u + a^\mu D_\mu &= a^\mu\hat{\nabla}_\mu(g) - R, && (48)\\
\hat{\nabla}_\mu &= \nabla_\mu - b^u\partial_{b^\mu} - b_\mu\partial_{b^u}, && (49)\\
R &= a^u[(a\partial_a) - \partial_u] + a^2\partial_{a^u}, && (50)
\end{aligned}
$$

act on ground states (45). These ground states can be characterized by the total symmetry in the argument and by the fact that they are annihilated by the following operators:

$$
\begin{aligned}
&\mid 0 >= e^{(s-2)u}h^{(s)}(x^\mu; b^\mu) && (51)\\
&\partial_{a^\mu} \mid 0 >= \partial_{a^u} \mid 0 >= \partial_{b^u} \mid 0 >= 0, && (52)\\
&R \mid 0 >= (2-s)a^u \mid 0 > . && (53)
\end{aligned}
$$

The operator of interest is

$$\left[e^{-u}(a,\hat{\nabla}) - e^{-u}R\right]^n, \qquad (54)$$

where in the sequel it is advantageous to write the operator R in the following way

$$R = a^u[(a\partial_a) + a^u\partial_{a^u} - \partial_u] + (a^2 - (a^u)^2)\partial_{a^u} \qquad (55)$$

with the following important algebraic relations:

$$
\begin{aligned}
&[(a\partial_a) + a^u\partial_{a^u}, R] = R, && (56)\\
&[(a\partial_a) + a^u\partial_{a^u}, (a,\hat{\nabla})] = (a,\hat{\nabla}), && (57)\\
&[R, e^{-u}(a,\hat{\nabla})] = 2e^{-u}a^u(a,\hat{\nabla}). && (58)
\end{aligned}
$$

we obtain

$$[(a, e^{-u}\hat{\nabla}) - e^{-u}R]^n \mid 0 >= e^{(s-2-n)u} \sum_{p=0}^{n} (-1)^p (a, \hat{\nabla})^{n-p}$$

$$\sum_{n-p \geq i_p \geq i_{p-1} \geq i_{p-2} \ldots \geq i_1 \geq 0} \phi_{i_p}\phi_{i_{p-1}} \ldots \phi_{i_2}\phi_{i_1} h^{(s)}(x^\mu; b^\mu), \tag{59}$$

where we have ϕ_{i_k} as a very simple "creation" operators

$$\phi_{i_k} = a^u[2(i_k + k) - s] + [a^2 - (a^u)^2]\partial_{a^u}. \tag{60}$$

Now we show how to perform summation in (59) and obtain wanted expansion on the power of a^u to contract after. Introducing notation

$$V^{p+1}(i_{p+1})h^{(s)}(x^\mu; b^\mu) = \sum_{i_{p+1} \geq i_p \geq i_{p-1} \geq i_{p-2} \ldots \geq i_1 \geq 0} \phi_{i_p}\phi_{i_{p-1}} \ldots \phi_{i_2}\phi_{i_1} h^{(s)}(x^\mu; b^\mu), \tag{61}$$

and performing summation over the labels $\{i_k\}|_{k=1}^p$ we should obtain a polynomial in a^u and (a^2) of the form

$$V^{p+1}(i_{p+1}) = \sum_{k=0}^{[\frac{p}{2}]} \xi_k^{p+1}(i_{p+1})(a^2)^k(a^u)^{p-2k}. \tag{62}$$

Considering the last expression as an ansatz for equation

$$V^{p+1}(i_{p+1}) = \sum_{i_{p=0}}^{i_{p+1}} \phi_{i_p} V^p(i_p) \tag{63}$$

we obtain the following recurrence relation for $2p-k$ order polynomials coefficients $\xi_k^{p+1}(i_{p+1}) \sim (i_{p+1})^{2p-k} + \dots$

$$\xi_k^{p+1}(j) = \sum_{i=0}^{j}(2i+p+1+2k-s)\xi_k^p(i) + \sum_{i=0}^{j}(p+1-2k)\xi_{k-1}^p(i) \tag{64}$$

This equation is easier to consider in "differential" form

$$\xi_k^{p+1}(i) - \xi_k^{p+1}(i-1) = (2i+p+1+2k-s)\xi_k^p(i) + (p+1-2k)\xi_{k-1}^p(i) \tag{65}$$

Investigating solutions of latter equation obtained by direct calculation of V^{p+1} for $p = 1, 2, 3, 4, \dots$, we arrive to the following important ansatz for $\xi_k^{p+1}(i)$

$$\xi_k^{p+1}(i) = \frac{1}{(p-2k)!}(i+1)_p(2k+2+i-s)_{p-2k}P_k(i) \tag{66}$$

where $P_k(i) \sim i^k + \dots$ is now *p- independent polynomial of order k* and we introduced Pochhammer symbols

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\dots(a+n-1) \tag{67}$$

Inserting (66) in equation (65) we obtain equation for $P_k(i)$:

$$(i+2k)P_k(i) - iP_k(i-1) = (i+2k-s)P_{k-1}(i) \tag{68}$$

Then after more convenient normalization of our polynomials with additional $2k$ order factor:

$$\mathcal{P}_k(i) \equiv (i+1)_{2k} P_k(i) \tag{69}$$

we arrive to the following simple equation with boundary condition:

$$\mathcal{P}_k(i) - \mathcal{P}_k(i-1) = (i+2k-1)(i+2k-s)\mathcal{P}_{k-1}(i) \tag{70}$$

$$\mathcal{P}_0(i) = P_0(i) = 1 \tag{71}$$

This we can solve in two way: first in the form of multiple sums:

$$\mathcal{P}_k(i) = \sum_{i \geq i_k \geq i_{k-1} \geq i_{k-2} \ldots \geq i_1 \geq 0} \prod_{n=1}^{k} (i_n + 2n - 1)(i_n + 2n - s) \tag{72}$$

or solving differential equation for generating function

$$\mathcal{P}_k(y) \equiv \sum_{i=0}^{\infty} \mathcal{P}_k(i) y^i \tag{73}$$

where we introduced formal variable y with $|y| < 1$ for production of the boundary condition:

$$\mathcal{P}_0(y) = \sum_{i=0}^{\infty} y^i = \frac{1}{1-y} \tag{74}$$

For this generation function, we obtain from recurrence relation the equation

$$(1-y)\mathcal{P}_k(y) = (y\frac{d}{dy} + 2k - 1)(y\frac{d}{dy} + 2k - s)\mathcal{P}_{k-1}(y) \tag{75}$$

Solving recursively and using (74) we can write the solution in the form:

$$\mathcal{P}_k(y) = y^{-(2k+1)} \left[\frac{y^4}{1-y}\frac{d}{dy}y^s\frac{d}{dy}y^{-s}\right]^k \frac{y^2}{1-y} \tag{76}$$

Finally, we can write $\xi_k^{p+1}(i)$ in term of $\mathcal{P}_k(i)$

$$\xi_k^{p+1}(i) = \frac{1}{(p-2k)!}(2k+i+1)_{p-2k}(2k+2+i-s)_{p-2k}\mathcal{P}_k(i) \tag{77}$$

Noncommutative algebra and b^u stripping

To extract exact dependence from b^u and obtain final expressions written directly through the AdS_{d+1} covariant derivatives ∇ we have to evaluate the remaining factors

$$(a,\hat{\nabla})^{n-p} = [(a,\nabla) - b^u(a,\partial_b) - (a,b)\partial_{b^u}]^{n-p}$$
$$= \sum_{\tilde{p}=0}^{n-p}(-1)^{\tilde{p}}\begin{pmatrix} n-p \\ \tilde{p} \end{pmatrix}(a,\nabla)^{n-p-\tilde{p}}(L^+ + L^-)^{\tilde{p}}, \tag{78}$$

where L^+, L^- generate a Lie algebra

$$L^+ = b^u(a,\partial_b), \quad L^- = (a,b)\partial_{b^u}, \tag{79}$$
$$[L^+, L^-] = H = a^2 b^u \partial_{b^u} - (a,b)(a,\partial_b), \tag{80}$$
$$[H, L^\pm] = \pm 2a^2 L^\pm. \tag{81}$$

Representations of this Lie algebra are created from an $(s+1)$-dimensional vector space of "null vectors" $\{\Phi_n(a;b)\}|_{n=0}^{s}$ of "level" n

$$\Phi_n(a;b) = h^{(s)}_{\mu_1,\mu_2,\ldots\mu_s} a^{\mu_1} a^{\mu_2} \ldots a^{\mu_n} b^{\mu_{n+1}} b^{\mu_{n+2}} \ldots b^{\mu_s}, \qquad L^-\Phi_n(a;b) = 0, \tag{82}$$

for any fixed tensor function h^s. From our algebra follows that starting from $\Phi_0(a;b)$ all $\Phi_n(a;b)$ can be produced by application of H

$$H\Phi_0(a;b) = -s(a,b)\Phi_1(a,b), \tag{83}$$
$$H^2\Phi_0(a;b) = [s]_2(a,b)^2\Phi_2(a;b) + sa^2(a,b)\Phi_1(a;b), \tag{84}$$
$$H^3\Phi_0(a;b) = -\{[s]_3(a,b)^3\Phi_3(a;b) + 3[s]_2a^2(a,b)^2\Phi_2(a;b) + s(a^2)^2(a,b)\Phi_1(a;b)\}. \tag{85}$$

The ansatz

$$H^n\Phi_0(a;b) = (-1)^n\sum_{r=1}^{n} A_r^{(n)}[s]_r(a^2)^{n-r}(a,b)^r\Phi_r(a;b), \tag{86}$$

leads to the recurrence relation

$$A_{r-1}^{(n)} + rA_r^{(n)} = A_r^{(n+1)}, \tag{87}$$
$$A_r^{(n)} = 0 \quad \text{for} \quad r > n. \tag{88}$$

The boundary conditions $A_{-1}^{(n)} = 0$ and $A_0^{(0)} = 1$ are assumed. Multiplying by x^r and introducing

$$P_n(x) = \sum_{r=0}^{\infty} A_r^{(n)}x^r \tag{89}$$

we obtain simple differential equation

$$x\frac{d}{dx}\left(e^xP_n(x)\right) = e^xP_{n+1}(x)\,. \tag{90}$$

which we can easily solve since $P_0(x) = 1$.

Iterating n times we find

$$e^x P_n(x) = \left(x \frac{d}{dx}\right)^n e^x \,, \tag{91}$$

or

$$P_n(x) = e^{-x} \left(x \frac{d}{dx}\right)^n e^x \,. \tag{92}$$

Evidently, $P_n(x)$ is a polynomial of order n, which means that $A_r^{(n)} = 0$ for $r > n$.
Finally, we can find a "double" generating function. Introducing

$$Q(x,t) = \sum_{n=0}^{\infty} P_n(x) \, \frac{t^n}{n!} \tag{93}$$

we see that

$$Q(x,t) = e^{-x} e^{tx\frac{d}{dx}} e^x = e^{x(e^t - 1)} \tag{94}$$

where we have explored the fact that the operator $e^{tx\frac{d}{dx}}$ rescales the variable x by the factor e^t. Expanding (94) in x and t we get

$$e^{x(e^t-1)} = \prod_{i=1}^{\infty} \sum_{k_i=0}^{\infty} \frac{x^{k_i} t^{ik_i}}{k_i!(i!)^{k_i}} \tag{95}$$

