On supermembranes and domain walls in D=4 supersymmetric theories

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based on arXiv:1905.02743 [hep-th] in collaboration with Stefano Lanza and Dmitri Sorokin and on arXiv:1906.09872 [hep-th].

SQS'19, Yerevan, August 26–31, 2019

Outline

- D=11 supermembrane [Bergshoeff+Sezgin+Townsend 1987], presently also known under the name of M2-brane, is one of the most important fundamental objects of the hypothetical underlying M-theory.
- Its consistency in curved superspace background subject this to the eqs of motion of the 11-dim supergravity, the low energy limit of the M-theory.

- The simpler 4D cousin of M2-brane also attracted an interest already in late 80-th [Achucarro+Gauntlett+Itoh+Townsend 89].
- Different aspects of its interaction with $\mathcal{N} = 1$ $D = 4$ supergravity and matter multiplets were the subject of study in [Ovrut+Waldram 97, Huebscher+Meessen+Ortin 2010, Bandos+ Meliveo 2010-2013, Kuzenko+Tartaglino-Mazzucchelli 2017, Bandos+ Farakos+ Lanza+Martucci+Sorokin 2018]
- The selfconsistency of interaction with supermembrane requires matter and SUGRA supermultiplets to include three form fields, thus leading naturally to the so-called variant superfield representations [Gates 80, Gates+Siegel 80, Binetruy+Pillon+Girardi +Grimm 96, Farrar+Gabadadze+Schwetz 98, Kuzenko+McCarthy 2005, Bandos+ Meliveo 2012, Farakos+Lanza+Martucci+Sorokin 2017].

- In this talk we begin by a brief review of the most general supersymmetric coupling of the supermembrane to 4D supergravity and matter multiplets making emphasis on the case of *open* supermembrane [I.B., arXiv:1906.09872 [hep-th]]
  - which is described by the sum of bulk action, worldvolume action $S_{p=2}$ and boundary term $S_{p=1}$,

$$S = S_{sugra+matter} + S_{p=2} + S_{p=1}$$

  - We will present the local fermionic κ–symmetry transformations leaving invariant interacting action and acting nontrivially on $S_{p=2}$ and $S_{p=1}$
  - which can be treated as actions of supermembrane and superstring at the boundary of supermembrane in the background of supergravity and matter multiplets.
- Then we discuss a particular case of the interaction of supermembrane with SYM and with Veneziano-Yankelovich (VY) effective model for the SYM [I.B.+ Stefano Lanza+Dmitri Sorokin, arXiv:1905.02743 [hep-th]].
- We will show that the account of supermembrane is necessary for consistent description of domain walls in such interacting theory.

- The domain walls in pure SYM theories and in super Quantum Chromodynamics (SQCD) have been under an extensive study [Townsend 1987, Abraham+Townsend 1990, Dvali+Shifman 1996, Smilga +Veselov 1997, Smilga 1997, Kovner+Shifman+Smilga 1997, Kogan+Kovner+Shifman 1997, Shifman+Yung 2009, Bashmakov+Benini+Benvenuti, +Bertolini 2018]
- However, still some strokes could be added.
- In this talk , following [I.B.+ Lanza+Sorokin, arXiv:1905.02743]
- we explicitly include into the 3d worldvolume theory of SYM domain walls the Goldstone d.o.f.-s by constructing the supersymmetric and κ-symmetric coupling of supermembrane to SYM and VY model
- and show that this solves the old standing problem that domain wall tension calculated in VY theory without membrane happened to be smaller than the BPS bound [Kogan+Kovner+Shifman 1997].

Outline

- The action for a supermembrane in a background of SUGRA or SUGRA and 3-form matter multiplet(s) can be written in the universal form

$$S_{p=2} = \int_{W^3} d^3\xi \sqrt{|h|}\, |\mathcal{Z}| + \int_{W^3} C_3 \; .$$

- In it $\xi^m = (\xi^0, \xi^1, \xi^2)$ are local coordinates on the worldvolume

$$W^3 \in \Sigma^{(4|4)} \quad : \qquad z^M = z^M(\xi) = (x^\mu(\xi), \theta^{\check{\alpha}}(\xi), \bar{\theta}^{\check{\dot{\alpha}}}(\xi)) \; .$$

- $h = \det h_{mn}$ is the determinant of the induced metric

$$h_{mn} = E_m^a \eta_{ab} E_n^b \; , \qquad E_m^a = \partial_m z^M(\xi) E_M^a(z(\xi))$$

  which is constructed from the pull-back $E^a(z(\xi)) = d\xi^m E_m^a$ to W^3 of the bosonic supervielbein of the curved $\mathcal{N} = 1$ superspace,

$$E^A(z) = (E^a, E^\alpha, \bar{E}^{\dot{\alpha}}) = dz^M E_M^A(z) \; ,$$

  which are subject to the (minimal) SUGRA constraints

$$T^a = \mathcal{D}E^a = -2i\sigma^a_{\alpha\dot{\alpha}} E^\alpha \wedge \bar{E}^{\dot{\alpha}} - \frac{1}{8} E^b \wedge E^c \varepsilon^a{}_{bcd} G^d \; , \qquad etc.$$

- Finally, in the first Dirac–Nambu–Goto term of the action, $\mathcal{Z}$ denotes the pull-back $\mathcal{Z}(z(\xi))$ of a covariantly chiral superfield $\mathcal{Z}(z)$ of a special type which we describe in a minute.
- Now we just notice that, as any covariantly chiral superfield, $\mathcal{Z}(z)$ obeys

$$\bar{\mathcal{D}}_{\dot{\alpha}}\mathcal{Z} = 0 \; ,$$

where $\bar{\mathcal{D}}_{\dot{\alpha}} = -(\mathcal{D}_{\alpha})^*$ is fermionic covariant derivative obeying

$$\{\mathcal{D}_{\alpha}, \bar{\mathcal{D}}_{\dot{\alpha}}\} = 2i\sigma^a_{\alpha\dot{\alpha}}\mathcal{D}_a \; .$$

- In the second, