Spin-Locality of Higher-Spin Theories

and

Star-Product Functional Classes

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Fields of the Nonlinear System

Infinite set of spins s:

one-forms $\omega(Y;K\,|x) = \sum\limits_s \omega_s(y,\bar{y};K\,|x)$

zero-forms $C(Y;K\,|x) = \sum\limits_s C_s(y,\bar{y};K\,|x)$

$$\omega_s = \sum_{n+m=2s} \frac{1}{n!m!}\omega_{\alpha_1\ldots\alpha_n,\dot{\alpha}_1\ldots\dot{\alpha}_m} y^{\alpha_1}\ldots y^{\alpha_n}\bar{y}^{\dot{\alpha}_1}\ldots\bar{y}^{\dot{\alpha}_m}\,,$$

$$C_s = \sum_{|n-m|=2s}^{\infty} \frac{1}{n!m!}C_{\alpha_1\ldots\alpha_n,\dot{\alpha}_1\ldots\dot{\alpha}_m} y^{\alpha_1}\ldots y^{\alpha_n}\bar{y}^{\dot{\alpha}_1}\ldots\bar{y}^{\dot{\alpha}_m}$$

Klein operators $K = (k,\bar{k})$: $\quad ky^\alpha = -y^\alpha k\,,\ \ \bar{k}\bar{y}^{\alpha'} = -\bar{y}^{\alpha'}\bar{k}\,,\ \ kk = \bar{k}\bar{k} = 1\,.$

Nonlinear equations via doubling of spinors

$$\omega(Y;K|x) \longrightarrow W(Z;Y;K|x)\,, \qquad C(Y;K|x) \longrightarrow B(Z;Y;K|x)$$

Nonlinear HS Equations

$$
\begin{cases}
\mathrm{d}W + W \star W = 0 \\
\mathrm{d}B + W \star B - B \star W = 0 \\
\mathrm{d}S + W \star S + S \star W = 0 \\
S \star B - B \star S = 0 \\
S \star S = i(dZ^A dZ_A + \eta dz^\alpha dz_\alpha B \star k \star \kappa + \bar{\eta} d\bar{z}^{\dot\alpha} d\bar{z}_{\dot\alpha} B \star \bar{k} \star \bar{\kappa})
\end{cases}
$$

Vasiliev 1992

η, $\bar{\eta}$-**free parameters**

HS star product

$$(f \star g)(Z,Y) = \int dS dT f(Z+S, Y+S) g(Z-T, Y+T) \exp -iS_A T^A$$

$$[Y_A, Y_B]_\star = -[Z_A, Z_B]_\star = 2iC_{AB}\,,$$

Inner Klein operators:

$\kappa = \exp i z_\alpha y^\alpha\,,\qquad \bar{\kappa} = \exp i\bar{z}_{\dot\alpha} \bar{y}^{\dot\alpha}\,,\qquad \kappa \star f(y,\bar{y}) = f(-y,\bar{y}) \star \kappa\,,\qquad \kappa \star \kappa = 1$

Perturbative Analysis

Linearization around some vacuum solution.

$$B_0 = 0\,, \qquad W_0 = w(Y;K|x)\,, \qquad S_0 = Z_A dZ^A\,, \qquad \mathrm{d}_x w + w\star w = 0\,.$$

$w(Y|x) = -\frac{i}{4}(w^{\alpha\beta}(x)y_\alpha y_\beta + \bar{w}^{\dot\alpha\dot\beta}(x)\bar{y}_{\dot\alpha}\bar{y}_{\dot\beta} + 2h^{\alpha\dot\beta}(x)y_\alpha\bar{y}_{\dot\beta})$ **describes** AdS_4.

$$[S_0\,, f]_\star = -2i\mathrm{d}_Z f\,, \qquad \mathrm{d}_Z = dZ^A\frac{\partial}{\partial Z^A}$$

First-order fluctuations

$$B_1 = C(Y)\,, \qquad S = S_0 + S_1\,, \qquad W = W_0(Y) + \omega(Y) + W_0(Y)C(Y)$$