It is not difficult to get a simple combinatorial formula for $A_r^{(n)}$. Let us denote by $\mathcal{P}(n,r)$ the set of partitions of n into r nonzero parts. The partitions are in one to one correspondence with Young diagrams with n boxes and r rows. An arbitrary partition λ may be represented as $\lambda = 1^{k_1} 2^{k_2} 3^{k_3} \cdots$, where the nonnegative integer k_i indicates the number of rows with length i. For example the partition $8 = 1 + 1 + 3 + 3$ is represented as $1^2 2^0 3^2$, hence $\{k_1, k_2, k_3\} = \{2, 0, 2\}$ and $k_4 = k_5 = \cdots = 0$. The corresponding Young diagram consists of two rows of length 3 and two rows of length 1. For a diagram $\lambda \in \mathcal{P}(n,r)$ let us arbitrarily distribute the integers $1, 2, \cdots n$ among boxes. Let us identify two configurations which differ from each other by permutations of numbers along rows or by permutation of entire rows of same lengths. Evidently, the number of non-equivalent distributions is given by

$$S(\lambda) = \frac{n!}{\prod_{i \geq 1} k_i! (i!)^{k_i}} \tag{96}$$

Now comparing with expansion of the solution of differential equation one easily gets

$$A_n^{(r)} = \sum_{\lambda \in \mathcal{P}(n,r)} S(\lambda) \tag{97}$$

With the help of the basis $\{\Phi_n(a;b)\}_{n=0}^{s}$ of null vectors the representation of the Lie algebra can be constructed as follows.

$$(L^+ + L^-)^{\tilde{p}}\Phi_0(b) = \sum_{\tilde{k}=1}^{[\frac{\tilde{p}}{2}]} (b^u)^{\tilde{p}-2\tilde{k}}(-1)^{\tilde{k}}(a,\partial_b)^{\tilde{p}-2\tilde{k}} W^{\tilde{k}}(a^2, H)\Phi_0(b) \tag{98}$$

Here we recognize that the whole basis $\{\Phi_n(a;b)\}$ of null vectors is produced from $\Phi_0(b)$ by the action of H. With the shorthand

$$\psi_i = iH + [i]_2\, a^2, \tag{99}$$

the result is

$$W^{\tilde{k}}(a^2, H, i_{\tilde{k}+1})\Phi_0(b) = \sum_{i_{\tilde{k}+1}\geq i_{\tilde{k}}\geq i_{\tilde{k}-1}\geq i_{\tilde{k}-2}\ldots\geq i_2\geq i_1\geq 1} \psi_{i_{\tilde{k}}-\tilde{k}+1}\psi_{i_{\tilde{k}-1}-\tilde{k}+2}\psi_{i_{\tilde{k}-2}-\tilde{k}+3}\ldots\psi_{i_2-1}\psi_{i_1}\Phi_0(b). \tag{100}$$

The sum is a homogeneous polynomial of H and a^2 of degree $\tilde{k}$, :

$$W^{\tilde{k}}(a^2, H, i_{\tilde{k}+1}) = \sum_{m=0}^{\tilde{k}} \eta_{\tilde{k}}^{m}(i_{\tilde{k}+1})(a^2)^m H^{\tilde{k}-m} \tag{101}$$

Using this ansatz and doing in the way similar to previous case we derive from

$$W^{\tilde{k}+1}(a^2, H, i_{\tilde{k}+2}) = \sum_{i_{\tilde{k}+1}=1}^{i_{\tilde{k}+2}} \psi_{i_{\tilde{k}+1}-\tilde{k}} W^{\tilde{k}}(a^2, H, i_{\tilde{k}+1}) \tag{102}$$

the following recurrence relation

$$\eta^m_{\tilde{k}+1}(j) = \sum_{i=1}^{j} \left[(i-\tilde{k})\eta^m_{\tilde{k}}(i) + (i-\tilde{k})(i-\tilde{k}-1)\eta^{m-1}_{\tilde{k}}(i) \right] \tag{103}$$

or without summation:

$$\eta^m_{\tilde{k}+1}(i) - \eta^m_{\tilde{k}+1}(i-1) = (i-\tilde{k})\eta^m_{\tilde{k}}(i) + (i-\tilde{k})(i-\tilde{k}-1)\eta^{m-1}_{\tilde{k}}(i) \tag{104}$$

