Nested Bethe ansatz for RTT–algebras of symplectic type

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Main result

We studied the highest weight representations of RTT–algebras for R–matrix $\mathrm{sp}(2n)$ and $\mathrm{so}(2n)$ types by the Nested algebraic Bethe ansatz.

The aim of my talk is to show the construction for $\mathrm{sp}_q(2n)$.

The formulation of the quantum inverse scattering method, or algebraic Bethe ansatz, by the Leningrad school provides eigenvectors and eigenvalues of the transfer matrix. The latter is the generating function of the conserved quantities of a large family of quantum integrable models. The transfer matrix eigenvectors are constructed from the representation theory of the RTT–algebras. In order to construct these eigenvectors, one should first prepare Bethe vectors, depending on a set of complex variables. The first formulation of Bethe vectors for $\mathrm{gl}(n)$-invariant models was given by P.P. Kulish and N.Yu. Reshetikhin in [4] where the nested algebraic Bethe ansatz was introduced. These vectors are given by recursion on the rank of the algebra and use of the embedding $\mathrm{gl}(n-1) \subset \mathrm{gl}(n)$ and for $\mathrm{gl}(2)$ case is well know. We will describe this construction for know case of RTT–algebra of $\mathrm{gl}(3)$ type. The crucial fact for this construction is that RTT–algebra of $\mathrm{gl}(2)$ type is RTT–subalgebra of RTT-algebra $\mathrm{gl}(3)$.

[1] P. P. Kulish, N. Yu. Reshetikhin, Diagonalization of GL(N) invariant transfer matrices and quantum N-wave system (Lee model), J. Phys. A: 16 (1983) L591 L596.

Let ua remember the normalized R–matrix of $\mathrm{gl}(n)$ type

$$\mathbf{R}(x,y) = \frac{x-y}{x-y+1}\,\mathbf{I}\otimes\mathbf{I} + \frac{1}{x-y+1}\,\mathbf{P} = \frac{1}{f(x,y)}\,\mathbf{I}\otimes\mathbf{I} + \frac{g(x,y)}{f(x,y)}\,\mathbf{P}\,, \tag{1}$$

where

$$\mathbf{I} = \sum_{i=1}^{n}\mathbf{E}_i^i\,, \qquad \mathbf{P} = \sum_{i,k=1}^{n}\mathbf{E}_k^i\otimes\mathbf{E}_i^k\,, \qquad g(x,y) = \frac{1}{x-y}\,, \qquad f(x,y) = \frac{x-y+1}{x-y}\,,$$

and then the R–matrix of $\mathrm{sp}(4)$ type,

$$\mathbf{R}(x,y) = \frac{1}{x-y+1}\left((x-y)\mathbf{I}\otimes\mathbf{I} + \mathbf{P} - \frac{x-y}{x-y+3}\mathbf{Q}\right) = \qquad (2)$$
$$= \frac{1}{f(x,y)}\Big(\mathbf{I}\otimes\mathbf{I} + g(x,y)\mathbf{P} - h(x,y)\mathbf{Q}\Big),$$

where

$$\mathbf{I} = \sum_{k=-2}^{2}\mathbf{E}_k^k\,, \qquad \mathbf{P} = \sum_{i,k=-2}^{2}\mathbf{E}_k^i\otimes\mathbf{E}_i^k\,, \qquad \mathbf{Q} = \sum_{i,k=-2}^{2}\theta_{i,k}\mathbf{E}_k^i\otimes\mathbf{E}_{-k}^{-i},$$
$$h(x,y) = \frac{1}{x-y+3}\,, \qquad \theta_{i,k} = \mathrm{sgn}(i)\cdot\mathrm{sgn}(k)\,.$$

In these formulas $\mathbf{E}_i^k$ are elementary matrices with 1 in the i–th row and k-th collum and 0 elsewhere. So $\mathbf{I}$ is the unit matrix and the relations $\mathbf{E}_k^i\mathbf{E}_s^r = \delta_s^i\mathbf{E}_k^r$ are hold.

These R–matrices fulfill the Yang–Baxter equation

$$\mathbf{R}_{1,2}(x,y)\mathbf{R}_{1,3}(x,z)\mathbf{R}_{2,3}(y,z) = \mathbf{R}_{2,3}(y,z)\mathbf{R}_{1,3}(x,z)\mathbf{R}_{1,2}(x,y) \tag{3}$$

The RTT–algebra is an associative algebra $\mathcal{A}$ with unity, which is generated by the elements $T^i_k(x)$, where $T^i_k(x)$ are defined by means of the monodromy matrix

$$\mathbf{T}(x) = \sum_{i,k} \mathbf{E}^k_i \otimes T^i_k(x)\,,$$

for which the RTT–equation

$$\mathbf{R}_{1,2}(x,y)\mathbf{T}_1(x)\mathbf{T}_2(y) = \mathbf{T}_2(y)\mathbf{T}_1(x)\mathbf{R}_{1,2}(x,y) \tag{4}$$

holds.

Next we define the elements

$$H(x) = \mathrm{Tr}\big(\mathbf{T}(x)\big) = \sum_i T_i^i(x)\,.$$

Since in both cases R–matrices are invertible, we obtain from the RTT–equation that the operators $H(x)$ and $H(y)$ commute for any x and y, i.e. for any x and y the relation $H(x)H(y) = H(y)H(x)$ is valid.

We assume that in the vector space of the representation of the RTT–algebra $\mathcal{A}$ there is an element ω, the vacuum vector for which is

$$T^i_k(x)\omega = 0 \quad \text{for} \quad i > k\,, \qquad \text{(or for } i < k\text{)}$$
$$T^i_i(x)\omega = \lambda_i(x)\omega\,.$$

In the vector space $\mathcal{W} = \mathcal{A}\omega$ we will search eigenvectors of the operators $H(x)$, i.e. non-zero elements $w \in \mathcal{W}$ which are for any x solution of the equation

$$H(x)w = E(x)w\,.$$

The RTT–equations (4) give us in the case of $\mathrm{gl}(n)$ relations

$$\begin{aligned} T^i_k(x)T^r_s(y) + g(x,y)T^r_k(x)T^i_s(y) &= T^r_s(y)T^i_k(x) + g(x,y)T^r_k(y)T^i_s(x) \\ T^i_k(x)T^r_s(y) + g(y,x)T^i_s(x)T^r_k(y) &= T^r_s(y)T^i_k(x) + g(y,x)T^i_s(y)T^r_k(x) \end{aligned} \tag{5}$$

It is easy to see that the RTT–algebra of $\mathrm{gl}(n-1)$ is a RTT–subalgebra of $\mathrm{gl}(n)$. This fact is used by Kulish and Reshetikhin for formulation of of The Nested Bethe ansatz in the paper [4].

In the case of the R–matrix of $\mathrm{sp}(4)$ type,the RTT–algebra $\mathcal{A}$ is generated by the elements $T^i_k(x)$, where $i,\ k = \pm 1,\ \pm 2$ and RTT–equations (4) give us

$$T^i_k(x)T^r_s(y) + g(x,y)T^r_k(x)T^i_s(y) + \delta^{i,-r}h(x,y)\sum_{p=-2}^{2}\theta_{p,r}T^p_k(x)T^{-p}_s(y) =$$
$$= T^r_s(y)T^i_k(x) + g(x,y)T^r_k(y)T^i_s(x) + \delta_{k,-s}h(x,y)\sum_{p=-2}^{2}\theta_{k,p}T^r_p(y)T^i_{-p}(x)$$
$$T^i_k(x)T^r_s(y) + g(y,x)T^i_s(x)T^r_k(y) + \delta_{k,-s}h(y,x)\sum_{p=-2}^{2}\theta_{s,p}T^i_p(x)T^r_{-p}(y) =$$
$$= T^r_s(y)T^i_k(x) + g(y,x)T^i_s(y)T^r_k(x) + h(y,x)\delta^{i,-r}\sum_{p=-2}^{2}\theta_{i,p}T^p_s(y)T^{-p}_k(x).$$

For this RTT–algebra we do not see any RTT-subalgebra structure similar to the $\mathrm{gl}(n)$ case but as we show latter we can still formulate the nested Bethe ansatz.

The applications to integrable systems are connected with the highest weight representations of RTT–algebras. For the RTT–algebra of $\mathrm{gl}(2)$ type the vacuum vector ω fulfils

$$T_1^2(x)\omega = 0\,, \qquad T_1^1(x)\omega = \lambda_1(x)\omega\,, \qquad T_2^2(x)\omega = \lambda_2(x)\omega\,.$$

We always denote the set of variables by a bar

$$\overline{u} = \{u_1, u_2, \ldots, u_N\}\,, \qquad \overline{u}_k = \overline{u} \setminus \{u_k\}\,.$$

For the highest weight representation the vector space of representation $\mathcal{W}$ is defined by the elements

$$\big|\overline{u}\big\rangle = T_2^1(\overline{u})\omega\,, \quad \text{where} \quad T_2^1(\overline{u}) = T_2^1(u_1)\,T_2^1(u_2)\ldots T_2^1(u_N)\,.$$

From the commutation relations (5) we will use

$$T_1^1(x)\,T_2^1(u) = f(u,x)\,T_2^1(u)\,T_1^1(x) - g(u,x)\,T_2^1(x)\,T_1^1(u)\,,$$
$$T_2^2(x)\,T_2^1(u) = f(x,u)\,T_2^1(u)\,T_2^2(x) - g(x,u)\,T_2^1(x)\,T_2^2(u)\,.$$

