$\mathcal{N}$-extended supersymmetric Calogero models

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Main result

We propose a new $\mathcal{N}$-extended supersymmetric A_n Calogero model whose supercharges have the standard form maximally cubic in the fermions. The complexity of the model is encoded in a non-canonical and nonlinear conjugation property of the fermions. Employing the new cubic supercharges, we apply a supersymmetric generalization of a "folding" procedure for $A_{2n-1} \oplus A_1$ to explicitly construct the supercharges and Hamiltonian for arbitrary even-$\mathcal{N}$ supersymmetric extensions of the B_n, C_n and D_n rational Calogero models. We demonstrate that all considered models possess a dynamical superconformal symmetry.
We also constructed $\mathcal{N} = 4$ hyperbolic/trigonometric Calogero models associated with the algebras A_n, B_n, C_n and D_n.

Plan

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Bosonic Calogero models

The Calogero model describes the system of identical particles on the line with inverse-square interaction potentials. Its Hamiltonian is

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{2}\sum_{i\neq j} \frac{g^2}{(x^i - x^j)^2},$$

To make the system bound, one has to include an external harmonic oscillator potential that confines the system:

$$H \to H + \gamma^2 \sum_{i=1}^{n} \left(x^i\right)^2$$

This additional term does not spoil the basic features of the system and leads to an integrable system.
The evident symmetries of the Calogero model are

- Translation invariance: $x^i \to x^i + \mathbf{a}, p_i \to p_i$
- Parity: $x^i \to -x^i, p_i \to -p_i$
- Permutation symmetry

Less trivial symmetry is the dynamical conformal symmetry. Indeed, one may check that the following conserved currents

$$K = \frac{1}{2}\sum_{i=1}^{n}\left(x^i\right)^2 - t\sum_{i=1}^{n} x^i p_i + t^2 H, \quad D = -\frac{1}{2}\sum_{i=1}^{n} x^i p_i + tH$$

form, together with the Hamiltonian H, $so(1,2)$ algebra

$$\{H,K\} = 2D, \quad \{H,D\} = H, \quad \{K,D\} = -K$$

This symmetry means that "real" Calogero model starts with 4-particles. Indeed,

- Due to the Translation invariance the center of mass decouples: $H_n \to H_{n-1}$
- Conformal symmetry dictates the following form of the Hamiltonian $H_{n-1} = \frac{1}{2}p_r^2 + \frac{H_{n-2}}{r^2}$

Thus, the generic conformal mechanical system splits into a “radial” and an “angular” (H_{n-2}) parts.
Note, that

$$\{H_{n-2}, H_{n-1}\} = 0, \quad \{r, H_{n-2}\} = \{p_r, H_{n-2}\} = 0.$$

There are several integrable modifications of the Calogero model:

- The spin-Calogero models with the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{2}\sum_{i\neq j}\frac{\mathbf{tr}\,(S_i S_j)}{(x^i - x^j)^2}.$$

  The S_i are a set of independent classical $U(n)$ spins with the proper Poisson brackets.

- B_n, C_n and D_n Calogero models

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{2}g^2\sum_{i\neq j}\left[\frac{1}{(x^i - x^j)^2} + \frac{1}{(x^i + x^j)^2}\right] + \frac{1}{2}\sum_{i=1}\frac{G^2}{(x^i)^2}$$

- The Sutherland model

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{2}\sum_{i\neq j}\frac{g^2}{\sin^2(x^i - x^j)},$$

- The hyperbolic Sutherland model

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{2}\sum_{i\neq j}\frac{g^2}{\sinh^2(x^i - x^j)}$$

Standard $\mathcal{N}=4$ supersymmetrization

The $N=4$ supersymmetrization of the n-particles Calogero model being the puzzle for a long time. In the standard approach, to construct the supercharges $Q^q, \overline{Q}_b,\; (a,b=1,2)$ forming an $\mathcal{N}=4, d=1$ Poincaré superalgebra

$$\left\{Q^a, Q^b\right\}=0, \quad \left\{\overline{Q}_a, \overline{Q}_b\right\}=0, \quad \left\{Q^a, \overline{Q}_b\right\}=-2\mathrm{i}\, H,$$

for n-dimensional systems one has introduce four fermionic degrees of freedom $\psi^{ai}, \bar{\psi}^j_b$ for each bosonic degrees of freedom x^i.

Let us remind that the irreducible $\mathcal{N}=4$ supermultiplets in $d=1$ have the structure $(k,4,4-k)$ - i.e. k - physical bosons, just four fermions and $4-k$ auxiliary components. Clearly, the natural choice for $\mathcal{N}=4$ supersymmetric n-particles Caogero model is to pick up n $(1,4,3)$ supermultiplets.

The most general ansatz for the standard supercharges reads:

$$Q^a = \sum_i p_i \psi^{ai} + \mathrm{i}\sum_i W_i \psi^{ai} + \mathrm{i}\sum_{i,j,k} F_{ijk}\, \psi^{bi}\, \psi^j_b\, \bar\psi^{ak} + \mathrm{i}\sum_{i,j,k} G_{ijk}\, \psi^{ai}\, \psi^{bj}\, \bar\psi^k_b,$$

$$\overline{Q}_a = \sum_i p_i \bar\psi^i_a - \mathrm{i}\sum_i W_i \bar\psi^i_a + \mathrm{i}\sum_{i,j,k} F_{ijk}\, \bar\psi^i_b\, \bar\psi^{bj}\, \psi^k_a + \mathrm{i}\sum_{i,j,k} G_{ijk}\, \bar\psi^i_a\, \bar\psi^j_b\, \psi^{bk}\ .$$