From the other hand we have already extracted b^u dependence and can calculate coefficients $\eta^m_{\tilde{k}}(i_{\tilde{k}+1})$ directly. We see that it is possible to write

$$\eta^m_{\tilde{k}}(\tilde{p}-\tilde{k}) = \eta^m_{\tilde{k}}(i_{\tilde{k}+1})|_{i_{\tilde{k}+1}=\tilde{p}-\tilde{k}} \tag{105}$$

in the following form:

$$
\eta^m_{\tilde{k}}(\tilde{p}-\tilde{k}) = \sum_{\tilde{p}-\tilde{k}\geq i_{\tilde{k}}\geq i_{\tilde{k}-1}\geq i_{\tilde{k}-2}\dots\geq i_2\geq i_1\geq 1} \quad \sum_{\tilde{k}\geq n_m\geq n_{m-1}\geq n_{m-2}\dots\geq n_2\geq n_1\geq 1}
$$

$$
\prod_{l_m=n_m+1}^{\tilde{k}} (i_{l_m} - l_k + 1)[i_{n_m} - n_m + 1]_2 \prod_{l_{m-1}=n_{m-1}+1}^{n_m-1} (i_{l_{m-1}} - l_{m-1} + 1)[i_{n_{m-1}} - n_{m-1} + 1]_2
$$

$$
\dots \prod_{l_2=n_2+1}^{n_3-1} (i_{l_2} - l_2 + 1)[i_{n_2} - n_2 + 1]_2 \prod_{l_1=n_1+1}^{n_2-1} (i_{l_1} - l_1 + 1)[i_{n_1} - n_1 + 1]_2 \prod_{l=1}^{n_1-1} (i_l - l + 1) \tag{106}
$$

This formula means that we should inside of expression for $\eta^0_{\tilde{k}}(\tilde{p}-\tilde{k})$:

$$
\eta^0_{\tilde{k}}(\tilde{p}-\tilde{k}) = \sum_{\tilde{p}-\tilde{k}\geq i_{\tilde{k}}\geq i_{\tilde{k}-1}\geq i_{\tilde{k}-2}\dots\geq i_2\geq i_1\geq 1} \prod_{l=1}^{\tilde{k}} (i_l - l + 1) \tag{107}
$$

replace m brackets $(i_{n_r} - n_r + 1)|^m_{r=1}$ with the m Pochhammers $\{[i_{n_r} - n_r + 1]_2\}|^m_{r=1}$ in all possible ways and then take sums.

