THE KONTSEVICH GRAPH ORIENTATION MORPHISM REVISITED.

Arthemy Kiselev (GRONINGEN · NL), R. BURING (Mainz · DE) SQS19 30/08/19 YEREVAN

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γ {[1710.00658] [1811.10638] $\vec{Or}$ {[1811.07878] [1904.13293] sym$\{\cdot,\cdot\}_{\mathcal{P}}$ {[1608.01710] [1712.05259]

★ {[1702.00681], [1907.00639].

⌈www.mathnet.ru/php/conference.phtml?option_lang= rus/eng & eventID=25& confid=1591⌋ (IUM, 13–15/05/2019; 10 hours)

① DGLA: $\Big(\text{Vect}(\gamma \leftarrow \text{unoriented graph(s)}) \big/ E(\gamma_\alpha) = e_1\wedge\ldots\wedge e_{\#\text{Edge}},\ [\,,\,]_{\text{Lie}}\,!\Big)$.

$\gamma_1 \vec{\circ} \gamma_2 := \sum_{v\in \text{Vert}(\gamma_2)} \Big($insert γ_1 into vertex v: $\times_v \rightsquigarrow (\gamma_1)$ in $\gamma_2\setminus\{v\}$, $E(\gamma_1)\wedge E(\gamma_2)\Big)$.

$[\gamma_1,\gamma_2]_{\text{Lie}} = \gamma_1\vec{\circ}\gamma_2 - (-)^{\#E(\gamma_1)\cdot\#E(\gamma_2)}\,\gamma_2\vec{\circ}\gamma_1.$

$(d = [\bullet\!\!-\!\!\bullet, \cdot])^2 \equiv 0$: differential = blow-up vertex $\times \rightsquigarrow$ ("edge").

Th. d-cocycles $\Big($[T. Willwacher, 2010–15] $(\#V = n, \#E = 2n-2)\Big)$: $\{\exists \geqslant$ countably many$\}$ $\overset{\text{Th.}}{\cong}$ $\mathfrak{grt} = \text{LIE}(GRT)$ (Drinfeld, 1990).

② Endomorphisms $\text{End}\Big(T^{\downarrow[1]}_{\text{polyvector}}(\mathcal{M}^{r=\dim}_{\text{affine}})\Big)$. ← Ex. $[\![\cdot,\cdot]\!]$ Schouten. ⓘ Natural n-ary?

Or. graphs: $\overleftarrow{\partial}/\partial\xi_i \otimes \vec{\partial}/\partial x^i \rightsquigarrow \bullet\!\xrightarrow{i}$.

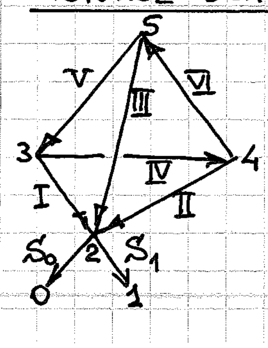
Ex. Bi-vector $\mathcal{P} = (\mathcal{P}^{ij}(\underline{x})) = \frac{1}{2}\mathcal{P}^{ij}(\underline{x})\,\xi_i\xi_j \rightsquigarrow$ ${}^i\!\swarrow\!\!\overset{\mathcal{P}^{ij}}{\wedge}\!\!\searrow^j$. Ex. $\boxed{\bullet\bullet}$ $=\text{Jac}(\mathcal{P})$.

Idea ($T_{\text{poly}} \circlearrowleft$ Endo): • Place multivector in each vertex of graph.
• Orient every edge ${}_a\xrightarrow{i}{}_b$ OR ${}_a\xleftarrow{i}{}_b$, sum up.

$\vec{Or}$ morphism: $\vec{Or}(\gamma)(p_1\otimes\cdots\otimes p_{\#V(\gamma)}) = \prod_{e_{ab}} \vec{\Delta}_{ab}(p_1\cdot\ldots\cdot p_{\#V(\gamma)})$.

$\vec{\Delta}_{ab} : {}_a\bullet\!\!-\!\!\bullet_b \mapsto \sum_{i=1}^{\dim M^r}\Big({}_a\xleftarrow{i}{}_b + {}_a\xrightarrow{i}{}_b\Big)$. $\sum_i\Big(\frac{\vec{\partial}}{\partial x^i_{(a)}}\otimes\frac{\vec{\partial}}{\partial\xi_i^{(b)}} + \frac{\vec{\partial}}{\partial\xi_i^{(a)}}\otimes\frac{\vec{\partial}}{\partial x^i_{(b)}}\Big)$ } odd edge

KONTSEVICH ORGRAPH: $(\swarrow\!\wedge\!\searrow$ = "BRICK").



Ex. $\Gamma = \begin{pmatrix} 01 & 24 & 25 & 23 \\ S_0S_1 & \text{I}\,\overline{\text{IV}} & \text{II}\,\overline{\text{VI}} & \text{III}\,\overline{\text{V}}\end{pmatrix}$. Let $p_\alpha := \mathcal{P} \leftarrow$ (Poisson) bi-vector.