Here, W_i, F_{ijk} and G_{ijk} are arbitrary, for the time being, real functions depending on n coordinates x^i. In addition, we assume that the functions F_{ijk} and G_{ijk} are symmetric and anti-symmetric over the first two indices, respectively:

$$F_{ijk} = F_{jik}, \qquad G_{ijk} = -G_{jik}.$$

The conditions that these supercharges span an $\mathcal{N}=4$ super Poincaré algebra result in the following equations on the functions involved:

- $G_{ijk} = 0, \qquad F_{ijk} - F_{ikj} = 0 \Rightarrow F_{ijk}$ is totally symmetric,
- $\partial_i F_{jkm} - \partial_j F_{ikm} = 0 \Rightarrow F_{ijk} = \partial_i \partial_j \partial_k F$,
- $\sum_p F_{ikp}F_{jmp} - \sum_p F_{jkp}F_{imp} = 0 \Rightarrow$ WDVV equation

and

- $\partial_i W_j - \partial_j W_i = 0 \qquad \Rightarrow \qquad W_i = \partial_i W$
- $\partial_i \partial_j W + \sum_k F_{ijk}\partial^k W = 0.$

Unfortunately, only 3-particles Calogero model fits this schema.

It seems that a guiding principle was missing for the construction of extended supersymmetric Calogero models. Indeed, while for $n \leq 3$ translation and (super-)conformal symmetry almost completely defines the system, the $n \geq 4$ cases admit a lot of freedom which cannot *a priori* be fixed. In the bosonic case, such a guiding principle exists. The Calogero model as well as its different extensions are closely related with matrix models and can be obtained from them by a reduction procedure (the first results in this direction have been obtained by A. Polychronakos). If we want to employ this principle also for finding extended supersymmetric Calogero models, then the two main steps are

- supersymmetrization of a matrix model
- supersymmetrization of the reduction procedure or proper gauge fixing.

Bosonic Calogero model from hermitian matrices

It is well known that the rational n-particle Calogero model can be obtained by Hamiltonian reduction from the hermitian matrix model. Adapted to our purposes, the procedure reads as follows. One starts from the $su(n)$ spin generalization of the standard Calogero model, as given by

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{2}\sum_{i\neq j}^{n} \frac{\ell_{ij}\ell_{ji}}{(x^i - x^j)^2} \ .$$

The particles are described by their coordinates x^i and momenta p_i together with their internal degrees of freedom encoded in the angular momenta $(\ell_{ij})^\dagger = \ell_{ji}$ with $\sum_i \ell_{ii} = 0$. The non-vanishing Poisson brackets are

$$\left\{x^i, p_j\right\} = \delta^i_j \qquad \text{and} \qquad \{\ell_{ij}, \ell_{km}\} = \mathrm{i}\left(\delta_{im}\ell_{kj} - \delta_{kj}\ell_{im}\right) \ .$$

To get the standard Calogero Hamiltonian one has to reduce the angular sector of the latter, in two steps. Firstly, one (weakly) imposes the constraints

$$\ell_{11} \approx \ell_{22} \approx \ldots \approx \ell_{nn} \approx 0 \ .$$

They commute with the Hamiltonian and with each other, hence are of first class. To resolve them one introduces auxiliary complex variables v_i and $\bar{v}_i = (v_i)^\dagger$ obeying the Poisson brackets

$$\{v_i, \bar{v}_j\} = -\mathrm{i}\, \delta_{ij}$$

and realizes the $su(n)$ generators ℓ_{ij} as

$$\hat{\ell}_{ij} = -v_i \bar{v}_j + \frac{1}{n} \delta_{ij} \sum_{k}^{n} v_k \bar{v}_k \ .$$

Secondly, passing to polar variables r_i and ϕ_i defined as

$$v_i = r_i e^{\mathrm{i}\phi_i} \qquad \text{and} \qquad \bar{v}_i = r_i e^{-\mathrm{i}\phi_i} \qquad \Rightarrow \qquad \{r_i, \phi_j\} = \frac{1}{2r_i} \delta_{ij} \ ,$$

the constraints are resolved by putting

$$r_1 \approx r_2 \approx \ldots \approx r_n \ .$$

Plugging this solution into the Hamiltonian one may additionally fix $n-1$ angles ϕ_i, say

$$\phi_1 \approx \phi_2 \approx \ldots \approx \phi_{n-1} \approx 0 \ .$$

At this stage the $2n$ variables $\{r_i, \phi_i\}$ are reduced to the two variables r_n and ϕ_n. However, the reduced Hamiltonian does not depend on ϕ_n and has the form

$$H_{\rm red} = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{2}\sum_{i\neq j}^{n} \frac{r_n^4}{(x^i - x^j)^2} \ .$$

Therefore

$$\{H_{\rm red}, r_n\} \approx 0 \qquad \text{and} \qquad r_n^2 \approx \text{const} =: g \ ,$$

and the reduced Hamiltonian $H_{\rm red}$ coincides with the standard n-particle rational Calogero Hamiltonian.

In what follows we will construct an $\mathcal{N}$-extended supersymmetric generalization of the $su(n)$ spin Calogero model Hamiltonian and perform the supersymmetric version of the reduction just discussed, finishing with an $\mathcal{N}$-extended supersymmetric Calogero model, for $\mathcal{N} = 2M$ and $M = 1, 2, 3, \ldots$.

$\mathcal{N}$-extended supersymmetric $su(n)$ Calogero model

First step is to clarify what is the minimal number of fermionic variables necessary to realize an $\mathcal{N}=2M$ supersymmetric extension of the $su(n)$ spin-Calogero model. Clearly, as partners to the bosonic coordinates x^i one needs $\mathcal{N}n$ fermions ψ_i^a and $\bar{\psi}_{i\,a}$ with $a=1,2,\ldots M$. However, this is not enough to construct $\mathcal{N}$ supercharges Q^a and $\overline{Q}_b$ which must generate the $\mathcal{N}=2M$ superalgebra

$$\left\{Q^a,\overline{Q}_b\right\}=-2\mathrm{i}\,\delta^a_b\,H \qquad \text{and} \qquad \left\{Q^a,Q^b\right\}=\left\{\overline{Q}_a,\overline{Q}_b\right\}=0\,.$$

The reason is simple: to generate the potential term $\sum_{i\neq j}^n \frac{\ell_{ij}\ell_{ji}}{(x^i-x^j)^2}$ in the Hamiltonian, the supercharges Q^a and $\overline{Q}_b$ must contain the terms

$$\mathrm{i}\sum_{i\neq j}^n \frac{\ell_{ij}\rho^a_{ji}}{x^i-x^j} \qquad \text{and} \qquad -\,\mathrm{i}\sum_{i\neq j}^n \frac{\ell_{ji}\bar{\rho}_{ij\,b}}{x^i-x^j}\,,$$

respectively, where ρ^a_{ij} and $\bar{\rho}_{ij\,a}$ are some additional fermionic variables. These fermions cannot be constructed from ψ_i^a or $\bar{\psi}_{i\,a}$. Hence, we are forced to introduce $\mathcal{N}n(n-1)$ further independent fermions ρ^a_{ij} and $\bar{\rho}_{ij\,a}$ subject to $\rho^a_{ii}=\bar{\rho}_{ii\,a}=0$ for each value of the index i.

In total, we thus utilize $\mathcal{N} n^2$ fermions of type ψ or ρ, which we demand to obey the following Poisson brackets,

$$\{\psi_i^a, \bar{\psi}_{j\,b}\} = -\mathrm{i}\,\delta_b^a \delta_{ij}\ ,\ \{\rho_{ij}^a, \bar{\rho}_{km\,b}\} = -\mathrm{i}\,\delta_b^a \delta_{im}\delta_{jk}\ ,\ \text{with}\ \left(\rho_{ij}^a\right)^\dagger = \bar{\rho}_{ji\,a} \text{ and } \rho_{ii}^a = \bar{\rho}_{ii\,a} = 0\ .$$

The next important ingredient of our construction is the composite object

$$\Pi_{ij} = \sum_{a=1}^{M}\Big[\left(\psi_i^a - \psi_j^a\right)\bar{\rho}_{ij\,a} + \left(\bar{\psi}_{i\,a} - \bar{\psi}_{j\,a}\right)\rho_{ij}^a + \sum_{k=1}^{n}\left(\rho_{ik}^a \bar{\rho}_{kj\,a} + \bar{\rho}_{ik\,a}\rho_{kj}^a\right)\Big] \qquad \Rightarrow \qquad (\Pi_{ij})^\dagger = \Pi_{ji}\ .$$

One may check that the Π_{ij} form an $su(n)$ algebra just like the ℓ_{ij},

$$\{\Pi_{ij}, \Pi_{km}\} = \mathrm{i}\left(\delta_{im}\Pi_{kj} - \delta_{kj}\Pi_{im}\right)\ .$$

It is a matter of straightforward calculation to check that the supercharges

$$Q^a = \sum_{i=1}^{n} p_i \psi_i^a + \mathrm{i} \sum_{i \neq j}^{n} \frac{(\ell_{ij} + \Pi_{ij})\, \rho_{ji}^a}{x^i - x^j} \qquad \text{and} \qquad \overline{Q}_b = \sum_{i=1}^{n} p_i \bar{\psi}_{i\,b} - \mathrm{i} \sum_{i \neq j}^{n} \frac{\bar{\rho}_{ij\,b}\, (\ell_{ji} + \Pi_{ji})}{x^i - x^j}$$

obey the $\mathcal{N}=2M$ superalgebra with the Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + \frac{1}{2} \sum_{i \neq j}^{n} \frac{(\ell_{ij} + \Pi_{ij})\,(\ell_{ji} + \Pi_{ji})}{(x^i - x^j)^2} \,,$$

modulo the first-class constraints

$$\chi_i := \ell_{ii} + \Pi_{ii} \approx 0 \quad \forall\, i \,,$$

with

$$\left\{Q^a, \chi_i\right\} \approx \left\{\overline{Q}_a, \chi_i\right\} \approx \left\{H, \chi_i\right\} \approx \left\{\chi_i, \chi_j\right\} \approx 0 \,.$$

These supercharges Q^a and $\overline{Q}_b$ and the Hamiltonian H describe the $\mathcal{N}=2M$ supersymmetric $su(n)$ spin-Calogero model.

$\mathcal{N}$-extended supersymmetric Calogero models

As we can see above, the supersymmetric analogs $\ell_{ii} + \Pi_{ii} \approx 0$ of the purely bosonic constraints $\ell_{ii} \approx 0$ appear automatically. In terms of the $2n$ polar variables r_i and ϕ_i defined previously, these constraints can be easily resolved as

$$r_k^2 \approx r_n^2 + \Pi_{kk} - \Pi_{nn} \qquad \text{for} \quad k = 1, \ldots, n{-}1 \ .$$

After fixing the residual gauge freedom as

$$\phi_1 \approx \phi_2 \approx \ldots \approx \phi_{n-1} \approx 0 \ ,$$

we obtain the supercharges and Hamiltonian which still obey the $\mathcal{N}{=}2M$ superalgebra and contain only the surviving pair (r_n, ϕ_n) of the originally $2n$ "angular" variables. One may check that the supercharges Q^a and $\overline{Q}_b$ and the Hamiltonian H, with the generators ℓ_{ij} replaced by $\hat{\ell}_{ij}$ and with the above partial solution of the constraints taken into account, perfectly commute with $r_n^2 - \Pi_{nn}$. Thus, the final step of the reduction is to impose the constraint

$$r_n^2 - \Pi_{nn} \approx \text{const} =: g \quad \text{and} \quad \phi_n \approx 0 \ .$$