$\omega(Y)$ **and** $C(Y)$ **to be identified with the one-forms and zero-forms of the original formulation of free fields belong to the** d_Z **cohomology.**

$$\begin{aligned}
\mathrm{d}_x\omega &= -\omega\star\omega + \Upsilon^\eta(\omega^2, C) + \Upsilon^{\eta^2}(\omega^2, C^2) + \Upsilon^{\bar\eta\eta}(\omega^2, C^2) + \ldots + (\eta\leftrightarrow\bar\eta)\,,\\
\mathrm{d}_x C &= -[\omega, C]_\star + \Upsilon^\eta(\omega C^2) + \Upsilon^{\eta^2}(\omega C^3) + \Upsilon^{\eta\bar\eta}(\omega C^3) + \ldots + (\eta\leftrightarrow\bar\eta)
\end{aligned}$$

$$\Upsilon^\eta(\omega^2, C) = \Upsilon^\eta_{\omega\omega C} + \Upsilon^\eta_{C\omega\omega} + \Upsilon^\eta_{\omega C\omega}\ \ldots$$

Reconstruction of Z^A Dependence

Perturbatively, equations containing S have the form

$$\mathsf{d}_Z U_n(Z;Y|\mathsf{d}Z) = V[U_{<n}](Z;Y|\mathsf{d}Z)\,, \qquad \mathsf{d}_Z V[U_{<n}](Z;Y|\mathsf{d}Z) = 0$$

can be solved as (discarding the $\bar{\theta}, \bar{z}$ sector for brevity)

$$U_n(z;Y|\theta) = \mathsf{d}_z^\star V[U_{<n}](Z;Y|\theta) + \mathbf{h(Y)} + \mathsf{d}_z \epsilon(Z;Y|\theta)\,, \qquad \theta := \mathsf{d}_z.$$

Shifted resolution operator $\mathsf{d}_z^\star = \Delta_q$

$$\Delta_q V(z;Y|dz) = (z^A + q^A)\frac{\partial}{\partial\theta^A}\int_0^1 dt \frac{1}{t} V\left(tz - (1-t)q; Y; t\theta\right)\,, \qquad \frac{\partial q^B}{\partial z^A} = 0\,.$$

gives the resolution of identity with the cohomology projector h_q

$$\{\mathsf{d}_z, \Delta_q\} = 1 - h_q\,, \qquad h_q f(z,\theta) = f(-q, 0)$$

Proper shift q $\Rightarrow$ local corrections $\Upsilon^\eta(\omega C^2) + \Upsilon^{\bar{\eta}}(\omega C^2)$ to field equations in the zero-form sector

Didenko, OG, Korybut, Vasiliev (2018)

β-shifted Resolutions

Local corrections $\Upsilon^{\eta\eta}(\omega^2 C^2)+\Upsilon^{\bar\eta\bar\eta}(\omega^2 C^2)$ **to field equations in the one-form sector demand** β**-shifted resolution:** $z \to z+q+\beta\frac{\partial}{\partial y} \qquad \beta \to -\infty$

Didenko, OG, Korybut, Vasiliev (2019)

$$\Delta_{q,\beta} J := \int \frac{\mathrm{d}^2 u \mathrm{d}^2 v}{(2\pi)^2} \exp(i u_\alpha v^\alpha) \int_0^1 \frac{\mathrm{d}\tau}{\tau} (z+q-v)^\alpha \frac{\partial}{\partial \theta^\alpha} J(\tau z-(1-\tau)(q-v); y+\beta u; \tau\theta)$$

Resolution of identity

$$\{\mathrm{d}_z, \Delta_{q,\beta}\} = 1 - h_{q,\beta}$$

the cohomology projector to the z,θ**-independent part**

$$h_{q;\beta} J(z;y;\theta) = \int \frac{\mathrm{d}^2 u\, \mathrm{d}^2 v}{(2\pi)^2} \exp(i u_\alpha v^\alpha) J(-q+v; y+\beta u; 0)$$

Normalized notations $\Delta'_{q,\beta} := \Delta_{q(1-\beta),\beta}\,, \qquad h'_{q;\beta} := h_{q(1-\beta);\beta}$

Pfaffian Locality Theorem

PLT holds for β-shifted resolutions (OG,Vasiliev, 2018,2019)

Order-n corrections in C (neglecting ω and θ)

$$\int d\tau C(y_1)\dots C(y_n)\rho(\tau,p,y,z)\,E_n^{\mathbf{p}}\,\bar{E}_n^{\bar{\mathbf{p}}}\Big|_{y_i=0}\,,\qquad \mathbf{p}=0,1$$

$$E_n^{\mathbf{p}}(T,A,B,P,p|z,y)=\exp i(Tz_\gamma y^\gamma-A_jp_\gamma^j z^\gamma-B_jp_\gamma^j y^\gamma+\frac{1}{2}P_{ij}p^{i\gamma}p^j{}_\gamma)k^{\mathbf{p}},\qquad p_\alpha^j=-i\frac{\partial}{\partial y_j^\alpha}$$