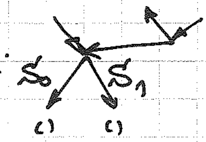
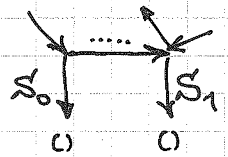
Def ⇒ Th. If $\Gamma \simeq \Gamma_0 \in \vec{Or}(\gamma_\alpha)$ ⇒ $\{\sigma : \Gamma\simeq\Gamma_0 \leftarrow$ orgraph iso$\}$

⇒ $\text{sign}(\Gamma) = (-)^{\sigma|_{\text{Edges}}}\cdot\text{sign}(\Gamma_0)$.

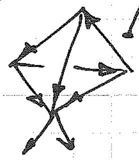
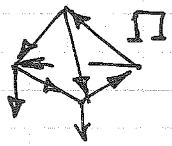
NB: "Millions of or. graphs" ⇒ ⓘ simple rules of sign(s): "odd" edges.

③ "SUPER-": "ξ" along fibre in ΠT^*M^r. ? mechanism "ξ odd"?

RULE 1] (normalize). $\Gamma \simeq \Gamma_0$ $\quad \text{Sign}\big(\Gamma = (S_0, A)(S_1, B)\cdots\big) = (-)\cdot \text{Sign}\big(\Gamma_0 = (S_0, B)(S_1, A)\cdots\big)$ if $A < B$

Def.  } Λ-shaped Kontsevich orgraph  } Π-shaped Kontsevich orgraph : source(S_0) $\neq$ source(S_1).

Rule 2] $\Gamma_1 \not\simeq \Gamma_2$ but $\Gamma_1, \Gamma_2 \in \text{Or}\vec{\ }(\gamma_\alpha \leftarrow$ single graph $\} \stackrel{\text{def}}{=}$ body of $\Gamma_1, \Gamma_2)$.

$\Gamma_1 =$  Λ $\not\simeq$ $\Gamma_2 =$  Π in $\text{Or}\vec{\ }($ $= \gamma_3 \in \ker(d))$.

- $\Pi \rightleftarrows \Pi$: If both Γ_1, Γ_2 are Π-shaped $\Rightarrow$ $\text{sign}(\Gamma_2) = (-)^{\#\{\text{reverses of arrows in body as } \Gamma_1 \Leftrightarrow \Gamma_2\}} \cdot \text{sign}(\Gamma_1)$. (*)
- $\Lambda \rightleftarrows \Pi$: $(-) \times$ formula (*).
- $\Lambda \rightleftarrows \Lambda$: formula (*) again.

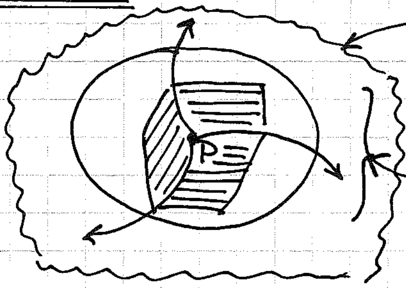
Conclusion = "Combinatorics encodes & SUPPORTS SUPERMATHEMATICS."

④ Quantum symmetries. $\left\{ \frac{\partial}{\partial t_\alpha}(P) = \text{Or}\vec{\ }(\gamma_\alpha)(P,\ldots,P) \ ; \frac{\partial}{\partial t_\beta}(P) \leftsquigarrow \gamma_\beta \right\}$

Th. $\text{Or}\vec{\ }([\gamma_\alpha, \gamma_\beta]_{\text{Lie}})(P,\ldots P) \stackrel{\checkmark}{=} \left[\frac{\partial}{\partial t_\alpha}, \frac{\partial}{\partial t_\beta}\right](P)$.

Chain of (iso)morphisms:

$\mathfrak{grt} = \text{LIE}(\text{GRT})$ ⌈Grothendieck–Teichmüller⌋ "$\simeq$" (T.W (2015)) Free Lie $\langle \gamma_3, \gamma_5, \gamma_7, \ldots \rangle / [\cdot,\cdot]_{\text{Lie}}$ (wheel cocycles) $\xrightarrow[\sim]{\text{Or}\vec{\ }}$ (Some) infinitesimal symmetries of Jacobi identity for P POISSON.

Conclusion = ("lower bound") 

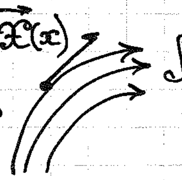
All bi-vectors near P POISSON

$\# = \infty$ (countably many)

Like a "chart" on an ∞-dim "manifold" of Poisson bi-vectors.

? (Also $\dot{P} = P$); Do there exist other natural $\frac{1}{\infty}$ sym$(\{\cdot,\cdot\}_{\text{Poisson}})$?

⑤ $(M^r_{\text{affine}}, P_{\text{POISSON}})$. If $\frac{\partial}{\partial t_\alpha}(P) = [\![P, \vec{\mathfrak{X}}]\!] \big(+ \underbrace{\nabla([\![P,P]\!], P)}_{\equiv 0 \text{ if } [\![P,P]\!] = 0}\big)$,

$\Rightarrow$ deformation $P \mapsto P + t_\alpha \cdot \frac{\partial}{\partial t_\alpha}(P) + \bar{o}(t_\alpha)$ realizes (non) linear $\tilde{x}(x) \rightleftarrows x(\tilde{x})$ along  $\vec{\mathfrak{X}}(x)$ $\int$ on M^r_{affine}!