The previous two relations are the supersymmetric analogs of $r_n^2 \approx \text{const} =: g$. We conclude that the full set of the reduction constraints reads

$$r_i^2 \approx g + \Pi_{ii} \qquad \text{and} \qquad \phi_i \approx 0 \qquad \text{for} \quad i = 1, \ldots, n \ .$$

With these constraints taken into account, our supercharges Q^a and $\overline{Q}_b$ and the Hamiltonian H acquire the form

$$
\begin{aligned}
\widehat{Q}^a &= \sum_{i=1}^{n} p_i \psi_i^a - \mathrm{i} \sum_{i \neq j}^{n} \frac{\left(\sqrt{g+\Pi_{ii}}\sqrt{g+\Pi_{jj}} - \Pi_{ij}\right) \rho_{ji}^a}{x^i - x^j} , \\
\widehat{\overline{Q}}_b &= \sum_{i=1}^{n} p_i \bar{\psi}_{i\, b} + \mathrm{i} \sum_{i \neq j}^{n} \frac{\bar{\rho}_{ij\, b} \left(\sqrt{g+\Pi_{ii}}\sqrt{g+\Pi_{jj}} - \Pi_{ji}\right)}{x^i - x^j} , \\
\widehat{H} &= \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{2}\sum_{i \neq j}^{n} \frac{\left(\sqrt{g+\Pi_{ii}}\sqrt{g+\Pi_{jj}} - \Pi_{ij}\right)\left(\sqrt{g+\Pi_{ii}}\sqrt{g+\Pi_{jj}} - \Pi_{ji}\right)}{\left(x^i - x^j\right)^2} .
\end{aligned}
$$

It is matter of quite lengthy and tedious calculations to check that these supercharges and Hamiltonian form an $\mathcal{N}{=}2M$ superalgebra. The main complication arises from the expressions $\sqrt{g+\Pi_{ii}}$ present in the supercharges and the Hamiltonian. Due to the nilpotent nature of Π_{ij}, the series expansion eventually terminates, but even in the two-particle case with $\mathrm{N}{=}4$ supersymmetry we encounter a lengthy expression,

$$
\sqrt{g+\Pi_{11}} = \sqrt{g}\left(1 + \tfrac{1}{2g}\Pi_{11} - \tfrac{1}{8g^2}\Pi_{11}^2 + \tfrac{1}{16g^3}\Pi_{11}^3 - \tfrac{5}{128g^4}\Pi_{11}^4\right) .
$$

For n particles the series will end with a term proportional to $(\Pi_{ii})^{\mathcal{N}(n-1)}$. Clearly, these terms will generate higher-degree monomials in the fermions, both for the supercharges and for the Hamiltonian.

New basis for the fermions

In terms of the fermions ρ^a_{ij} and $\bar\rho_{ij\,a}$, the supercharges Q^a, $\overline{Q}_a$ and the Hamiltonian H have a rather complicated structure due to the presence of the square roots $\sqrt{g+\Pi_{ii}}$. It is possible to redefine the fermionic components in such a way as to bring the supercharges and Hamiltonian to the standard form, with the fermions appearing maximally cubicly in the supercharges and quartically in the Hamiltonian. The price to pay for this simplicity is a more complicated conjugation rule for the fermions.
The advertized transformation defines new fermions ξ^a_{ij} and $\bar\xi_{ij\,a}$ as follows,

$$\xi^a_{ij} = \frac{1}{\sqrt{g+\Pi_{ii}}}\,\rho^a_{ij}\,\sqrt{g+\Pi_{jj}} \qquad \text{and} \qquad \bar\xi_{ij\,a} = \frac{1}{\sqrt{g+\Pi_{ii}}}\,\bar\rho_{ij\,a}\,\sqrt{g+\Pi_{jj}}.$$

The corresponding fermionic bilinears read

$$\Pi^\xi_{ij} = \sum_{a=1}^{\mathcal{N}/2}\sum_{k=1}^{n}\left(\xi^a_{ik}\bar\xi_{kj\,a} + \bar\xi_{ik\,a}\xi^a_{kj}\right) = \frac{1}{\sqrt{g+\Pi_{ii}}}\,\Pi_{ij}\,\sqrt{g+\Pi_{jj}},$$

so that the diagonal terms remain unchanged,

$$\Pi^\xi_{ii} = \sum_{a=1}^{\mathcal{N}/2}\sum_{k=1}^{n}\left(\xi^a_{ik}\bar\xi_{ki\,a} + \bar\xi_{ik\,a}\xi^a_{ki}\right) = \sum_{a=1}^{\mathcal{N}/2}\sum_{k=1}^{n}\left(\rho^a_{ik}\bar\rho_{ki\,a} + \bar\rho_{ik\,a}\rho^a_{ki}\right) = \Pi_{ii}.$$

The supercharges Q^a and $\overline{Q}_a$, when rewritten in terms of ξ^a_{ij} and $\bar{\xi}_{ij\,a}$, acquire the standard structure

$$Q^a = \sum_{i=1}^n p_i \xi^a_{ii} - \mathrm{i} \sum_{i\neq j}^n \frac{\left(g+\Pi^\xi_{jj}-\Pi^\xi_{ij}\right)\xi^a_{ji}}{x_i - x_j}, \overline{Q}_a = \sum_{i=1}^n p_i \bar{\xi}_{ii\,a} + \mathrm{i} \sum_{i\neq j}^n \frac{\left(g+\Pi^\xi_{ii}-\Pi^\xi_{ji}\right)\bar{\xi}_{ij\,a}}{x_i - x_j} .$$