Pullback of the main term of cubic self-interaction

Now we can write expression for the whole main interaction term

$$
\mathcal{L}_I^{main} = \int due^{(d+2s-4)u} d^{d+1}x\sqrt{g} \sum_{\substack{\alpha,\beta,\gamma \\ \alpha+\beta+\gamma=s}} \binom{s}{\alpha,\beta,\gamma} \sum_{m_1,p_1,k_1,\tilde{p}_1,\tilde{k}_1}^{\gamma,\alpha,[\frac{p_1}{2}],\alpha-p_1,[\frac{\tilde{p}_1}{2}]} \sum_{m_2,p_2,k_2,\tilde{p}_2,\tilde{k}_2}^{\alpha,\beta,[\frac{p_2}{2}],\beta-p_2,[\frac{\tilde{p}_2}{2}]} \sum_{m_3,p_3,k_3,\tilde{p}_3,\tilde{k}_3}^{\beta,\gamma,[\frac{p_3}{2}],\gamma-p_3,[\frac{\tilde{p}_3}{2}]}
$$
$$
\sum_{n_1,n_2,n_3=0}^{\gamma+\alpha,\alpha+\beta,\beta+\gamma} \frac{(-1)^{n_1+n_2+n_3}}{\binom{\gamma+\alpha}{n_1}\binom{\alpha+\beta}{n_2}\binom{\beta+\gamma}{n_3}} *_{a^u}^{n_1} *_{b^u}^{n_2} *_{c^u}^{n_3} *_{a^\mu}^{\gamma+\alpha-n_1} *_{b^\mu}^{\alpha+\beta-n_2} *_{c^\mu}^{\beta+\gamma-n_3}
$$
$$
(a^u)^{p_1-2k_1+m_1}(b^u)^{\tilde{p}_1-2\tilde{k}_1-m_1}(a,\partial_b)^{\gamma+\tilde{p}_1-2\tilde{k}_1-m_1}(a,\nabla)^{\alpha-p_1-\tilde{p}_1}\Theta[\gamma,\alpha,m_1,p_1,k_1,\tilde{p}_1,\tilde{k}_1,a^2,H_1]h^{(s)}(b^\mu)
$$
$$
(b^u)^{p_2-2k_2+m_2}(c^u)^{\tilde{p}_2-2\tilde{k}_2-m_2}(b,\partial_c)^{\alpha+\tilde{p}_2-2\tilde{k}_2-m_2}(b,\nabla)^{\beta-p_2-\tilde{p}_2}\Theta[\alpha,\beta,m_2,p_2,k_2,\tilde{p}_2,\tilde{k}_2,b^2,H_2]h^{(s)}(c^\mu)
$$
$$
(c^u)^{p_3-2k_3+m_3}(a^u)^{\tilde{p}_3-2\tilde{k}_3-m_3}(c,\partial_a)^{\beta+\tilde{p}_3-2\tilde{k}_3-m_3}(c,\nabla)^{\gamma-p_3-\tilde{p}_3}\Theta[\beta,\gamma,m_3,p_3,k_3,\tilde{p}_3,\tilde{k}_3,c^2,H_3]h^{(s)}(a^\mu) \tag{108}
$$

Now we can contract all non AdS_{d+1} components a^u, b^u, c^u using corresponding "u"-stars from second line of (108). This leads to the following constraints for summation indices:

$$
p_1 - 2k_1 + m_1 = \tilde{p}_3 - 2\tilde{k}_3 - m_3 = n_1 \tag{109}
$$
$$
p_2 - 2k_2 + m_2 = \tilde{p}_1 - 2\tilde{k}_1 - m_1 = n_2 \tag{110}
$$
$$
p_3 - 2k_3 + m_3 = \tilde{p}_2 - 2\tilde{k}_2 - m_2 = n_3 \tag{111}
$$

So we can take summation over $m_i, i = 1,2,3$ with remaining constraints on other variables :

$$
p_1 + \tilde{p}_1 = n_1 + n_2 + 2(k_1 + \tilde{k}_1) \tag{112}
$$
$$
p_2 + \tilde{p}_2 = n_2 + n_3 + 2(k_2 + \tilde{k}_2) \tag{113}
$$
$$
p_3 + \tilde{p}_3 = n_3 + n_1 + 2(k_3 + \tilde{k}_3) \tag{114}
$$

Then to understand better the structure of the derivativesof interaction we can take into account constraints latter constraints and rearrange the summations in the following way

$$\sum_{n_3\geq 0}\sum_{n_2\geq 0}\sum_{n_1\geq 0}(-1)^{n_1+n_2+n_3} = \sum_{N\geq 0}(-1)^N \sum_{\substack{n_1,n_2,n_3\\ \sum n_i=N}}, \tag{115}$$

$$\sum_{\substack{\{p_i,k_i,\tilde{p}_i,\tilde{k}_i\}_{i=1,2,3}\\ p_i+\tilde{p}_i=n_i+n_{i+1}+2(k_i+\tilde{k}_i)}} = \sum_{K\geq 0}\sum_{\substack{\{P_i,K_i\}_{i=1,2,3}\\ P_i=n_i+n_{i+1}+2K_i\\ \sum K_i=K}}\sum_{\substack{\{p_i,k_i,\tilde{p}_i,\tilde{k}_i\}_{i=1,2,3}\\ p_i+\tilde{p}_i=P_i;k_i+\tilde{k}_i=K_i}} \tag{116}$$