By induction it is possible to obtain formulas

$$T_1^1(x)\,T_2^1(\overline{u}) = F(\overline{u},x)\,T_2^1(\overline{u})\,T_1^1(x) - \sum_{u_k\in\overline{u}} g(u_k,x)F(\overline{u}_k,u_k)\,T_2^1\left(\{\overline{u}_k,x\}\right)T_1^1(u_k)\,,$$
$$T_2^2(x)\,T_2^1(\overline{u}) = F(x,\overline{u})\,T_2^1(\overline{u})\,T_2^2(x) - \sum_{u_k\in\overline{u}} g(x,u_k)F(u_k,\overline{u}_k)\,T_2^1\left(\{\overline{u}_k,x\}\right)T_2^2(u_k)\,,$$

where

$$F(\overline{u},x) = \prod_{u_k\in\overline{u}} f(u_k,x)\,,\qquad F(x,\overline{u}) = \prod_{u_k\in\overline{u}} f(x,u_k)\,.$$

If we apply these relations to the vacuum vector ω, we get relationships

$$T_1^1(x)\big|\overline{u}\big\rangle = \lambda_1(x)F(\overline{u},x)\big|\overline{u}\big\rangle - \sum_{u_k\in\overline{u}}\lambda_1(u_k)g(u_k,x)F(\overline{u}_k,u_k)\big|\{\overline{u}_k,x\}\big\rangle\,,$$

$$T_2^2(x)\big|\overline{u}\big\rangle = \lambda_2(x)F(x,\overline{u})\big|\overline{u}\big\rangle - \sum_{u_k\in\overline{u}}\lambda_2(u_k)g(x,u_k)F(u_k,\overline{u}_k)\big|\{\overline{u}_k,x\}\big\rangle\,,$$

$$H(x)\big|\overline{u}\big\rangle = \Big(\lambda_1(x)F(\overline{u},x) + \lambda_2(x)F(x,\overline{u})\Big)\big|\overline{u}\big\rangle + \\ + \sum_{u_k\in\overline{u}} g(x,u_k)\Big(\lambda_1(u_k)F(\overline{u}_k,u_k) - \lambda_2(u_k)F(u_k,\overline{u}_k)\Big)\big|\{\overline{u}_k,x\}\big\rangle$$

Theorem Let for any $u_k \in \overline{u}$ the Bethe conditions

$$\lambda_1(u_k)F(\overline{u}_k, u_k) = \lambda_2(u_k)F(u_k, \overline{u}_k)$$

are fulfilled. Then for any x is vector $|\overline{u}\rangle$ eigenvectors of operators $H(x)$ with eigenvalue

$$E(x; \overline{u}) = \lambda_1(x)F(\overline{u}, x) + \lambda_2(x)F(x, \overline{u}) \,.$$

To fix the notation, we will take the vacuum vector ω for highest weight representation of RTT–algebra of $\mathrm{gl}(3)$ type as

$$T_1^2(x)\omega = 0\,,\qquad T_2^3(x)\omega = 0\,,\qquad T_1^3(x)\omega = 0\,,$$
$$T_1^1(x)\omega = \lambda_1(x)\omega\,,\qquad T_2^2(x)\omega = \lambda_2(x)\omega\,,\qquad T_3^3(x)\omega = \lambda_3(x)\omega\,.$$

For further calculation we denote the R–matrix of $\mathrm{gl}(3)$ restricted to indices 1, 2 as $\tilde{\mathbf{R}}(x, y)$. Namely,

$$\tilde{\mathbf{R}} = \frac{1}{f(x,y)}\left(\tilde{\mathbf{I}} \otimes \tilde{\mathbf{I}} + g(x,y)\tilde{\mathbf{P}}\right), \quad \text{where} \quad \tilde{\mathbf{I}} = \sum_{a=1}^{2} \mathbf{E}_a^a, \quad \tilde{\mathbf{P}} = \sum_{a,b=1}^{2} \mathbf{E}_b^a \otimes \mathbf{E}_a^b .$$

Similarly, $\tilde{\mathbf{T}}(x)$ denotes the monodromy matrix

$$\tilde{\mathbf{T}}(x) = \sum_{a,b=1}^{2} \mathbf{E}_a^b \otimes T_b^a(x) .$$

Since the R–matrix $\tilde{\mathbf{R}}(x, y)$ evidently fulfils thr Yang–Baxter equation, we can define by means of RTT–equation for $\tilde{\mathbf{T}}(x)$ RTT–subalgebra $\tilde{\mathcal{A}}$. This subalgebra is RTT–algebra of the $\mathrm{gl}(2)$ type.

The idea of nested Bethe ansatz formulated in [4] is to take the eigenvectors of operators $H(x) = T_1^1(x) + T_2^2(x) + T_3^3(x)$, in the form

$$\sum_{a_1,a_2,\dots,a_M=1}^{2} T_3^{a_1}(v_1)T_3^{a_2}(v_2)\dots T_3^{a_M}(v_M)\Phi_{a_1,a_2,\dots,a_M}, \qquad (6)$$

where $\vec{v} = (v_1, v_2, \dots, v_M)$ is ordered set of different numbers and $\Phi_{a_1,a_2,\dots,a_M} \in \tilde{\mathcal{A}}\omega = \mathcal{W}_0$.

The matrices $\mathbf{E}_b^a$ can be identify with operators on vector space $\mathcal{V}$ using the relation

$$\mathbf{E}_b^a \mathbf{e}_c = \mathbf{E}_b^a \mathbf{E}_c^3 = \delta_c^a \mathbf{E}_b^3 = \delta_c^a \mathbf{e}_b$$

For these operators the relations $\mathbf{E}_b^a \mathbf{E}_d^c = \delta_d^a \mathbf{E}_d^c$ are hold.
As usual, we define in the dual space $\mathcal{V}^*$ linear operators $\left(\mathbf{E}_b^a\right)^* = \mathbf{F}_a^b$ using the relationship

$$\left\langle \mathbf{E}_a^b \mathbf{e}_c, \mathbf{f}^d \right\rangle = \mathrm{Tr}\left(\mathbf{E}_a^b \mathbf{E}_c^3 \mathbf{E}_3^d\right) = \mathrm{Tr}\left(\mathbf{E}_c^3 \mathbf{E}_3^d \mathbf{E}_a^b\right) = \left\langle \mathbf{e}_c, \mathbf{F}_a^b \mathbf{f}^d \right\rangle,$$

i.e. $\mathbf{F}_a^b \mathbf{f}^d = \delta_a^d \mathbf{f}^b$. Contrary to relation for operators $\mathbf{E}_b^a$, the operators $\mathbf{F}_b^a$ fulfill relations

$$\mathbf{F}_a^b \mathbf{F}_c^d = \delta_a^d \mathbf{F}_c^b \,.$$

The shape (6) for the eigenvectors can then be write then as

$$\left\langle \mathbf{B}_{1,\dots,M}(\vec{v}), \Phi \right\rangle,$$

where $\mathbf{B}(v) = \sum_{a=1}^{2} \mathbf{e}_a \otimes T_3^a(v)$ and

$$\mathbf{B}_{1,\dots,M}(\vec{v}) = \mathbf{B}_1(v_1)\mathbf{B}_2(v_2)\dots\mathbf{B}_M(v_M) =$$

$$= \sum_{a_1,\dots,a_M=1}^{2} \mathbf{e}_{a_1} \otimes \dots \otimes \mathbf{e}_{a_M} \otimes T_3^{a_1}(v_1)\dots T_3^{a_M}(v_M)$$

$$\Phi = \sum_{b_1,\dots,b_M=1}^{2} \mathbf{f}^{b_1} \otimes \mathbf{f}^{b_2} \otimes \dots \otimes \mathbf{f}^{b_M} \otimes \Phi_{b_1,\dots,b_M}.$$

What we will do now. We will apply on the vector(6) elements $T_3^3(x)$, $T_2^2(x)$ and $T_1^1(x)$. To do this purpose we use the commutation relations (5). So let me prepare some systematic way. For calculation is easy to show that the next lemma is true.

Lemma
(a) For all $\Phi \in \mathcal{W}_0$ the relation

$$T_1^3(x)\Phi = T_2^3(x)\Phi = 0\,, \qquad T_3^3(x)\Phi = \lambda_3(x)\Phi$$

is hold.

(b) The following relations are true:

$$\mathbf{B}_1(x)\mathbf{B}_2(y) = \tilde{\mathbf{R}}_{2,1}(y,x)\mathbf{B}_2(y)\mathbf{B}_1(x)\,, \qquad \mathbf{B}_1(x)\mathbf{B}_2(y) = \tilde{\mathbb{R}}_{1,2}\mathbf{B}_2(x)\mathbf{B}_1(y)\,,$$

where

$$\tilde{\mathbb{R}}_{1,2} = \tilde{\mathbf{R}}_{1,2}(x,x) = \tilde{\mathbf{P}}_{1,2} = \sum_{a,b=1}^{2} \mathbf{E}_b^a \otimes \mathbf{E}_a^b\,.$$

We can write the first relation as

$$\Big\langle \mathbf{B}_1(x)\mathbf{B}_2(y), \mathbf{f}^{b_1} \otimes \mathbf{f}^{b_2} \Big\rangle = \Big\langle \mathbf{B}_2(y)\mathbf{B}_1(x), \tilde{\mathbf{R}}^*_{2^*,1^*}(y,x)\big(\mathbf{f}^{b_1} \otimes \mathbf{f}^{b_2}\big) \Big\rangle,$$

where

$$\tilde{\mathbf{R}}^*_{2^*,1^*}(y,x) = \frac{1}{f(y,x)}\Big(\mathbf{I}^* \otimes \mathbf{I}^* + g(y,x)\sum_{a,b=1}^{2} \mathbf{F}_b^a \otimes \mathbf{F}_a^b\Big), \qquad \mathbf{I}^* = \sum_{a=1}^{2} \mathbf{F}_a^a\,.$$