τ : **integration parameters over some compact domain,**

T, $A^i(\tau)$, $B^i(\tau)$, $P^{ij}(\tau)$: **polynomials in τ, $\rho(\tau,p,y,z)$ is a polynomial**

Even-class holomorphic exponentials $E_n^{\mathbf{p}}$ satisfy: $\mathbf{p}=n|_2$,

$$\sum_{j=1}^n(-1)^jA_j=-T\,,\qquad \sum_{j=1}^n(-1)^jB_j=0\,,\qquad \sum_{i=1}^n(-1)^iP_{ij}=B_j$$

Even resolution: if $q=\sum v^jp_j+p\omega$, **with** $\sum_{j=1}^n(-1)^jv_j=0$ **then** $\forall\beta<1$

$$\Delta'_{q,\beta}:\qquad E_n^{\mathbf{p}}(T,A,B,P,p|z,y)\quad\to\quad E_n^{\mathbf{p}}(T',A',B',P'|z,y)$$

leaves even class invariant.

(Anti-)Holomorphic $\Upsilon(\omega,\omega,\underbrace{C,\ldots,C}_{n})$ **: Even class** ,

Even resolutions reduce the degree of non-locality in all (anti-)holomorphic vertices $\Upsilon(\omega,\omega,\underbrace{C,C,\ldots,C}_{n})$

By PLT local (anti-)holomorphic vertex $\Upsilon(\omega,\omega,\underbrace{C,C,\ldots,C}_{n})$ **has to be ultra-local containing no dependence on** y **or** $\bar{y}$ **in** $C(Y|x)$.

y **or** $\bar{y}$ **independence of bilinear in** C **vertex** $\Upsilon(\omega,\omega,C,C)$ **implies ultra-locality**

$$\sum_{i=1}^{2}(-1)^{i}P_{ij}=B_{j}: \qquad B_{j}=0\rightarrow P_{ij}=0$$

Star-Product Functions

Analysis of locality is most convenient for p-forms in θ $\quad f_p(z,y,\theta)$

$$\mathcal{H}_p \ni f_p(z,y,\theta) = \int_0^1 d\tau \phi(\tau z, (1-\tau)y, \tau\theta, \tau)\exp[i\tau z_\alpha y^\alpha]$$ M.A. Vasiliev (2015)

$$\phi(\tau z,(1-\tau)y,\tau\theta,\tau) = \frac{1-\tau}{\tau}\psi_1(\tau z,(1-\tau)y,\frac{\tau\theta}{1-\tau},\tau)+\delta(\tau)\psi_2(y)+\delta(1-\tau)\delta^2(\theta)\psi_3(z)$$

The poles at τ and $(1-\tau)$ are fictitious:

$$\psi_1(0,y,0,0)=0\,,\qquad \psi_1(z,0,\theta,1)=0\,.$$

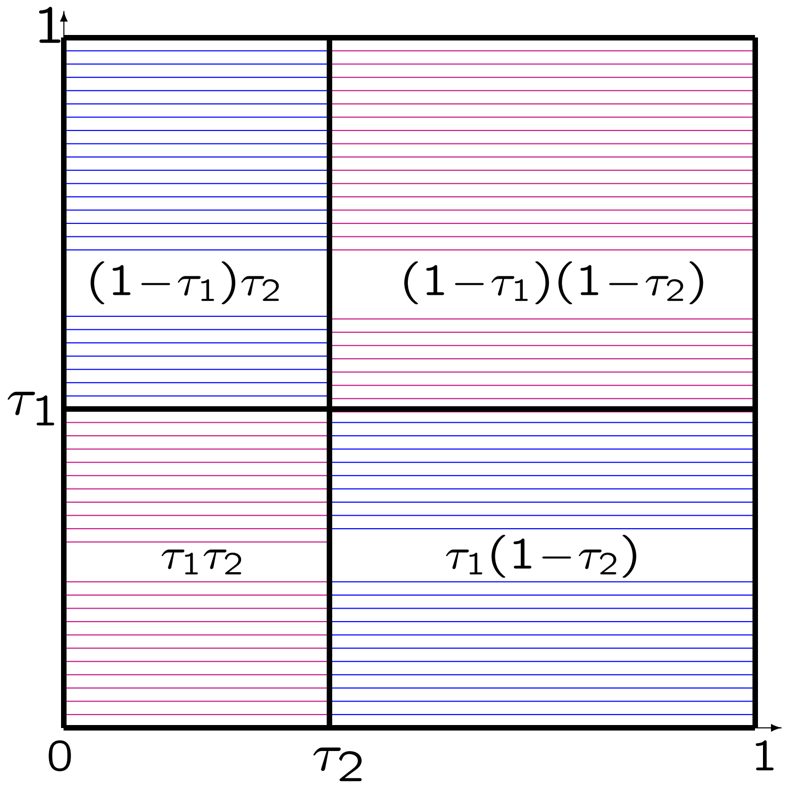
$$\mathcal{H} := \oplus_{p=0}^2 \mathcal{H}_p$$