The rewritten Hamiltonian H also contains maximally four-fermion terms,

$$H = \frac{1}{2}\sum_{i=1}^n p_i^2 + \frac{1}{2}\sum_{i\neq j}^n \frac{\left(g+\Pi^\xi_{jj}-\Pi^\xi_{ij}\right)\left(g+\Pi^\xi_{ii}-\Pi^\xi_{ji}\right)}{\left(x_i - x_j\right)^2} .$$

The complicated structure of the $\mathcal{N}$-extended Calogero models is now hidden in the conjugation rules for the new fermions and, consequentially, for Π^ξ_{ij}. The previous rules

$$\left(\rho^a_{ij}\right)^\dagger = \bar{\rho}_{ji\,a} \qquad \text{and} \qquad \left(\Pi_{ij}\right)^\dagger = \Pi_{ji}$$

induce on ξ^a_{ij}, $\bar{\xi}_{ij\,a}$ and Π^ξ_{ij} the conjugation rules

$$\left(\xi^a_{ij}\right)^\dagger = \frac{g+\Pi^\xi_{jj}}{g+\Pi^\xi_{ii}}\,\bar{\xi}_{ji\,a} \qquad \text{and} \qquad \left(\Pi^\xi_{ij}\right)^\dagger = \frac{g+\Pi^\xi_{jj}}{g+\Pi^\xi_{ii}}\,\Pi^\xi_{ji}.$$

With respect to these new conjugation rules the supercharges Q^a, $\overline{Q}_b$ are conjugated to each other, and the Hamiltonian H is a Hermitian one for an $\mathcal{N}$-extended supersymmetric Calogero model with $A_{n-1} \oplus A_1$ symmetry.

B_n, C_n and D_n supersymmetric Calogero models

One may apply a supersymmetric generalization of the "folding" procedure, which relates the $A_{2n-1} \oplus A_1$ Calogero model with the B_n, C_n and D_n ones. In the bosonic case this reduction imposes the following identification of the $2n$ coordinates (x_i, x_{n+i}) and momenta (p_i, p_{n+i}) for $i = 1, \dots, n$:

$$x_{2n+1-i} = -x_i \qquad \text{and} \qquad p_{2n+1-i} = -p_i \qquad \text{for} \quad i \le n.$$

The fermionic equations of motion suggest that the bosonic constraints should be supplemented by the fermionic identifications

$$\begin{aligned} &\xi^a_{2n+1-i,j} = -\xi^a_{i,2n+1-j} = \xi^a_{2n+1-i,2n+1-j} = -\xi^a_{i,j} \qquad \text{and} \\ &-\bar\xi_{2n+1-i,j\,a} = \bar\xi_{i,2n+1-j\,a} = \bar\xi_{2n+1-i,2n+1-j\,a} = -\bar\xi_{i,j\,a} \qquad \text{for} \quad i,j \le n. \end{aligned}$$

As a consequence, besides the Π^ξ_{ij} new fermionic bilinears $\widetilde\Pi^\xi_{ij}$ appeared

$$\widetilde\Pi^\xi_{ij} = \sum_{a=1}^{\mathcal{N}/2} \sum_{k=1}^{n} \left(\xi^a_{ik}\bar\xi_{kj\,a} - \bar\xi_{ik\,a}\xi^a_{kj} \right).$$

One may check that Π^ξ_{ij} and $\widetilde\Pi^\xi_{ij}$ form an $s(u(n) \oplus u(n))$ algebra (remember that $\sum_i \Pi^\xi_{ii} = 0$),

$$\left\{\Pi^\xi_{ij}, \Pi^\xi_{km}\right\} = \left\{\widetilde\Pi^\xi_{ij}, \widetilde\Pi^\xi_{km}\right\} = \mathrm{i}\left(\delta_{im}\Pi^\xi_{kj} - \delta_{kj}\Pi^\xi_{im}\right) \qquad \text{and} \qquad \left\{\Pi^\xi_{ij}, \widetilde\Pi^\xi_{km}\right\} = \mathrm{i}\left(\delta_{im}\widetilde\Pi^\xi_{kj} - \delta_{kj}\widetilde\Pi^\xi_{im}\right).$$

After taking care of these subtleties, one finally arrives at the supercharges

$$
\begin{aligned}
\mathcal{Q}^a &= \sum_i^n p_i \xi_{ii}^a - \mathrm{i} \sum_{i\neq j}^n \frac{\left(\Pi_{jj}^\xi - \Pi_{ij}^\xi\right) \xi_{ji}^a}{x_i - x_j} + \mathrm{i} \sum_{i,j}^n \frac{\left(\Pi_{jj}^\xi + \widetilde{\Pi}_{ij}^\xi\right) \xi_{ji}^a}{x_i + x_j} + \\
&\quad \mathrm{i}\, \frac{g}{2} \sum_{i\neq j}^n \left(\frac{\xi_{ij}^a - \xi_{ji}^a}{x_i - x_j} + \frac{\xi_{ij}^a + \xi_{ji}^a}{x_i + x_j} \right) + \mathrm{i}\, g' \sum_i^n \frac{\xi_{ii}^a}{x_i} \qquad \text{and} \\
\overline{\mathcal{Q}}_a &= \sum_i^n p_i \bar{\xi}_{ii\,a} - \mathrm{i} \sum_{i\neq j}^n \frac{\left(\Pi_{jj}^\xi - \Pi_{ij}^\xi\right) \bar{\xi}_{ji\,a}}{x_i - x_j} - \mathrm{i} \sum_{i,j}^n \frac{\left(\Pi_{jj}^\xi + \widetilde{\Pi}_{ij}^\xi\right) \bar{\xi}_{ji\,a}}{x_i + x_j} + \\
&\quad \mathrm{i}\, \frac{g}{2} \sum_{i\neq j}^n \left(\frac{\bar{\xi}_{ij\,a} - \bar{\xi}_{ji\,a}}{x_i - x_j} - \frac{\bar{\xi}_{ij\,a} + \bar{\xi}_{ji\,a}}{x_i + x_j} \right) - \mathrm{i}\, g' \sum_i^n \frac{\bar{\xi}_{ii\,a}}{x_i}
\end{aligned}
$$