(c) The relations

$$
\begin{aligned}
T_3^3(x)\mathbf{B}(v) &= f(x,v)\mathbf{B}(v)\,T_3^3(x) - g(x,v)\mathbf{B}(x)\,T_3^3(v)\,,\\
\tilde{\mathbf{T}}_0(x)\Big\langle \mathbf{B}_1(v), \mathbf{I}\otimes\mathbf{f}^b\Big\rangle_1 &= f(v,x)\Big\langle \mathbf{B}_1(v), \widehat{\mathbf{T}}_{0,1}(x;v)\big(\mathbf{I}\otimes\mathbf{f}^b\big)\Big\rangle_1 - \\
&\qquad -g(v,x)\Big\langle \mathbf{B}_1(x), \widehat{\mathbb{T}}_{0,1}(v)\big(\mathbf{I}\otimes\mathbf{f}^b\big)\Big\rangle_1,
\end{aligned}
$$

where

$$
\widehat{\mathbf{T}}_{0,1}(x;v) = \widehat{\mathbf{R}}_{0,1^*}(x,v)\tilde{\mathbf{T}}_0(x)\,,
$$

$$
\widehat{\mathbf{R}}_{0,1^*}(x;v) = \frac{1}{f(v,x)}\Big(\mathbf{I}\otimes\mathbf{I}^* + g(v,x)\sum_{a,b=1}^{2}\mathbf{E}_b^a\otimes\mathbf{F}_a^b\Big)
$$

$$
\widehat{\mathbb{T}}_{0,1}(v) = \widehat{\mathbf{T}}_{0,1}(v;v) = \widehat{\mathbb{R}}_{0,1^*}\tilde{\mathbf{T}}_0(v)\,, \qquad \widehat{\mathbb{R}}_{0,1^*} = \widehat{\mathbf{R}}_{0,1^*}(v,v) = \sum_{a,b=1}^{2}\mathbf{E}_b^a\otimes\mathbf{F}_a^b\,,
$$

are valid.

(d) If we denote

$$\mathbf{B}_{k;1,\ldots,M}(x,\vec{v}_k) = \mathbf{B}_k(x)\mathbf{B}_1(v_1)\ldots\mathbf{B}_{k-1}(v_{k-1})\mathbf{B}_{k+1}(v_{k+1})\ldots\mathbf{B}_N(v_M)\,,$$

$$\tilde{\mathbf{R}}^*_{k;1,\ldots,k}(\vec{v}) = \tilde{\mathbf{R}}^*_{k,1}(v_k,v_1)\ldots\tilde{\mathbf{R}}^*_{k,k-1}(v_k,v_{k-1})\,,\qquad \tilde{\mathbf{R}}_{1;1\ldots,1}(\vec{v}) = \mathbf{I}^*$$

we have

$$\left\langle \mathbf{B}_{1,\ldots,M}(\vec{v}),\mathbf{f}^{b_1}\otimes\ldots\otimes\mathbf{f}^{b_M}\right\rangle = \left\langle \mathbf{B}_{k;1,\ldots,M}(v_k,\vec{v}_k),\tilde{\mathbf{R}}^*_{k;1,\ldots,k}(\vec{v})\big(\mathbf{f}^{b_1}\otimes\ldots\otimes\mathbf{f}^{b_M}\big)\right\rangle.$$

Further we define

$$\widehat{\mathbf{T}}_{0,1,\ldots,M}(x;\vec{v}) = \widehat{\mathbf{R}}_{0,1}(x,v_1)\widehat{\mathbf{R}}_{0,2}(x,v_2)\ldots\widehat{\mathbf{R}}_{0,M}(x,v_M)\tilde{\mathbf{T}}_0(x)\,,$$
$$\widehat{\mathbb{T}}_{k;0,1,\ldots,M}(\vec{v}) = \widehat{\mathbf{T}}_{0,1,\ldots,M}(v_k;\vec{v})\,.$$

Now we will formulate the theorem which can found in the original work by Kulish and Reshetikhin [4].

Theorem Let Φ be an eigenvector of the operator

$$\widehat{H}_{1,\ldots,M}(x;\vec{v}) = \left(\widehat{\mathbf{T}}_{0,1,\ldots,M}(x;\vec{v})\right)_1^1 + \left(\widehat{\mathbf{T}}_{0,1,\ldots,M}(x;\vec{v})\right)_2^2,$$

i.e. $\widehat{H}_{1,\ldots,M}(x;\vec{v})\Phi = \mu(x;\vec{v})\Phi$ is valid. If for any $v_k \in \overline{v}$ the relations

$$\lambda_3(v_k)F(v_k,\overline{v}_k) = \mu(v_k;\vec{v})F(\vec{v}_k,v_k),$$

is hold, the vector $\langle \mathbf{B}_{1,\ldots,M}(\vec{v}),\Phi\rangle$ is eigenvectors of operator $H(x) = T_1^1(x) + T_2^2(x) + T_3^3(x)$ with eigenvalue

$$E(x;\vec{v},\mu) = \lambda_3(x)F(x,\overline{v}) + \mu(x;\vec{v})F(\overline{v},x).$$

The proof of the theorem follows from following lemma

Lemma For any $\Phi = \sum_{b_1,\ldots,b_M=1}^{2} \mathbf{f}^{b_1} \otimes \ldots \otimes \mathbf{f}^{b_M} \otimes \Phi_{b_1,\ldots,b_M}$, where $\Phi_{b_1,\ldots,b_M} \in \mathcal{W}_0$, it is true

$$\text{(a)}\qquad T_3^3(x)\Big\langle \mathbf{B}_{1,\ldots,M}(\vec{v}), \Phi \Big\rangle = \lambda_3(x) F(x,\overline{v}) \Big\langle \mathbf{B}_{1,\ldots,M}(\vec{v}), \Phi \Big\rangle - \\ - \sum_{v_k\in\overline{v}} \lambda_3(v_k) g(x,v_k) F(v_k,\overline{v}_k) \Big\langle \mathbf{B}_{k;1,\ldots,M}(x,\vec{v}_k), \tilde{\mathbf{R}}^*_{k;1,\ldots,k}(\vec{v})\Phi \Big\rangle,$$

$$\text{(b)}\qquad \tilde{\mathbf{T}}_0(x)\Big\langle \mathbf{B}_{1,\ldots,M}(\vec{v}), \Phi \Big\rangle = F(\overline{v},x) \Big\langle \mathbf{B}_{1,\ldots,M}(\vec{v}), \widehat{\mathbf{T}}_{0,1,\ldots,M}(x;\vec{v})\Phi \Big\rangle - \\ - \sum_{v_k\in\overline{v}} g(v_k,x) F(\overline{v}_k,v_k) \Big\langle \mathbf{B}_{k;1,\ldots,M}(x,\vec{v}_k), \tilde{\mathbf{R}}^*_{k;1,\ldots,k}(\vec{v}) \widehat{\mathbb{T}}_{k;0,1,\ldots,M}(\vec{v})\Phi \Big\rangle,$$

Now we will study the eigenvectors of $\widehat{H}_{1,\ldots,M}(x;\vec{v})$. The following lemma converts this problem to the $\mathrm{gl}(2)$ case.

Lemma The operators $\widehat{\mathbf{T}}_{0,1,\ldots,M}(x;\vec{v})$ satisfy the RTT–equation

$$\tilde{\mathbf{R}}_{0,0'}(x,y)\widehat{\mathbf{T}}_{0,1,\ldots,M}(x;\vec{v})\widehat{\mathbf{T}}_{0',1,\ldots,M}(y;\vec{v}) = \widehat{\mathbf{T}}_{0',1,\ldots,M}(y;\vec{v})\widehat{\mathbf{T}}_{0,1,\ldots,M}(x;\vec{v})\tilde{\mathbf{R}}_{0,0'}(x,y)\,.$$

PROOF. From the definition of the operators $\widehat{\mathbf{T}}_{0,1,\dots,M}(x;\vec{v})$ we obtain

$$\widehat{\mathbf{T}}_{0,1,\dots,M}(x;\vec{v})\widehat{\mathbf{T}}_{0',1,\dots,M}(y;\vec{v}) = \widehat{\mathbf{R}}_{0,1}(x;v_1)\dots\widehat{\mathbf{R}}_{0,M}(x;v_M)\tilde{\mathbf{T}}_0(x)$$
$$\widehat{\mathbf{R}}_{0',1}(y;v_1)\dots\widehat{\mathbf{R}}_{0',M}(y;v_M)\tilde{\mathbf{T}}_{0'}(y) =$$
$$= \Big(\widehat{\mathbf{R}}_{0,1}(x;v_1)\widehat{\mathbf{R}}_{0',1}(y;v_1)\Big)\dots\Big(\widehat{\mathbf{R}}_{0,M}(x;v_M)\widehat{\mathbf{R}}_{0',M}(y;v_M)\Big)\Big(\tilde{\mathbf{T}}_0(x)\tilde{\mathbf{T}}_{0'}(y)\Big)$$

Since for the operators $\tilde{\mathbf{T}}_0(x)$ the RTT–equation

$$\tilde{\mathbf{R}}_{0,0'}(x,y)\tilde{\mathbf{T}}_0(x)\tilde{\mathbf{T}}_{0'}(y) = \tilde{\mathbf{T}}_{0'}(y)\tilde{\mathbf{T}}_0(x)\tilde{\mathbf{R}}_{0,0'}(x,y)$$

are valid, it is enough to show that for each $v_k \in \overline{v}$ the relationship holds

$$\tilde{\mathbf{R}}_{0,0'}(x,y)\widehat{\mathbf{R}}_{0,k}(x;v_k)\widehat{\mathbf{R}}_{0',k}(y;v_k) = \widehat{\mathbf{R}}_{0',k}(y;v_k)\widehat{\mathbf{R}}_{0,k}(x;v_k)\tilde{\mathbf{R}}_{0,0'}(x,y)\,,$$

which can be verified by direct calculation.

Now we must only find the vacuum vectors of $\widehat{H}_{1,\ldots,M}(x;\vec{v})$. The following lemma will give us the answer.