$$\begin{aligned} f_1 \star f_2 \quad &= \int_0^1 d\tau_1 \int_0^1 d\tau_2 \int d^2s d^2t \exp i[\tau_1\circ\tau_2 z_\alpha y^\alpha + s_\alpha t^\alpha] \\ &\times\phi_1(\tau_1((1-\tau_2)z-\tau_2 y+s),(1-\tau_1)((1-\tau_2)y-\tau_2 z+s),\tau_1\theta,\tau_1) \\ &\times\phi_2(\tau_2((1-\tau_1)z+\tau_1 y-t),(1-\tau_2)((1-\tau_1)y+\tau_1 z+t),\tau_2\theta,\tau_2)\,, \end{aligned}$$

$$\tau_1\circ\tau_2 = \tau_1(1-\tau_2)+\tau_2(1-\tau_1)\,.$$

The product law $\circ$ is commutative and associative.

Magic Square



$$1 - \tau_1 \circ \tau_2 = \tau_1\tau_2 + (1-\tau_2)(1-\tau_1) \leq 1$$

$$\tau_1 \circ \tau_2 = \tau_1(1-\tau_2) + (1-\tau_1)\tau_2 \leq 1$$

Algebra $\mathcal{H}$

Space $\mathcal{H}$ forms an algebra under the star product.

$$f_1 \star f_2 = \int_0^1 d\mathcal{T} \exp i[\mathcal{T} z_\alpha y^\alpha] \phi_{1,2}(\mathcal{T} z, (1-\mathcal{T})y, \mathcal{T}\theta, \mathcal{T})$$

$$\begin{aligned}\phi_{1,2} = & \int_0^1 d\tau_1 \int_0^1 d\tau_2 \delta(\mathcal{T} - \tau_1 \circ \tau_2) \int d^2s d^2t \exp i[s_\alpha t^\alpha] \\ & \phi_1(\alpha_{12}\mathcal{T} z - \alpha_{11}(1-\mathcal{T})y + \tau_1 s, \alpha_{22}(1-\mathcal{T})y + \alpha_{21}\mathcal{T} z + (1-\tau_1)s, \tau_1\theta, \tau_1) \\ & \phi_2(\alpha_{21}\mathcal{T} z + \alpha_{11}(1-\mathcal{T})y - \tau_2 t, \alpha_{22}(1-\mathcal{T})y + \alpha_{12}\mathcal{T} z + (1-\tau_2)t, \tau_2\theta, \tau_2)\,.\end{aligned}$$

Since

$$\int_0^1 d\tau_1 \int_0^1 d\tau_2 \delta(\tau - \tau_1 \circ \tau_2) = -\frac{1}{2}\log((1-2\tau)^2)$$

$\log((1-2\tau)^2)$ **has simple zeroes both at** $\tau \to 0$ **and at** $\tau \to 1$:

$$f_1,\ f_2 \in \mathcal{H} \Rightarrow f_1 \star f_2 \in \mathcal{H}$$

Classes $\mathcal{H}^{+0}$, $\mathcal{H}^{0+}$ and Ideal $\mathcal{I}$

$$f^p \in \mathcal{H}_p^{+0} : \qquad \lim_{\tau\to 0} \tau^{1-p+\varepsilon}\phi(w,u,\theta,\tau) = 0 \ , \qquad\qquad \exists\varepsilon > 0 \, ,$$

$$f^p \in \mathcal{H}_p^{0+} : \qquad \lim_{\tau\to 1} (1-\tau)^{-1+p+\varepsilon}\phi(w,u,\theta,\tau) = 0 \ , \qquad\qquad \exists\varepsilon > 0$$