which, together with the Hamiltonian

$$
\begin{aligned}
\mathcal{H} &= \frac{1}{2} \sum_i^n p_i^2 + \frac{1}{2} \sum_{i\neq j}^n \left[\frac{\left(g + \Pi_{jj}^\xi - \Pi_{ij}^\xi\right)\left(g + \Pi_{ii}^\xi - \Pi_{ji}^\xi\right)}{(x_i - x_j)^2} + \frac{\left(g + \Pi_{jj}^\xi + \widetilde{\Pi}_{ij}^\xi\right)\left(g + \Pi_{ii}^\xi + \widetilde{\Pi}_{ji}^\xi\right)}{(x_i + x_j)^2} \right] \\
&\quad + \frac{1}{8} \sum_i^n \frac{\left(2g' + \Pi_{ii}^\xi + \widetilde{\Pi}_{ii}^\xi\right)\left(2g' + \Pi_{ii}^\xi + \widetilde{\Pi}_{ii}^\xi\right)}{x_i^2}\,, \qquad (1)
\end{aligned}
$$

generate an $\mathcal{N}$-extended super-Poincaré algebra.

The bosonic part of this Hamiltonian has the standard form for the B_n, C_n and D_n Calogero models,

$$\mathcal{H}_{\text{bos}} = \frac{1}{2}\sum_{i}^{n} p_i^2 + \frac{g^2}{2}\sum_{i\neq j}^{n}\left[\frac{1}{(x_i - x_j)^2} + \frac{1}{(x_i + x_j)^2}\right] + \frac{g'^2}{2}\sum_{i}^{n}\frac{1}{x_i^2}.$$

The constructed systems possess the dynamical superconformal symmetries: $Osp(\mathcal{N}/2|1)$ for the A_n models and $su(1,1|\mathcal{N}/2)$ for B_n, C_n and D_n models. Summarizing, we developed the Hamiltonian description of the $\mathcal{N}$-extended supersymmetric rational Calogero models. The crucial new feature is a particular redefinition of the fermionic matrix degrees of freedom ρ_{ij}^a and $\bar{\rho}_{ij\,a}$ accompanying the n bosonic coordinates of the rational Calogero models. In terms of the new fermions ξ_{ij}^a and $\bar{\xi}_{ij\,a}$ the supercharges forming an $\mathcal{N}$-extended super-Poincaré algebra are at most cubic in the fermions, i.e. they acquire the standard structure common to almost all known supersymmetric mechanics. The complicated structure of the initial supercharges and Hamiltonian got traded for a quite complicated conjugation rule, which is an almost symbolic price to pay for the drastic simplification.

In this simplest case the supercharges forming $\mathcal{N}=2$ Poincaré superalgebra have a quite simple structure

$$
\begin{aligned}
Q_{hyp} &= \sum_{i=1}^{n} p_i \xi_{ii} - \mathrm{i} \sum_{i\neq j}^{n} \left[\frac{g+\Pi_{jj}}{\sinh(x_i-x_j)} - \frac{\Pi_{ij}}{\tanh(x_i-x_j)} \right] \xi_{ji} \,, \\
\overline{Q}_{hyp} &= \sum_{i=1}^{n} p_i \bar{\xi}_{ii} + \mathrm{i} \sum_{i\neq j}^{n} \left[\frac{g+\Pi_{ii}}{\sinh(x_i-x_j)} - \frac{\Pi_{ji}}{\tanh(x_i-x_j)} \right] \bar{\xi}_{ij} \,.
\end{aligned}
$$

They form the $\mathcal{N}=2$-extended super-Poincaré algebra together with the Hamiltonian

$$
H_{hyp} = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{2}\sum_{i\neq j}^{n} \left[\left(\frac{g+\Pi_{jj}}{\sinh(x_i-x_j)} - \frac{\Pi_{ij}}{\tanh(x_i-x_i)} \right) \left(\frac{g+\Pi_{ii}}{\sinh(x_i-x_j)} - \frac{\Pi_{ji}}{\tanh(x_i-x_j)} \right) - \Pi_{ij}\Pi
$$

Thus, these supercharges and the Hamiltonian describe an $\mathcal{N}=2$-extended supersymmetric hyperbolic Calogero model of type $A_{n-1}\oplus A_1$.
It should be noted that when checking the $\mathcal{N}=2$ superalgebra it is not enough to know the brackets between Π_{ij} and the fermions $\xi_{ij}, \bar{\xi}_{ij}$. Instead, the explicit expressions for Π_{ij} have to be substitute in the supercharges. This makes the calculations slightly more complicated as comparing to the previously considered cases.

It is quite expectable that in the $\mathcal{N}=2$ supersymmetric case passing from hyperbolic to trigonometric case goes straightforwardly. Indeed, one may easily check that the following supercharges

$$
\begin{aligned}
Q_{trig} &= \sum_{i=1}^{n} p_i \xi_{ii} - \mathrm{i} \sum_{i\neq j}^{n} \left[\frac{g+\Pi_{jj}}{\sin(x_i - x_j)} - \frac{\Pi_{ij}}{\tan(x_i - x_j)} \right] \xi_{ji} \,, \\
\overline{Q}_{trig} &= \sum_{i=1}^{n} p_i \bar{\xi}_{ii} + \mathrm{i} \sum_{i\neq j}^{n} \left[\frac{g+\Pi_{ii}}{\sin(x_i - x_j)} - \frac{\Pi_{ji}}{\tan(x_i - x_j)} \right] \bar{\xi}_{ij} \,.
\end{aligned}
$$

form the $\mathcal{N}=2$-extended super-Poincaré algebra together with the Hamiltonian

$$
H_{trig} = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{2}\sum_{i\neq j}^{n} \left[\left(\frac{g+\Pi_{jj}}{\sin(x_i - x_j)} - \frac{\Pi_{ij}}{\tan(x_i - x_j)} \right) \left(\frac{g+\Pi_{ii}}{\sin(x_i - x_j)} - \frac{\Pi_{ji}}{\tan(x_i - x_j)} \right) + \Pi_{ij}\Pi_{ji} \right] .
$$

Thus, these supercharges and the Hamiltonian describe an $\mathcal{N}=2$-extended supersymmetric trigonometric Calogero model of type $A_{n-1} \oplus A_1$.