Lemma For the vector

$$\widehat{\Omega} = \underbrace{\mathbf{f}^2 \otimes \ldots \otimes \mathbf{f}^2}_{M\times} \otimes\, \omega$$

we have

$$\widehat{T}_1^2(x;\vec{v})\widehat{\Omega} = 0\,, \qquad \widehat{T}_1^1(x;\vec{v})\widehat{\Omega} = \mu_1(x;\vec{v})\widehat{\Omega}\,, \qquad \widehat{T}_2^2(x;\vec{v})\widehat{\Omega} = \mu_2(x;\vec{v})\widehat{\Omega}\,,$$

where

$$\mu_1(x;\vec{v}) = \frac{\lambda_1(x)}{F(\overline{v},x)}\,, \qquad \mu_2(x;\vec{v}) = \lambda_2(x)\,.$$

The proof of this lemma will omit.

So we can now formulate the final theorem for the $\mathrm{gl}(3)$ case.

Theorem Let for any $u_i \in \overline{u}$ and $v_k \in \overline{v}$ the Bethe conditions

$$\lambda_1(u_i)F(\overline{u}_i, u_i) = \lambda_2(u_i)F(\overline{v}, u_i)F(u_i, \overline{u}_i)\,,$$
$$\lambda_3(v_k)F(v_k, \overline{v}_k) = \lambda_2(v_k)F(\overline{v}_k, v_k)F(v_k, \overline{u})$$

be valid. Then $\left|\vec{v}, \overline{u}\right\rangle = \left\langle \mathbf{B}_{1,\ldots,M}(\vec{v}); \Phi(\overline{u}; \vec{v})\right\rangle$ is eigenvector of $H(x)$ with eigenvalue

$$E(x; \overline{u}; \vec{v}) = \lambda_1(x)F(\overline{u}, x)F(\overline{v}, x) + \lambda_2(x)F(x, \overline{u})F(\overline{v}, x) + \lambda_3(x)F(x, \overline{v})$$

PROOF: According to lemma 4.3, apply to operators $\widehat{\mathbf{T}}_{0,1,\ldots,M}(x;\vec{v})$ RTT–algebra of the $\mathrm{gl}(2)$ type. The vector $\widehat{\Omega}$ is the vacuum vector with weights

$$\mu_1(x;\vec{v}) = \frac{\lambda_1(x)}{F(\overline{v},x)}\,, \qquad \mu_2(x;\vec{v}) = \lambda_2(x)$$

According of Theorem 3.1, if for any $u_i \in \overline{u}$ the relation

$$\mu_1(u_i;\vec{v})F(\overline{u}_i,u_i) = \mu_2(u_i;\vec{v})F(u_i,\overline{u}_i)$$

is true, the vector $\Phi(\overline{u};\vec{v}) = \widehat{T}_1^2(\overline{u};\vec{v})\widehat{\Omega}$ is an eigenvector of the operators

$$\mathrm{Tr}_0\big(\widehat{\mathbf{T}}_{0,1,\ldots}(x;\vec{v})\big) = \widehat{H}_{1,\ldots,M}(x;\vec{v})$$

with eigenvalue

$$\widehat{E}(x;\overline{u};\vec{v}) = \mu_1(x;\vec{v})F(\overline{u},x) + \mu_2(x;\vec{v})F(x,\overline{u})\,,$$

Further, according to Theorem 4.1, the vector $\left|\vec{v},\overline{u}\right\rangle$ is an eigenvector of the operator $H(x)$, if the condition

$$\lambda_3(v_k)F(v_k,\overline{v}_k) = \mu(v_k;\vec{v})F(\vec{v}_k,v_k) = \lambda_2(v_k)F(v_k,\overline{u})F(\vec{v}_k,v_k)$$

is met.

Eigenvectors and Bethe condition for $U_q\big(\mathrm{sp}(2n)\big)$ case.

We will consider the R-matrix of $U_q(\mathrm{sp}(2n))$ which has the shape

$$\mathbf{R}(x) = \frac{1}{\alpha(x)}\Big(\sum_{i,k;\, i\neq\pm k}\mathbf{E}_i^i\otimes\mathbf{E}_k^k + f(x)\sum_i\mathbf{E}_i^i\otimes\mathbf{E}_i^i + f(x^{-1}q^{-n-1})\sum_i\mathbf{E}_i^i\otimes\mathbf{E}_{-i}^{-i} +$$
$$+g(x)\sum_{k<i}\mathbf{E}_k^i\otimes\mathbf{E}_i^k - g(x^{-1})\sum_{i<k}\mathbf{E}_k^i\otimes\mathbf{E}_i^k -$$
$$-g(xq^{n+1})\sum_{k<i}q^{k-i}\epsilon_i\epsilon_k\mathbf{E}_k^i\otimes\mathbf{E}_{-k}^{-i} + g(x^{-1}q^{-n-1})\sum_{i<k}q^{k-i}\epsilon_i\epsilon_k\mathbf{E}_k^i\otimes\mathbf{E}_{-k}^{-i}\Big)$$

where $\epsilon_i = \mathrm{sign}(i)$ and

$$f(x) = \frac{xq - x^{-1}q^{-1}}{x - x^{-1}}, \qquad g(x) = \frac{x(q-q^{-1})}{x - x^{-1}}, \qquad \alpha(x) = 1 + \frac{q - q^{-1}}{x - x^{-1}}.$$

This R-Matrix satisfies the Yang-Baxter equation

$$\mathbf{R}_{1,2}(x)\mathbf{R}_{1,3}(xy)\mathbf{R}_{2,3}(y) = \mathbf{R}_{2,3}(y)\mathbf{R}_{1,3}(xy)\mathbf{R}_{1,2}(x)$$

and is invertible.
By using RTT-equation

$$\mathbf{R}_{1,2}(xy^{-1})\mathbf{T}_1(x)\mathbf{T}_2(y) = \mathbf{T}_2(y)\mathbf{T}_1(x)\mathbf{R}_{1,2}(xy^{-1})\,.$$

we can define RTT–algebra $\mathcal{A}$ of $U_q\big(\mathrm{sp}(2n)\big)$ type.
From invertibility of R–matrix we have :
If we define

$$H(x) = \mathrm{Tr}\big(\mathbf{T}(x)\big) = \sum_{i=-n}^{n} T_i^i(x)$$

fulfills the equation $H(x)H(y) = H(y)H(x)$ for any x and y.

We suppose that in the representation space $\mathcal{W}$ of the RTT–algebra $\mathcal{A}$ there exists a vacuum vector $\omega \in \mathcal{W}$, for which $\mathcal{W} = \mathcal{A}\omega$ and

$$T^i_k(x)\omega = 0 \quad \text{pro} \quad i < k\,, \qquad T^i_i(x)\omega = \lambda_i(x)\omega \quad \text{pro} \quad i = \pm 1,\, \pm 2,\, \ldots,\, \pm n\,.$$

In the vector space $\mathcal{W} = \mathcal{A}\omega$, we will look for eigenvectors for which

$$H(x)w = E(x)w\,.$$

In the RTT–algebra $\mathcal{A}$, we have the RTT–subalgebras $\mathcal{A}^{(+)}$ and $\mathcal{A}^{(-)}$, that are generated by the elements $T^i_k(x)$ and $T^{-i}_{-k}(x)$, where $i,\ k = 1,\ 2,\ \ldots,\ n$.
First, we will study the subspace

$$\mathcal{W}_0 = \mathcal{A}^{(+)}\mathbf{A}^{(-)}\omega \subset \mathcal{W} = \mathcal{A}\omega\,.$$

Lemma 1. For any $i,\ k = 1,\ 2,\ \ldots,\ n$ and any $\Omega \in \mathcal{W}_0$ $T_k^{-i}(x)\Omega = 0$ is valid.

Lemma 2. If we denote

$$\mathbf{T}^{(+)}(x) = \sum_{i,k=1}^{n} \mathbf{E}_i^k \otimes T^i_k(x)\,, \qquad \mathbf{T}^{(-)}(x) = \sum_{i,k=1}^{n} \mathbf{E}_{-i}^{-k} \otimes T^{-i}_{-k}(x)\,,$$

then on the space $\mathcal{W}_0$ for any $\epsilon_1,\ \epsilon_2 = \pm$

$$\mathbf{R}_{1,2}^{(\epsilon_1,\epsilon_2)}(xy^{-1})\mathbf{T}_1^{(\epsilon_1)}(x)\mathbf{T}_2^{(\epsilon_2)}(y) = \mathbf{T}_2^{(\epsilon_2)}(y)\mathbf{T}_1^{(\epsilon_1)}(x)\mathbf{R}_{1,2}^{(\epsilon_1,\epsilon_2)}(xy^{-1}) \qquad (7)$$

where

$$\mathbf{R}_{1,2}^{(+,+)}(x) = \frac{1}{f(x)}\Big(\sum_{i,k=1;\, i\neq k}^{n} \mathbf{E}_i^i \otimes \mathbf{E}_k^k + f(x)\sum_{i=1}^{n} \mathbf{E}_i^i \otimes \mathbf{E}_i^i +$$
$$+g(x)\sum_{1\leq k<i\leq n} \mathbf{E}_k^i \otimes \mathbf{E}_i^k - g(x^{-1})\sum_{1\leq i<k\leq n} \mathbf{E}_k^i \otimes \mathbf{E}_i^k\Big)$$

$$\mathbf{R}_{1,2}^{(-,-)}(x) = \frac{1}{f(x)}\Big(\sum_{i,k=1;\, i\neq k}^{n} \mathbf{E}_{-i}^{-i} \otimes \mathbf{E}_{-k}^{-k} + f(x)\sum_{i=1}^{n} \mathbf{E}_{-i}^{-i} \otimes \mathbf{E}_{-i}^{-i} +$$
$$+g(x)\sum_{1\leq i<k\leq n} \mathbf{E}_{-k}^{-i} \otimes \mathbf{E}_{-i}^{-k} - g(x^{-1})\sum_{1\leq k<i\leq n} \mathbf{E}_{-k}^{-i} \otimes \mathbf{E}_{-i}^{-k}\Big)$$