Any $f \in \mathcal{H}$ decomposes as $f = f^{0+} + f^{+0}$ by virtue of

$$\tau + (1-\tau) = 1$$

Easy to check:

$$\mathcal{H}^{0+} \star \mathcal{H}^{0+} \subset \mathcal{H}^{0+} \ , \qquad\qquad \mathcal{H}^{+0} \star \mathcal{H}^{+0} \subset \mathcal{H}^{0+} \, ,$$

$$\mathcal{H}^{0+} \star \mathcal{H}^{+0} \subset \mathcal{H}^{+0} \, , \qquad\qquad \mathcal{H}^{+0} \star \mathcal{H}^{0+} \subset \mathcal{H}^{+0} \, .$$

$\mathcal{H}^{0+}$ **forms a subalgebra of $\mathcal{H}$**

$\mathcal{I} = \mathcal{H}^{+0} \cap \mathcal{H}^{0+}$ **forms a two-sided ideal in $\mathcal{H}$**

Factorization Lemma and Ultra-Locality Theorem

Factorization Lemma

$$h'_{q,-\infty}(\mathcal{H}^{+0}) = 0 \ , \qquad h'_{q,-\infty} := \lim_{\beta \to -\infty} h'_{q,\beta}.$$

Ultra-Locality Theorem:

$\Delta'_{q_1,-\infty}(\Delta'_{q_2,-\infty}(\mathcal{H}_2^{+0}) \star \mathcal{H}_0)$ **is well defined and** y**-independent**

The limit $\beta \to -\infty$ is highly important

Structure Relation

The equation on S_2 in the holomorphic sector

$$2i\mathsf{d}_z S_2 = S_1 \star S_1 - i\eta B_2 \star \gamma \sim \Delta_{a,0}\Delta_{b,0}(\gamma) \star \gamma - \Delta_{a,0}(\gamma) \star \Delta_{b,0}(\gamma).$$

Structure relation:

$$\Delta_{a,0}\Delta_{b,0}(\gamma) \star \gamma - \Delta_{a,0}(\gamma) \star \Delta_{b,0}(\gamma) \in \mathcal{H}_2^{+0}\,.$$

Hence the second-order part of W_2 generated by S_2 is well defined in the limit $\beta \to -\infty$. Note that

$$S_1 \star S_1 \notin \ \mathcal{H}_2^{+0}\,, \qquad \eta B_2 \star \gamma \notin \ \mathcal{H}_2^{+0}$$

Each leads to divergency, their difference does not

Conclusions

Proper classes of star-product functions compatible with the limiting resolution action are introduced

PLT is extended to shifted resolution formalism

$\beta \to -\infty$ leads to ultra-local vertices by virtue of Ultra-locality Theorem and Factorization Lemma provided that the right-hand-sides of the HS equations in the θ^2 sector is in $\mathcal{H}_2^{+0}$. Structure relation is proven, showing that this is the case.

Analysis of the mixed $\eta\bar{\eta}$ vertices turns out to be β-independent, also leading to local $\omega^2 C^2$ vertices Didenko, OG, Korybut, Vasiliev (2019)

For explicit form of local vertices see Slava's talk

Odd-class exponentials $E_n^{\mathbf{p}}$ **satisfy** $\mathbf{p} = (n+1)|_2$

$$\sum_{j=1}^{n} (-1)^j A_j = 0\,, \qquad \sum_{j=1}^{n} (-1)^j B_j = 1 - T\,, \qquad \sum_{i=1}^{n} (-1)^i P_{ij} = -A_j$$

Odd resolution: If $q = \sum v^j p_j$, **with** $\sum_{j=1}^{n} (-1)^j v_j = 1$ **then** $\forall \beta$

$$\Delta'_{q,\beta}: \quad E_n^{\mathbf{p}}(T, A, B, P, p|z, y) \quad \to \quad E_n^{\mathbf{p}}(T', A', B', P'|z, y) k^{\mathbf{p}}$$

maps odd class to odd .

$$\Upsilon(\omega, \underbrace{C, \ldots, C}_{n}) \ \textbf{: Odd class}$$

Odd resolutions reduce the degree of non-locality in all (anti-)holomorphic vertices $\Upsilon(\omega, \underbrace{C, C, \ldots, C}_{n})$