where in last equation $\{n_i\} = n_1, n_2, n_3$ with cyclic property $n_4 = n_1$ After that we should introduce instead of α, β, γ new summation variables

$$\tilde{\alpha} = \alpha - n_1 - n_2 - 2K_1 = \alpha - P_1, \tag{117}$$

$$\tilde{\beta} = \beta - n_2 - n_3 - 2K_2 = \beta - P_2, \tag{118}$$

$$\tilde{\gamma} = \gamma - n_3 - n_1 - 2K_3 = \gamma - P_3. \tag{119}$$

with corresponding summation limits and constraints

$$0 \leq \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} \leq s - 2(N + K), \tag{120}$$

$$\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} = s - 2(N + K), \tag{121}$$

$$N = \sum_i n_i; \quad K = \sum_i K_i = \sum_i (k_i + \tilde{k}_i). \tag{122}$$

These transformations lead to the following formula:

Result

These transformations lead to the following formula:

$$\mathcal{L}_I^{main} = \int due^{(d+2s-4)u} d^{d+1}x\sqrt{g} \sum_{N\geq 0}\sum_{K\geq 0} \frac{(-1)^N s!}{(s-2(N+K))!} \sum_{\substack{\tilde\alpha,\tilde\beta,\tilde\gamma \\ \tilde\alpha+\tilde\beta+\tilde\gamma=s-2(N+K)}} \binom{s-2(N+K)}{\tilde\alpha,\tilde\beta,\tilde\gamma}$$
$$\sum_{\substack{\{n_i\}_{i=1,2,3} \\ \sum n_i=N}} \sum_{\substack{\{P_i,K_i\}_{i=1,2,3} \\ P_i=n_i+n_{i+1}+2K_i \\ \sum K_i=K}} \sum_{\substack{\{p_i,k_i,\tilde p_i,\tilde k_i\}_{i=1,2,3} \\ p_i+\tilde p_i=P_i; k_i+\tilde k_i=K_i}} \frac{*_{a^\mu}^{\tilde\gamma+\tilde\alpha+N+2(K_3+K_1)} *_{b^\mu}^{\tilde\alpha+\tilde\beta+N+2(K_1+K_2)} *_{c^\mu}^{\tilde\beta+\tilde\gamma+N+2(K_2+K_3)}}{\binom{\tilde\gamma+\tilde\alpha+N+2(K_3+K_1)+n_1}{n_1}\binom{\tilde\alpha+\tilde\beta+N+2(K_1+K_2)+n_2}{n_2}\binom{\tilde\beta+\tilde\gamma+N+2(K_2+K_3)+n_3}{n_3}}$$
$$(a,\partial_b)^{\tilde\gamma+N+2K_3}(a,\nabla)^{\tilde\alpha}\,\Xi^{2K_1}[\tilde\gamma,\tilde\alpha,n_2,p_1,k_1,\tilde p_1,\tilde k_1,a^2,H_1]h^{(s)}(b^\mu)$$
$$(b,\partial_c)^{\tilde\alpha+N+2K_1}(b,\nabla)^{\tilde\beta}\,\Xi^{2K_2}[\tilde\alpha,\tilde\beta,n_3,p_2,k_2,\tilde p_2,\tilde k_2,b^2,H_2]h^{(s)}(c^\mu)$$
$$(c,\partial_a)^{\tilde\beta+N+2K_2}(c,\nabla)^{\tilde\gamma}\,\Xi^{2K_3}[\tilde\beta,\tilde\gamma,n_1,p_3,k_3,\tilde p_3,\tilde k_3,c^2,H_3]h^{(s)}(a^\mu) \tag{123}$$