$$\mathbf{R}_{1,2}^{(+,-)}(x) = \sum_{i,k=1;\, i\neq k}^{n} \mathbf{E}_i^i \otimes \mathbf{E}_{-k}^{-k} + f(x^{-1}q)\sum_{i=1}^{n} \mathbf{E}_i^i \otimes \mathbf{E}_{-i}^{-i} -$$
$$-g(xq^{-1})\sum_{1\leq k<i\leq n} q^{k-i}\mathbf{E}_k^i \otimes \mathbf{E}_{-k}^{-i} + g(x^{-1}q)\sum_{1\leq i<k\leq n} q^{k-i}\mathbf{E}_k^i \otimes \mathbf{E}_{-k}^{-i}$$

$$\mathbf{R}_{1,2}^{(-,+)}(x) = \sum_{i,k=1;\, i\neq k}^{n} \mathbf{E}_{-i}^{-i} \otimes \mathbf{E}_k^k + f(x^{-1}q^{-n-1})\sum_{i=1}^{n} \mathbf{E}_{-i}^{-i} \otimes \mathbf{E}_i^i -$$
$$-g(xq^{n+1})\sum_{1\leq i<k\leq n} q^{i-k}\mathbf{E}_{-k}^{-i} \otimes \mathbf{E}_k^i + g(x^{-1}q^{-n-1})\sum_{1\leq k<i\leq n} q^{i-k}\mathbf{E}_{-k}^{-i} \otimes \mathbf{E}_k^i$$

is valid.

Proposition 1.
If we define

$$\tilde{\mathbf{R}}_{1,2}(x) = \mathbf{R}_{1,2}^{(+,+)}(x) + \mathbf{R}_{1,2}^{(+,-)}(x) + \mathbf{R}_{1,2}^{(-,+)}(x) + \mathbf{R}_{1,2}^{(-,-)}(x)$$
$$\tilde{\mathbf{T}}(x) = \mathbf{T}^{(+)}(x) + \mathbf{T}^{(-)}(x)$$

on the space $\mathcal{W}_0$, the RTT–equation

$$\tilde{\mathbf{R}}_{1,2}(xy^{-1})\tilde{\mathbf{T}}_1(x)\tilde{\mathbf{T}}_2(y) = \tilde{\mathbf{T}}_2(y)\tilde{\mathbf{T}}_1(x)\tilde{\mathbf{R}}_{1,2}(xy^{-1})$$

is valid.
Also, the R–matrix $\tilde{\mathbf{R}}(x)$ fulfills the Yang–Baxter equation

$$\tilde{\mathbf{R}}_{1,2}(x)\tilde{\mathbf{R}}_{1,3}(xy)\tilde{\mathbf{R}}_{2,3}(y) = \tilde{\mathbf{R}}_{2,3}(y)\tilde{\mathbf{R}}_{1,3}(xy)\tilde{\mathbf{R}}_{1,2}(x)$$

and has the inverse matrix

$$\left(\tilde{\mathbf{R}}_{1,2}(x)\right)^{-1} = \left(\mathbf{R}_{1,2}^{(+,+)}(x)\right)^{-1} + \left(\mathbf{R}_{1,2}^{(+,-)}(x)\right)^{-1} + \left(\mathbf{R}_{1,2}^{(-,+)}(x)\right)^{-1} + \left(\mathbf{R}_{1,2}^{(-,-)}(x)\right)^{-1}$$

.
So the R–matrix $\tilde{\mathbf{R}}(x)$ defines the RTT-algebra as $\tilde{\mathcal{A}}_n$.

We find out by the standard procedure from the RTT–equation

$$\tilde{\mathbf{T}}_1(x)\tilde{\mathbf{T}}_2(y) = \left(\tilde{\mathbf{R}}_{1,2}(xy^{-1})\right)^{-1}\tilde{\mathbf{T}}_2(y)\tilde{\mathbf{T}}_1(x)\tilde{\mathbf{R}}_{1,2}(xy^{-1})$$

that the operators $\tilde{H}(x)$ and $\tilde{H}(y)$, where

$$\tilde{H}(x) = \mathrm{Tr}_{(+,-)}\left(\tilde{\mathbf{T}}(x)\right) = \mathrm{Tr}_{+}\left(\mathbf{T}^{(+)}(x)\right) + \mathrm{Tr}_{-}\left(\mathbf{T}^{(-)}(x)\right) = \sum_{i=1}^{n}\left(T_i^i(x) + T_{-i}^{-i}(x)\right)$$

commutate with each other. Futher in the RTT–algebra $\tilde{\mathcal{A}}_n$ for any $\varepsilon_1,\,\varepsilon_2 = \pm$ the RTT–equations

$$\left(\mathbf{R}_{1,2}^{(\varepsilon_1,\varepsilon_2)}(xy^{-1})\right)\mathbf{T}_1^{(\varepsilon_1)}(x)\mathbf{T}^{(\varepsilon_2)}(y) = \mathbf{T}_2^{(\varepsilon_2)}(y)\mathbf{T}^{(\varepsilon_1)}(x)\mathbf{R}_{1,2}^{(\varepsilon_1,\varepsilon_2)}(xy^{-1})$$

are valid for any $\varepsilon_1,\,\varepsilon_2 = \pm$.
It follows that in the RTT–algebra $\tilde{\mathcal{A}}_n$ all operators $\tilde{H}^{(\pm)}(x)$ a $\tilde{H}^{(\pm)}(y)$, where

$$\tilde{H}^{(+)}(x) = \mathrm{Tr}_{+}\left(\mathbf{T}^{(+)}(x)\right) = \sum_{i=1}^{n} T_i^i(x)\,, \qquad \tilde{H}^{(-)}(x) = \mathrm{Tr}_{-}\left(\mathbf{T}^{(-)}(x)\right) = \sum_{i=1}^{n} T_{-i}^{-i}(x),$$

commute.

We denote by $\vec{u} = (u_1, u_2, \ldots, u_M)$ an ordered set of mutually different complex numbers.
We will look for eigenvectors in the form

$$\mathfrak{V}(\vec{u}) = \sum_{i_1,\ldots,i_M,k_1,\ldots,k_M=1}^{n} T^{i_1}_{-k_1}(u_1) T^{i_2}_{-k_2}(u_2) \ldots T^{i_M}_{-k_M}(u_M) \Phi^{k_1,k_2,\ldots,k_M}_{i_1,i_2,\ldots i_M}$$

where $\Phi^{k_1,k_2,\ldots,k_M}_{i_1,i_2,\ldots i_M} \in \mathcal{W}_0$.
We selec

$$\mathbf{B}(u) = \sum_{i,k=1}^{n} \mathbf{e}_i \otimes \mathbf{f}^{-k} \otimes T^{i}_{-k}(u) \in \mathcal{V}_+ \otimes \mathcal{V}^*_- \otimes \mathcal{A}$$

where $\mathbf{e}_i$ is the basis of the space $\mathcal{V}_+$ and $\mathbf{f}^{-k}$ is the basis of the space $\mathcal{V}^*_-$ and define

$$\mathbf{B}_{1,\ldots,M}(\vec{u}) = \mathbf{B}_1(u_1) \otimes \mathbf{B}_2(u_2) \otimes \ldots \otimes \mathbf{B}_M(u_M) =$$

$$= \sum_{i_1,\ldots,i_M,k_1,\ldots,k_M}^{n} \mathbf{e}_{i_1} \otimes \ldots \otimes \mathbf{e}_{i_M} \otimes \mathbf{f}^{-k_1} \otimes \ldots \otimes \mathbf{f}^{-k_M} \otimes T^{i_1}_{-k_1}(u_1) \ldots T^{i_M}_{-k_M}(u_M)$$

On the space $\mathcal{V}_0 \otimes \mathcal{V}^*_{1_+} \otimes \mathcal{V}_{1-} \otimes \mathcal{A}$ we define

$$\widehat{\mathbf{T}}^{(+)}_{0;1}(x;u) = \left(\widehat{\mathbf{R}}^{(+,+)}_{0,1^*}(xu^{-1})\right)^{-1}\mathbf{T}^{(+)}_0(x)\widehat{\mathbf{R}}^{(+,-)}_{0,1}(xu^{-1})$$
$$\widehat{\mathbf{T}}^{(-)}_{0;1}(x;u) = \left(\widehat{\mathbf{R}}^{(-,+)}_{0,1^*}(xu^{-1})\right)^{-1}\mathbf{T}^{(-)}_0(x)\widehat{\mathbf{R}}^{(-,-)}_{0,1}(xu^{-1})$$

where

$$\left(\widehat{\mathbf{R}}^{(+,+)}_{0,1^*}(x)\right)^{-1} = \frac{1}{f(x^{-1})}\Big(\sum_{i,k=1;\, i\neq k}^{n}\mathbf{E}^i_i\otimes\mathbf{F}^k_k\otimes\mathbf{I}_- + f(x^{-1})\sum_{i=1}^{n}\mathbf{E}^i_i\otimes\mathbf{F}^i_i\otimes\mathbf{I}_- +$$
$$+g(x^{-1})\sum_{1\le i<k\le n}\mathbf{E}^i_k\otimes\mathbf{F}^k_i\otimes\mathbf{I}_- - g(x)\sum_{1\le k<i\le n}\mathbf{E}^i_k\otimes\mathbf{F}^k_i\otimes\mathbf{I}_-\Big)$$

$$\left(\widehat{\mathbf{R}}^{(-,+)}_{0,1^*}(x)\right)^{-1} = \sum_{i,k=1;\, i\neq k}^{n}\mathbf{E}^{-i}_{-i}\otimes\mathbf{F}^k_k\otimes\mathbf{I}_- + f(xq)\sum_{i=1}^{n}\mathbf{E}^{-i}_{-i}\otimes\mathbf{F}^i_i\otimes\mathbf{I}_- +$$
$$+g(xq)\sum_{1\le i<k\le n}q^{k-i}\mathbf{E}^{-i}_{-k}\otimes\mathbf{F}^i_k\otimes\mathbf{I}_- - g(x^{-1}q^{-1})\sum_{1\le k<i\le n}q^{k-i}\mathbf{E}^{-i}_{-k}\otimes\mathbf{F}^i_k\otimes\mathbf{I}_-$$