$\Upsilon^{\eta}(\omega C^2)+$ **and** $\Upsilon^{\bar{\eta}}(\omega C^2)$ **are local** Didenko, OG, Korybut, Vasiliev (2018)

Inequalities

$$\alpha_{11}(\tau) := \frac{\tau_1\tau_2}{1-\tau_1\circ\tau_2}\,, \qquad \alpha_{22}(\tau) := \frac{(1-\tau_1)(1-\tau_2)}{1-\tau_1\circ\tau_2}\,, \quad \alpha_{11}(\tau)+\alpha_{22}(\tau)=1\,,$$

$$\alpha_{12}(\tau) := \frac{\tau_1(1-\tau_2)}{\tau_1\circ\tau_2}\,, \quad \alpha_{21}(\tau) := \frac{(1-\tau_1)\tau_2}{\tau_1\circ\tau_2}\,, \qquad \alpha_{12}(\tau)+\alpha_{21}(\tau)=1\,.$$

A part is smaller than entire $\quad 0 \le \alpha_{ij}(\tau) \le 1$

Important inequalities

$$0 \le \tau_1(1-\tau_1) \le \tau_1\circ\tau_2(1-\tau_1\circ\tau_2) \le \tau_1\circ\tau_2\circ\tau_3(1-\tau_1\circ\tau_2\circ\tau_3) \le \ldots \le 1$$

Limiting Resolution and Ultra-Locality

In $\beta \to -\infty$ **limit**

$$\Delta'_{q,-\infty}(f^p) = \int d^2v d^2u d^3\tau \vartheta(\tau_1)\vartheta(\tau_2)\vartheta(\tau_3)\delta(1-\sum_{i=1}^{3}\tau_i)\exp i[v_\alpha u^\alpha + \tau_1 z_\alpha y^\alpha + \tau_2 q_\beta y^\beta]$$

$$\left[\frac{\tau_1}{\tau_1+\tau_3}\right]^{p-1} \frac{\tau_1(z^\beta - q^\beta) + \tau_3(u^\beta - q^\beta)}{\tau_1+\tau_3} \frac{\partial}{\partial\theta^\beta}\phi\left(\tau_1 z - \frac{\tau_2\tau_3}{\tau_1+\tau_3}u + \tau_2 q, v + \tau_3 y, \theta, \frac{\tau_1}{\tau_1+\tau_3}\right)$$

Since

$$\forall n > 1 \qquad \int_0^{1-\tau_1} d\tau_3 \frac{\tau_1^n}{(\tau_3+\tau_1)^n} = \frac{1}{n-1}\tau_1(1-\tau_1^{n-1}) \sim \tau_1 \ \text{at} \ \tau_1 \to 0$$

y**-dependent part is in** $\mathcal{H}^{+0}$:

$$\int_0^{1-\tau_1} d\tau_3 \frac{\tau_3\tau_1^n}{(\tau_3+\tau_1)^n} \sim (\tau_1)^2 \ \text{at} \ \tau_1 \to 0,$$

Hence $\beta \to -\infty$ **limit eliminates** y. **For** $\Upsilon(\omega,\omega,C,C)$ **-vertices this leads to ultra-locality by PLT and Factorization Lemma explained below**

Factorization Lemma

$$h'_{q,\beta}(f) = \int_0^1 d\tau \frac{1}{(1-\beta\tau)^2} \int d^2v d^2u \,\mathrm{exp}\, i[v_\alpha u^\alpha] \,\mathrm{exp}\, i\left(\tau y_\alpha (1-\beta) q^\alpha \frac{1}{(1-\beta\tau)}\right)$$
$$\phi\left(\tau(\beta u + (1-\beta)q)\frac{1}{(1-\beta\tau)}, (1-\tau)\left(v + y\frac{1}{(1-\beta\tau)}\right); \tau\right)$$

$$\varepsilon = \frac{1}{|\beta|} : \qquad \int_0^1 \frac{1}{(1-\beta\tau)^{2+n}} = (n+1)^{-1}\varepsilon + O(\varepsilon^2)\,.$$

Extra degree in τ yields zero:

$$h'_{q,-\infty}(f) = \int_0^1 d\sigma \int \frac{d^2v d^2u}{(2\pi)^2} \,\mathrm{exp}\, i[v_\alpha u^\alpha + \sigma y_\alpha q^\alpha] \phi\Big(\sigma(q-u), v+(1-\sigma)y, \theta, \tau\Big)\Big|_{\theta=\tau=0}\,.$$