The evident generalization of the $\mathcal{N}=2$ supercharges to the $\mathcal{N}=4$ supersymmetric case is the following one

$$
\begin{aligned}
Q^a_{hyp} &= \sum_{i=1}^n p_i \xi^a_{ii} - \mathrm{i} \sum_{i\neq j}^n \Big[\frac{g+\Pi_{jj}}{\sinh(x_i-x_j)} - \frac{\Pi_{ij}}{\tanh(x_i-x_j)} \Big] \xi^a_{ji} , \\
\overline{Q}_{hyp\, a} &= \sum_{i=1}^n p_i \bar\xi_{ii\, a} + \mathrm{i} \sum_{i\neq j}^n \Big[\frac{g+\Pi_{ii}}{\sinh(x_i-x_j)} - \frac{\Pi_{ji}}{\tanh(x_i-x_j)} \Big] \bar\xi_{ij\, a} .
\end{aligned}
$$

The ansatz just mimics the structure of the supercharges of the $\mathcal{N}$-extended supersymmetric Calogero model where all information about extended supersymmetry was hidden in the structure of the object Π_{ij}. However, this guess is not correct and these supercharges do not form the $\mathcal{N}=4$ superalgebra. However, all differences consist in the four-fermionic terms without any dependence on the coordinates x_i. The possible modification of the supercharges looks as follows

$$
\mathcal{Q}^a_{hyp} = Q^a_{hyp} - \mathrm{i}\,\alpha\, Q^a_0, \qquad \overline{\mathcal{Q}}_{hyp\, a} = \overline{Q}_{hyp\, a} + \mathrm{i}\,\alpha\, \overline{Q}_{0\, a},
$$

where

$$
Q^a_0 = \sum_{i,j}^n \xi^a_{ij} \widetilde{\Pi}_{ji}, \quad \overline{Q}^a_0 = \sum_{i,j}^n \bar\xi_{ij\, a} \widetilde{\Pi}_{ji}.
$$

The closure of the algebra imposed the final constraint

$$
\alpha^2 = 1.
$$

The situation with the $\mathcal{N}=4$ supersymmetric trigonometric Calogero model is similar. One has to start with the evident ansatz for the $\mathcal{N}=4$ supercharges

$$
\begin{aligned}
Q^a_{trig} &= \sum_{i=1}^n p_i \rho^a_{ii} - \mathrm{i} \sum_{i\neq j}^n \left[\frac{g+\Pi_{jj}}{\sin(x_i-x_j)} - \frac{\Pi_{ij}}{\tan(x_i-x_j)} \right] \rho^a_{ji} \,, \\
\overline{Q}_{trig\,a} &= \sum_{i=1}^n p_i \bar\rho_{ii\,a} + \mathrm{i} \sum_{i\neq j}^n \left[\frac{g+\Pi_{ii}}{\sin(x_i-x_j)} - \frac{\Pi_{ji}}{\tan(x_i-x_j)} \right] \bar\rho_{ij\,a} \,.
\end{aligned}
$$

These supercharges again do not form closed superalgebra and again the following modification

$$
\mathcal{Q}^a_{trig} = Q^a_{trig} - \mathrm{i}\,\beta\, Q^a_0, \qquad \overline{\mathcal{Q}}_{trig\,a} = \overline{Q}_{trig\,a} + \mathrm{i}\,\beta\, \overline{Q}_{0\,a},
$$

with the same additional supercharges $Q_0, \overline{Q}_0$ cure the situation. The resulting Hamiltonian reads

$$
H_{trig} = \frac{1}{2}\sum_{i=1}^n p_i^2 + \frac{1}{2}\sum_{i\neq j}^n \left[\left(\frac{g+\Pi_{jj}}{\sin(x_i-x_j)} - \frac{\Pi_{ij}}{\tan(x_i-x_j)} \right) \left(\frac{g+\Pi_{ii}}{\sin(x_i-x_j)} - \frac{\Pi_{ji}}{\tan(x_i-x_j)} \right) + \Pi_{ij}\Pi_{ji} \right] .
$$

However, the parameter β now has to obey the equation $\beta^2=-1$. The Hamiltonian H_{trig} does not depend on the sign of $\beta=\pm\mathrm{i}$ and it is perfectly Hermitian. However, the supercharges do depend on the parameter β and they conjugated as follows

$$
\left(\mathcal{Q}^a_{trig}|_{\beta=\mathrm{i}} \right)^\dagger = \overline{\mathcal{Q}}_{trig\,a}|_{\beta=-\mathrm{i}}.
$$

It is not completely clear whether such properties are satisfying.

Trigonometric

$$
\begin{aligned}
Q^a &= \sum_{i=1}^n p_i \rho^a_{ii} - \mathrm{i}\sum_{i\neq j}^n \left[\frac{g+\Pi_{jj}-\cos(x_i-x_j)\,\Pi_{ij}}{\sin(x_i-x_j)} - \frac{g+\Pi_{jj}+\cos(x_i+x_j)\,\widetilde{\Pi}_{ij}}{\sin(x_i+x_j)} \right] \rho^a_{ji} + \\
&\quad \mathrm{i}\sum \frac{g'+\Pi_{ij}+\cos(2x_i)\,\widetilde{\Pi}_{ii}}{\sin(2x_i)} \rho^a_{ii}, \\
\overline{Q}_a &= \sum_{i=1}^n p_i \bar\rho_{ii\,a} - \mathrm{i}\sum_{i\neq j}^n \left[\frac{g+\Pi_{jj}-\cos(x_i-x_j)\,\Pi_{ij}}{\sin(x_i-x_j)} + \frac{g+\Pi_{jj}+\cos(x_i+x_j)\,\widetilde{\Pi}_{ij}}{\sin(x_i+x_j)} \right] \bar\rho_{ji\,a} - \\
&\quad \mathrm{i}\sum \frac{g'+\Pi_{ij}+\cos(2x_i)\,\widetilde{\Pi}_{ii}}{\sin(2x_i)} \bar\rho_{ii\,a},
\end{aligned}
$$

$$
\begin{aligned}
H &= \frac{1}{2}\sum_{i=1}^n p_i^2 + \frac{1}{2}\sum_{i\neq j}^n \left[\frac{\left(g+\Pi_{jj}-\cos(x_i-x_j)\Pi_{ij}\right)\left(g+\Pi_{ii}-\cos(x_i-x_j)\Pi_{ji}\right)}{\sin^2(x_i-x_j)} + \right. \\
&\quad \left. \frac{\left(g+\Pi_{jj}+\cos(x_i+x_j)\widetilde{\Pi}_{ij}\right)\left(g+\Pi_{ii}+\cos(x_i+x_j)\widetilde{\Pi}_{ji}\right)}{\sin^2(x_i+x_j)} \right] + \frac{1}{2}\sum_{i,j}^n \left(\Pi_{ij}\Pi_{ji} + \widetilde{\Pi}_{ij}\widetilde{\Pi}_{ji}\right) + \\
&\quad \frac{1}{2}\sum_i^n \frac{\left(g'+\Pi_{ii}+\cos(2x_i)\widetilde{\Pi}_{ii}\right)\left(g'+\Pi_{ii}+\cos(2x_i)\widetilde{\Pi}_{ii}\right)}{\sin^2(2x_i)}.
\end{aligned}
$$

Hyperbolic

$$
\begin{aligned}
Q^a &= \sum_{i=1}^{n} p_i \rho^a_{ii} - \mathrm{i} \sum_{i\neq j}^{n} \left[\frac{g+\Pi_{jj} - \cosh(x_i - x_j)\,\Pi_{ij}}{\sinh(x_i - x_j)} - \frac{g+\Pi_{jj} + \cosh(x_i + x_j)\,\widetilde{\Pi}_{ij}}{\sinh(x_i + x_j)} \right] \rho^a_{ji} + \\
&\mathrm{i} \sum \frac{g' + \Pi_{ij} + \cosh(2x_i)\,\widetilde{\Pi}_{ii}}{\sinh(2x_i)} \rho^a_{ii}, \\
\overline{Q}_a &= \sum_{i=1}^{n} p_i \bar{\rho}_{ii\,a} - \mathrm{i} \sum_{i\neq j}^{n} \left[\frac{g+\Pi_{jj} - \cosh(x_i - x_j)\,\Pi_{ij}}{\sinh(x_i - x_j)} + \frac{g+\Pi_{jj} + \cosh(x_i + x_j)\,\widetilde{\Pi}_{ij}}{\sinh(x_i + x_j)} \right] \bar{\rho}_{ji\,a} - \\
&\mathrm{i} \sum \frac{g' + \Pi_{ij} + \cosh(2x_i)\,\widetilde{\Pi}_{ii}}{\sinh(2x_i)} \bar{\rho}_{ii\,a},
\end{aligned}
$$

$$
\begin{aligned}
H &= \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{2}\sum_{i\neq j}^{n} \left[\frac{\left(g+\Pi_{jj} - \cosh(x_i - x_j)\Pi_{ij}\right)\left(g+\Pi_{ii} - \cosh(x_i - x_j)\Pi_{ji}\right)}{\sinh^2(x_i - x_j)} + \right. \\
&\left. \frac{\left(g+\Pi_{jj} + \cosh(x_i + x_j)\widetilde{\Pi}_{ij}\right)\left(g+\Pi_{ii} + \cosh(x_i + x_j)\widetilde{\Pi}_{ji}\right)}{\sinh^2(x_i + x_j)} \right] - \frac{1}{2}\sum_{i,j}^{n} \left(\Pi_{ij}\Pi_{ji} + \widetilde{\Pi}_{ij}\widetilde{\Pi}_{ji}\right) + \\
&\frac{1}{2}\sum_{i}^{n} \frac{\left(g' + \Pi_{ii} + \cosh(2x_i)\widetilde{\Pi}_{ii}\right)\left(g' + \Pi_{ii} + \cosh(2x_i)\widetilde{\Pi}_{ii}\right)}{\sinh^2(2x_i)}.
\end{aligned}
$$

Conclusion

We constructed a wide class of $\mathcal{N}$-extended supersymmetric rational/trigonometric/hyperbolic Calogero models. The main features of our models are

- a huge number of fermionic coordinates, namely $\mathcal{N}n^2$ in number rather than the $\mathcal{N}n$ to be expected
- the supercharges and the Hamiltonian contain terms which a fermionic power much larger than three.

Clearly, these features merit a more careful and detailed analysis.
Let us list possible further developments:

- Exceptional Lie algebras: extension of our analysis to models associated with G_2, F_4 or E_n.
- Trigonometric models: One should clarify the situation with $\mathcal{N} = 4$ supersymmetric trigonometric A_n Calogero model.