$$\widehat{\mathbf{R}}^{(+,-)}_{0,1}(x) = \sum_{i,k=1;\, i\neq k}^{n}\mathbf{E}^i_i\otimes\mathbf{I}^*_+\otimes\mathbf{E}^{-k}_{-k} + f(x^{-1}q)\sum_{i=1}^{n}\mathbf{E}^i_i\otimes\mathbf{I}^*_+\otimes\mathbf{E}^{-i}_{-i} +$$
$$+g(x^{-1}q)\sum_{1\le i<k\le n}q^{k-i}\mathbf{E}^i_k\otimes\mathbf{I}^*_+\otimes\mathbf{E}^{-i}_{-k} - g(xq^{-1})\sum_{1\le k<i\le n}q^{k-i}\mathbf{E}^i_k\otimes\mathbf{I}^*_+\otimes\mathbf{E}^{-i}_{-k}$$

$$\widehat{\mathbf{R}}^{(-,-)}_{0,1}(x) = \frac{1}{f(x)}\Big(\sum_{i,k=1;\, i\neq k}^{n}\mathbf{E}^{-i}_{-i}\otimes\mathbf{I}^*_+\otimes\mathbf{E}^{-k}_{-k} + f(x)\sum_{i=1}^{n}\mathbf{E}^{-i}_{-i}\otimes\mathbf{I}^*_+\otimes\mathbf{E}^{-i}_{-i} +$$
$$+g(x)\sum_{1\le i<k\le n}\mathbf{E}^{-i}_{-k}\otimes\mathbf{I}^*_+\otimes\mathbf{E}^{-k}_{-i} - g(x^{-1})\sum_{1\le k<i\le n}\mathbf{E}^{-i}_{-k}\otimes\mathbf{I}^*_+\otimes\mathbf{E}^{-k}_{-i}\Big)$$

Introduce operators

$$\widehat{\mathbf{T}}^{(+)}_{0;1,\ldots,M}(x;\vec{u}) = \left(\widehat{\mathbf{R}}^{(+,+)}_{0,1^*}(xu_1^{-1})\right)^{-1}\ldots\left(\widehat{\mathbf{R}}^{(+,+)}_{0,M^*}(xu_M^{-1})\right)^{-1}\mathbf{T}^{(+)}_0(x)$$
$$\widehat{\mathbf{R}}^{(+,-)}_{0,M}(xu_M^{-1})\ldots\widehat{\mathbf{R}}^{(+,-)}_{0,1}(xu_1^{-1})$$

$$\widehat{\mathbf{T}}^{(-)}_{0;1,\ldots,M}(x;\vec{u}) = \left(\widehat{\mathbf{R}}^{(-,+)}_{0,1^*}(xu_1^{-1})\right)^{-1}\ldots\left(\widehat{\mathbf{R}}^{(-,+)}_{0,M^*}(xu_M^{-1})\right)^{-1}\mathbf{T}^{(-)}_0(x)$$
$$\widehat{\mathbf{R}}^{(-,-)}_{0,M}(xu_M^{-1})\ldots\widehat{\mathbf{R}}^{(-,-)}_{0,1}(xu_1^{-1})$$

$$\mathbf{B}_{k;1,\ldots,M}(x;\vec{u}_k) = \mathbf{B}_k(x)\otimes\mathbf{B}_1(u_1)\otimes\ldots\otimes\mathbf{B}_{k-1}(u_{k-1})\otimes\mathbf{B}_{k+1}(u_{k+1})\otimes\ldots\otimes\mathbf{B}_M(u_M)$$

The following statement, which gives part of the Bethe conditions, follows from the previous part.

Theorem 1. Let Φ be common eigenvector operators

$$\widehat{H}^{(+)}_{1,\dots,M}(x;\vec{u}) = \mathrm{Tr}_0\Big(\widehat{\mathbf{T}}^{(+)}_{0;1,\dots,M}(x;\vec{u})\Big) \quad \text{a} \quad \widehat{H}^{(-)}_{1,\dots,M}(x;\vec{u}) = \mathrm{Tr}_0\Big(\widehat{\mathbf{T}}^{(-)}_{0;1,\dots,M}(x;\vec{u})\Big),$$

which means that

$$\widehat{H}^{(+)}_{1,\dots,M}(x;\vec{u})\Phi = \widehat{E}^{(+)}_{1,\dots,M}(x;\vec{u})\Phi\,, \qquad \widehat{H}^{(-)}_{1,\dots,M}(x;\vec{u})\Phi = \widehat{E}^{(-)}_{1,\dots,M}(x;\vec{u})\Phi$$

If the relations for each $u_k \in \overline{u}$

$$\widehat{E}^{(+)}_{1,\dots,M}(u_k;\vec{u})F(u_k^{-1};\overline{u}_k) = \widehat{E}^{(-)}_{1,\dots,M}(u_k;\vec{u})F(u_k;\overline{u}_k^{-1})$$

are true
then $\Big\langle \mathbf{B}_{1,\dots,M}(\vec{u}),\Phi \Big\rangle$ is the eigenvector of the operator $H(x) = H^{(+)}(x) + H^{(-)}(x)$, where $H^{(\pm)}(x) = \mathrm{Tr}\big(\mathbf{T}^{(\pm)}_0(x)\big)$, and

$$H(x)\Big\langle \mathbf{B}_{1,\dots,M}(\vec{u}),\Phi \Big\rangle = \Big(\widehat{E}^{(+)}_{1,\dots,M}(x;\vec{u})F(x^{-1};\overline{u}) + \widehat{E}^{(-)}_{1,\dots,M}(x;\vec{u})F(x;\overline{u}^{-1})\Big)\Big\langle \mathbf{B}_{1,\dots,M}(\vec{u}),\Phi \Big\rangle$$

are valid.

So we can find the proper vectors of the $H(x)$ common eigenvectors of the operators $\widehat{H}^{(+)}_{1,\dots,M}(x;\vec{u})$ and $\widehat{H}^{(-)}_{1,\dots,M}(x;\vec{u})$.

Theorem 2. The operators $\widehat{\mathbf{T}}^{(\pm)}_{0;1,\dots,M}(x;\vec{u})$ fulfill RTT–equation

$$\mathbf{R}^{(\epsilon,\epsilon')}_{0,0'}(xy^{-1})\widehat{\mathbf{T}}^{(\epsilon)}_{0;1,\dots,M}(x;\vec{u})\widehat{\mathbf{T}}^{(\epsilon')}_{0';1,\dots,M}(y;\vec{u}) = \widehat{\mathbf{T}}^{(\epsilon')}_{0';1,\dots,M}(y;\vec{u})\widehat{\mathbf{T}}^{(\epsilon)}_{0;1,\dots,M}(x;\vec{u})\mathbf{R}^{(\epsilon,\epsilon')}_{0,0'}(xy^{-1})$$

for any $\vec{u}$ and $\epsilon,\ \epsilon' = \pm$ and thus generate RTT - algebra $\tilde{A}_n$.
This RTT–equation we couldn't write the equation for $M = 1$ using a matrix $\tilde{\mathbf{B}}(u)$.
Therefore, we prefer to use the formulation using spaces $\mathcal{V}_\pm$ a $\mathcal{V}^*_\pm$.

Theorem 3. The vector

$$\Omega = \underbrace{\mathbf{f}^1 \otimes \ldots \otimes \mathbf{f}^1}_{M\times} \otimes \underbrace{\mathbf{e}_{-1} \otimes \ldots \otimes \mathbf{e}_{-1}}_{M\times} \otimes \omega$$

is a vacuum vector for representing RTT–algebra $\tilde{\mathcal{A}}_n$ with the weights

$$\mu_1(x;\vec{u}) = \lambda_1(x)F(x^{-1}q;\overline{u}) \qquad \mu_k(x;\vec{u}) = \lambda_k(x)F(xq^{-1};\overline{u}^{-1}) \qquad k = 2,\ldots,n$$

$$\mu_{-1}(x;\vec{u}) = \lambda_{-1}(x)F(xq;\overline{u}^{-1}) \qquad \mu_{-k}(x;\vec{u}) = \lambda_{-k}(x)F(x^{-1}q^{-1};\overline{u}) \qquad k = 2,\ldots,n\,.$$

So to find our own vectors of the operator $H(x)$ for the RTT-algebra of $U_q\big(\mathrm{sp}(2n)\big)$ type, just formulate the Bethe ansatz for the RTT–algebra $\tilde{\mathcal{A}}_n$.