where

$$\Xi^{2K_1}[\tilde\gamma,\tilde\alpha,n_2,P_3,p_1,k_1,\tilde p_1,\tilde k_1,a^2,H_1] = \frac{(\tilde\alpha+\tilde p_1)!(a^2)^{k_1}}{(\tilde\gamma+P_3-\tilde p_1+2\tilde k_1+n_2)!}\binom{\tilde p_1-2\tilde k_1}{n_2}\xi_{k_1}^{p_1+1}(\tilde\alpha+\tilde p_1)W^{\tilde k_1}(a^2,H_1)\,, \tag{124}$$
$$\Xi^{2K_2}[\tilde\alpha,\tilde\beta,n_3,P_1,p_2,k_2,\tilde p_2,\tilde k_2,b^2,H_2] = \frac{(\tilde\beta+\tilde p_2)!(a^2)^{k_2}}{(\tilde\alpha+P_1-\tilde p_2+2\tilde k_2+n_3)!}\binom{\tilde p_2-2\tilde k_2}{n_3}\xi_{k_2}^{p_2+1}(\tilde\beta+\tilde p_2)W^{\tilde k_2}(b^2,H_2)\,, \tag{125}$$
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Finalizing our consideration we can write direct $(a^2),(b)^2,(c)^2$ expansion of corresponding Ξ^{2K_i} terms using second recurrence relation

$$(a^2)^{k_1} W^{\tilde{k}_1}(a^2,H_1)h^{(s)}(b^\mu)=\sum_{t_1=0}^{\tilde{k}_1}(-1)^{t_1}\sum_{r_1=1}^{\tilde{k}_1-t_1}\eta^{t_1}_{\tilde{k}_1}(\tilde{p}_1-\tilde{k}_1)A^{\tilde{k}_1-t_1}_{r_1}[s]_{r_1}(a^2)^{K_1-r_1}(a,b)^{r_1}\Phi_{r_1}(a,b) \tag{127}$$

$$(b^2)^{k_2} W^{\tilde{k}_2}(b^2,H_2)h^{(s)}(c^\mu)=\sum_{t_2=0}^{\tilde{k}_2}(-1)^{t_2}\sum_{r_2=1}^{\tilde{k}_2-t_2}\eta^{t_2}_{\tilde{k}_2}(\tilde{p}_2-\tilde{k}_2)A^{\tilde{k}_2-t_2}_{r_2}[s]_{r_2}(b^2)^{K_2-r_2}(b,c)^{r_2}\Phi_{r_2}(b,c) \tag{128}$$

$$(c^2)^{k_3} W^{\tilde{k}_3}(a^3,H_3)h^{(s)}(c^\mu)=\sum_{t_3=0}^{\tilde{k}_3}(-1)^{t_3}\sum_{r_3=1}^{\tilde{k}_3-t_3}\eta^{t_3}_{\tilde{k}_3}(\tilde{p}_3-\tilde{k}_3)A^{\tilde{k}_3-t_3}_{r_3}[s]_{r_3}(c^2)^{K_3-r_3}(c,a)^{r_3}\Phi_{r_3}(c,a) \tag{129}$$

So we see that Ξ^{2K_i} really behave like $a^{2K_1}, b^{2K_2}, c^{2K_3}$ as they should for correct contractions of indices.

Conclusion

- We have constructed all *AdS* corrections including trace and divergence terms to the main term of the cubic self-interaction by a slightly modified method of radial pullback (reduction) proposed in *arXiv:1210.7227* where all quantum fields are carried by a real AdS space and corresponding interaction terms expressed through the covariant *AdS* derivatives.

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- For given spin s and $\Delta_{min} = s$ we derived all curvature correction terms (123) in the form of series of terms with numbers $s - 2(N + K)$ of derivatives, where $0 \leq N + K \leq \frac{s}{2}$. The latter is the number of seized pair of derivatives replaced by corresponding power of $1/L^2$ and K is the sum of power of a^2, b^2, c^2 terms connected with trace and divergent correction terms produced from the main term of interaction after pullback.

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- Now we can expect that the same method can be used for the derivation of the AdS corrections to traces and deDonder terms connected with the main term by Noether's procedure derived for the flat case in *R. M,. K. Mkrtchyan and W. Rühl arXiv:1003.2877*.

Thank you for your attention