2. Common eigenvectors of the operators $\tilde{H}^{(+)}(x)$ and $\tilde{H}^{(-)}(x)$ in the RTT–algebra $\tilde{\mathcal{A}}_2$

Specific forms of commutation relations in RTT - type algebra $\tilde{\mathcal{A}}_2$ are listed in Appendix. Among other things, for every x and y

$$\tilde{T}_1^2(x)\tilde{T}_1^2(y) = \tilde{T}_1^2(y)\tilde{T}_1^2(x)\,, \qquad \tilde{T}_{-2}^{-1}(x)\tilde{T}_{-2}^{-1}(y) = \tilde{T}_{-2}^{-1}(y)\tilde{T}_{-2}^{-1}(x)\,,$$
$$\tilde{T}_1^2(x)\tilde{T}_{-2}^{-1}(y) = \tilde{T}_{-2}^{-1}(y)\tilde{T}_1^2(x)$$

holds.
Let $\tilde{\omega}$ be a vacuum vector for representation of the RTT–algebra $\tilde{\mathcal{A}}_2$, so we have

$$\tilde{T}_2^1(x)\tilde{\omega} = \tilde{T}_{-1}^{-2}(x)\tilde{\omega} = 0\,, \qquad \tilde{T}_{\pm i}^{(\pm i)}(x)\tilde{\omega} = \mu_{\pm i}(x)\tilde{\omega} \quad i = 1, 2\,.$$

Common eigenvectors of the operators $\tilde{H}^{(+)}(x)$ and $\tilde{H}^{(-)}(x)$ will be searched for in the form

$$\big|\overline{v};\overline{w}\big\rangle = \tilde{T}_1^2(v_1)\tilde{T}_1^2(v_2)\ldots\tilde{T}_1^2(v_P)\tilde{T}_{-2}^{-1}(w_1)\tilde{T}_{-2}^{-1}(w_2)\ldots\tilde{T}_{-2}^{-1}(w_Q)\tilde{\omega} = \tilde{T}_1^2(\overline{v})\tilde{T}_{-2}^{-1}(\overline{w})\tilde{\omega}\,,$$

where $\overline{v}$ and $\overline{w}$ are the sets $\overline{v} = \big\{v_1, v_2, \ldots, v_P\big\}$ and $\overline{w} = \big\{w_1, w_2, \ldots, w_Q\big\}$.

Propositin 2. In the RTT–algebra $\tilde{\mathcal{A}}_2$ we have

$$\tilde{T}_1^1(x)\tilde{T}_1^2(\overline{v}) = F(x;\overline{v}^{-1})\tilde{T}_1^2(\overline{v})\tilde{T}_1^1(x) - \sum_{v_r\in\overline{v}} g(xv_r^{-1})F(v_r;\overline{v}_r^{-1})\tilde{T}_1^2(x,\overline{v}_r)\tilde{T}_1^1(v_r)$$

$$\tilde{T}_2^2(x)\tilde{T}_1^2(\overline{v}) = F(x^{-1};\overline{v})\tilde{T}_1^2(\overline{v})\tilde{T}_2^2(x) + \sum_{v_r\in\overline{v}} g(xv_r^{-1})F(v_r^{-1};\overline{v}_r)\tilde{T}_1^2(x,\overline{v}_r)\tilde{T}_2^2(v_r)$$

$$\tilde{T}_{-1}^{-1}(x)\tilde{T}_{-2}^{-1}(\overline{w}) = F(x^{-1};\overline{w})\tilde{T}_{-2}^{-1}(\overline{w})\tilde{T}_{-1}^{-1}(x) + \\ + \sum_{w_s\in\overline{w}} g(xw_s^{-1})F(w_s^{-1};\overline{w}_s)\tilde{T}_{-2}^{-1}(x;\overline{w}_s)\tilde{T}_{-1}^{-1}(w_s)$$

$$\tilde{T}_{-2}^{-2}(x)\tilde{T}_{-2}^{-1}(\overline{w}) = F(x;\overline{w}^{-1})\tilde{T}_{-2}^{-1}(\overline{w})\tilde{T}_{-2}^{-2}(x) - \sum_{w_s\in\overline{w}} g(xw_s^{-1})F(w_s;\overline{w}_s^{-1})\tilde{T}_{-2}^{-1}(x,\overline{w}_s)\tilde{T}_{-2}^{-2}(w$$

$$\tilde{T}_1^1(x)\tilde{T}_{-2}^{-1}(\overline{w}) = F(xq^{-2};\overline{w}^{-1})\tilde{T}_{-2}^{-1}(\overline{w})\tilde{T}_1^1(x) + \\ + \sum_{w_s\in\overline{w}} g(xw_s^{-1}q^{-2})F(w_s;\overline{w}_s^{-1})\tilde{T}_1^2(x)\tilde{T}_{-2}^{-1}(\overline{w}_s)\tilde{T}_{-2}^{-2}(w_s)$$

$$\tilde{T}_2^2(x)\tilde{T}_{-2}^{-1}(\overline{w}) = F(x^{-1}q^2;\overline{w})\tilde{T}_{-2}^{-1}(\overline{w})\tilde{T}_2^2(x) - \\ - \sum_{w_s\in\overline{w}} g(xw_s^{-1}q^{-2})F(w_s^{-1};\overline{w}_s)\tilde{T}_1^2(x)\tilde{T}_{-2}^{-1}(\overline{w}_s)\tilde{T}_{-1}^{-1}(w_s)$$

$$\tilde{T}_{-1}^{-1}(x)\tilde{T}_1^2(\overline{v}) = F(x^{-1}q^{-2};\overline{v})\tilde{T}_1^2(\overline{v})\tilde{T}_{-1}^{-1}(x) - \\ - \sum_{v_r\in\overline{v}} g(xv_r^{-1}q^2)F(v_r^{-1};\overline{v}_r)\tilde{T}_1^2(\overline{v}_r)\tilde{T}_{-2}^{-1}(x)\tilde{T}_2^2(v_r)$$

$$\tilde{T}_{-2}^{-2}(x)\tilde{T}_1^2(\overline{v}) = F(xq^2;\overline{v}^{-1})\tilde{T}_1^2(\overline{v})\tilde{T}_{-2}^{-2}(x) + \sum_{v_r\in\overline{v}} g(xv_r^{-1}q^2)F(v_r;\overline{v}_r^{-1})\tilde{T}_1^2(\overline{v}_r)\tilde{T}_{-2}^{-1}(x)T_1^1(v_r)$$

Using Proposition 2 we obtain next proposition

Proposition 3. For any x, $\overline{v}$ and $\overline{w}$ we have

$$\tilde{T}_1^1(x)\big|\overline{v};\overline{w}\big\rangle = \mu_1(x)F(x;\overline{v}^{-1})F(xq^{-2};\overline{w}^{-1})\big|\overline{v};\overline{w}\big\rangle -$$

$$-\sum_{v_r\in\overline{v}}\mu_1(v_r)g(xv_r^{-1})F(v_r;\overline{v}_r^{-1})F(v_rq^{-2};\overline{w}^{-1})\big|x,\overline{v}_r;\overline{w}\big\rangle +$$

$$+\sum_{w_s\in\overline{w}}\mu_{-2}(w_s)g(xw_s^{-1}q^{-2})F(w_sq^2;\overline{v}^{-1})F(w_s;\overline{w}_s^{-1})\big|x,\overline{v};\overline{w}_s\big\rangle$$

$$\tilde{T}_2^2(x)\big|\overline{v};\overline{w}\big\rangle = \mu_2(x)F(x^{-1};\overline{v})F(x^{-1}q^2;\overline{w})\big|\overline{v};\overline{w}\big\rangle +$$

$$+\sum_{v_r\in\overline{v}}\mu_2(v_r)g(xv_r^{-1})F(v_r^{-1};\overline{v}_r)F(v_r^{-1}q^2;\overline{w})\big|x,\overline{v}_r;\overline{w}\big\rangle -$$

$$-\sum_{w_s\in\overline{w}}\mu_{-1}(w_s)g(xw_s^{-1}q^{-2})F(w_s^{-1}q^{-2};\overline{v})F(w_s^{-1};\overline{w}_s)\big|x,\overline{v};\overline{w}_s\big\rangle$$

$$\tilde{T}_{-1}^{-1}(x)\big|\overline{v};\overline{w}\big\rangle = \mu_{-1}(x)F(x^{-1}q^{-2};\overline{v})F(x^{-1};\overline{w})\big|\overline{v};\overline{w}\big\rangle -$$

$$-\sum_{v_r\in\overline{v}}\mu_2(v_r)g(xv_r^{-1}q^2)F(v_r^{-1};\overline{v}_r)F(v_r^{-1}q^2;\overline{w})\big|\overline{v}_r;x,\overline{w}\big\rangle +$$

$$+\sum_{w_s\in\overline{w}}\mu_{-1}(w_s)g(xw_s^{-1})F(w_s^{-1}q^{-2};\overline{v})F(w_s^{-1};\overline{w}_s)\big|\overline{v};x,\overline{w}_s\big\rangle$$

$$\tilde{T}_{-2}^{-2}(x)\big|\overline{v};\overline{w}\big\rangle = \mu_{-2}(x)F(xq^2;\overline{v}^{-1})F(x;\overline{w}^{-1})\big|\overline{v};\overline{w}\big\rangle +$$

$$+\sum_{v_r\in\overline{v}}\mu_1(v_r)g(xv_r^{-1}q^2)F(v_r;\overline{v}_r^{-1})F(v_rq^{-2};\overline{w}^{-1})\big|\overline{v}_r;x,\overline{w}\big\rangle -$$

$$-\sum_{w_s\in\overline{w}}\mu_{-2}(w_s)g(xw_s^{-1})F(w_sq^2;\overline{v}^{-1})F(w_s;\overline{w}_s^{-1})\big|\overline{v};x,\overline{w}_s\big\rangle .$$

From this statement we obtain for the action of the operators $\tilde{H}^{(\pm)}(x)$

$$\tilde{H}^{(+)}(x)\big|\overline{v};\overline{w}\big\rangle = \tilde{T}_1^1(x)\big|\overline{v};\overline{w}\big\rangle + \tilde{T}_2^2(x)\big|\overline{v};\overline{w}\big\rangle =$$

$$= \Big(\mu_1(x)F(x;\overline{v}^{-1})F(xq^{-2};\overline{w}^{-1}) + \mu_2(x)F(x^{-1};\overline{v})F(x^{-1}q^2;\overline{w})\Big)\big|\overline{v};\overline{w}\big\rangle -$$

$$-\sum_{v_r\in\overline{v}} g(xv_r^{-1})\Big(\mu_1(v_r)F(v_r;\overline{v}_r^{-1})F(v_rq^{-2};\overline{w}^{-1}) -$$

$$-\mu_2(v_r)F(v_r^{-1};\overline{v}_r)F(v_r^{-1}q^2;\overline{w})\Big)\big|x,\overline{v}_r;\overline{w}\big\rangle -$$

$$-\sum_{w_s\in\overline{w}} g(xw_s^{-1}q^{-2})\Big(\mu_{-1}(w_s)F(w_s^{-1}q^{-2};\overline{v})F(w_s^{-1};\overline{w}_s) -$$

$$-\mu_{-2}(w_s)F(w_sq^2;\overline{v}^{-1})F(w_s;\overline{w}_s^{-1})\Big)\big|x,\overline{v};\overline{w}_s\big\rangle$$

$$\tilde{H}^{(-)}(x)\big|\overline{v};\overline{w}\big\rangle = \tilde{T}_{-1}^{-1}(x)\big|\overline{v};\overline{w}\big\rangle + \tilde{T}_{-2}^{-2}(x)\big|\overline{v};\overline{w}\big\rangle =$$

$$= \Big(\mu_{-1}(x)F(x^{-1}q^{-2};\overline{v})F(x^{-1};\overline{w}) + \mu_{-2}(x)F(xq^2;\overline{v}^{-1})F(x;\overline{w}^{-1})\Big)\big|\overline{v};\overline{w}\big\rangle +$$

$$+\sum_{v_r\in\overline{v}} g(xv_r^{-1}q^2)\Big(\mu_1(v_r)F(v_r;\overline{v}_r^{-1})F(v_rq^{-2};\overline{w}^{-1}) -$$

$$-\mu_2(v_r)F(v_r^{-1};\overline{v}_r)F(v_r^{-1}q^2;\overline{w})\Big)\big|\overline{v}_r;x,\overline{w}\big\rangle +$$

$$+\sum_{w_s\in\overline{w}} g(xw_s^{-1})\Big(\mu_{-1}(w_s)F(w_s^{-1}q^{-2};\overline{v})F(w_s^{-1};\overline{w}_s) -$$

$$-\mu_{-2}(w_s)F(w_sq^2;\overline{v}^{-1})F(w_s;\overline{w}_s^{-1})\Big)\big|\overline{v};x,\overline{w}_s\big\rangle$$

The following statement is

Proposition 4. If for each $v_r \in \overline{v}$ and $w_s \in \overline{w}$ the Bethe conditions are fulfilled

$$
\begin{aligned}
\mu_1(v_r)F(v_r;\overline{v}_r^{-1})F(v_r q^{-2};\overline{w}^{-1}) &= \mu_2(v_r)F(v_r^{-1};\overline{v}_r)F(v_r^{-1}q^2;\overline{w}) \\
\mu_{-1}(w_s)F(w_s^{-1}q^{-2};\overline{v})F(w_s^{-1};\overline{w}_s) &= \mu_{-2}(w_s)F(w_s q^2;\overline{v}^{-1})F(w_s;\overline{w}_s^{-1})
\end{aligned}
\tag{8}
$$

the vectors $|\overline{v},\overline{w}\rangle = \tilde{T}_1^2(\overline{u})\tilde{T}_{-2}^{-1}(\overline{w})\tilde{\omega}$ are common eigenvectors of the operators $\tilde{H}^{(+)}(x)$ and $\tilde{H}^{(-)}(x)$ with eigenvalues

$$
\begin{aligned}
\tilde{E}^{(+)}(x;\overline{v};\overline{w}) &= \mu_1(x)F(x;\overline{v}^{-1})F(xq^{-2};\overline{w}^{-1}) + \mu_2(x)F(x^{-1};\overline{v})F(x^{-1}q^2;\overline{w}) \\
\tilde{E}^{(-)}(x;\overline{v};\overline{w}) &= \mu_{-1}(x)F(x^{-1}q^{-2};\overline{v})F(x^{-1};\overline{w}) + \mu_{-2}(x)F(xq^2;\overline{v}^{-1})F(x;\overline{w}^{-1})
\end{aligned}
$$

3. Bethe conditons and Bethe eigenvectors for the RTT–algebra of $U_q\big(\mathrm{sp}(4)\big)$ type

We have mentioned that the operators $\widehat{T}^i_k(x;\vec{u})$ and $\widehat{T}^{-i}_{-k}(x;\vec{u})$, where $i,\ k=1,\ 2$, generate each $\vec{u}$ RTT–algebra $\tilde{\mathcal{A}}_2$ and the vector $\tilde{\Omega}$ is for this algebra a vacuum vector with weights

$$\mu_1(x;\overline{u})=\lambda_1(x)F(x^{-1}q;\overline{u}) \qquad \mu_2(x;\overline{u})=\lambda_2(x)F(xq^{-1};\overline{u}^{-1})$$
$$\mu_{-1}(x;\overline{u})=\lambda_{-1}(x)F(xq;\overline{u}^{-1}) \qquad \mu_{-2}(x;\overline{u})=\lambda_{-2}(x)F(x^{-1}q^{-1};\overline{u})$$

Propositin 4 says if for every $v_r \in \overline{v}$ and $w_s \in \overline{w}$ the Bethe conditions are fulfilled

$$\mu_1(v_r;\overline{u})F(v_r;\overline{v}_r^{-1})F(v_r q^{-2};\overline{w}^{-1}) = \mu_2(v_r;\overline{u})F(v_r^{-1};\overline{v}_r)F(v_r^{-1}q^2;\overline{w})$$
$$\mu_{-1}(w_s;\overline{u})F(w_s^{-1};\overline{w}_s)F(w_s^{-1}q^{-2};\overline{v}) = \mu_{-2}(w_s;\overline{u})F(w_s;\overline{w}_s^{-1})F(w_s q^2;\overline{v}^{-1})$$

the vectors

$$\Phi(\vec{u};\overline{v};\overline{w}) = \widehat{T}_1^2(\vec{u};\overline{v})\widehat{T}_{-2}^{-1}(\vec{u};\overline{w})\Omega$$

are common eigenvectors of the operators $\widehat{H}^{(+)}_{1,\ldots,M}(x;\vec{u})$ and $\widehat{H}^{(-)}_{1,\ldots,M}(x;\vec{u})$ with eigenvalues

$$\widehat{E}^{(+)}_{1,\ldots,M}(x;\vec{u};\overline{v};\overline{w}) = \mu_1(x;\overline{u})F(x;\overline{v}^{-1})F(xq^{-2};\overline{w}^{-1}) + \mu_2(x;\overline{u})F(x^{-1};\overline{v})F(x^{-1}q^2;\overline{w})$$
$$\widehat{E}^{(-)}_{1,\ldots,M}(x;\vec{u};\overline{v};\overline{w}) = \mu_{-1}(x;\overline{u})F(x^{-1}q^{-2};\overline{v})F(x^{-1};\overline{w}) + \mu_{-2}(x;\overline{u})F(xq^2;\overline{v}^{-1})F(x;\overline{w}^{-1})$$

From Proposition 1 it follows that if for every $u_k \in \overline{u}$ we have

$$F(u_k^{-1};\overline{u}_k)\widehat{E}^{(+)}_{1,\ldots,M}(u_k;\vec{u};\overline{v};\overline{w}) = F(u_k;\overline{u}_k^{-1})\widehat{E}^{(-)}_{1,\ldots,M}(u_k;\vec{u};\overline{v};\overline{w})$$

then the vector

$$\mathfrak{V}(\vec{u};\overline{v};\overline{w}) = \Big\langle \mathbf{B}_{1,\ldots,M})(\vec{u}), \Phi(\vec{u};\overline{v};\overline{w})\Big\rangle \qquad (9)$$

is the eigenvector of the operator $H(x)$. From this we obtain the following theorem:

Theorem. Let the Bethe condition

$$\lambda_1(u_k)F(u_k^{-1};\overline{u}_k)F(u_k^{-1}q;\overline{u}_k)F(u_k;\overline{v}^{-1})F(u_kq^{-2};\overline{w}^{-1}) = \\ = \lambda_{-1}(u_k)F(u_k;\overline{u}_k^{-1})F(u_kq;\overline{u}_k^{-1})F(u_k^{-1}q^{-2};\overline{v})F(u_k^{-1};\overline{w})$$

$$\lambda_1(v_r)F(v_r^{-1}q;\overline{u})F(v_r;\overline{v}_r^{-1})F(v_rq^{-2};\overline{w}^{-1}) = \\ = \lambda_2(v_r)F(v_rq^{-1};\overline{u}^{-1})F(v_r^{-1};\overline{v}_r)F(v_r^{-1}q^2;\overline{w})$$

$$\lambda_{-1}(w_s)F(w_sq;\overline{u}^{-1})F(w_s^{-1}q^{-2};\overline{v})F(w_s^{-1};\overline{w}_s) = \\ = \lambda_{-2}(w_s)F(w_s^{-1}q^{-1};\overline{u})F(w_sq^2;\overline{v}^{-1})F(w_s;\overline{w}_s^{-1})$$

be fulfilled for any for any $u_k \in \overline{u}$, $v_r \in \overline{v}$ and $w_w \in \overline{w}$, then the vectors (9) are eigenvectors of $H(x)$ with eigenvalues

$$E(x;\overline{u};\overline{v};\overline{w}) = \lambda_1(x)F(x^{-1};\overline{u})F(x^{-1}q;\overline{u})F(x;\overline{v}^{-1})F(xq^{-2};\overline{w}^{-1}) + \\ +\lambda_2(x)F(x^{-1};\overline{u})F(xq^{-1};\overline{u}^{-1})F(x^{-1};\overline{v})F(x^{-1}q^2;\overline{w}) + \\ +\lambda_{-1}(x)F(x;\overline{u}^{-1})F(xq;\overline{u}^{-1})F(x^{-1}q^{-2};\overline{v})F(x^{-1};\overline{w}) + \\ +\lambda_{-2}(x)F(x;\overline{u}^{-1})F(x^{-1}q^{-1};\overline{u})F(xq^2;\overline{v}^{-1})F(x;\overline{w}^{-1})$$

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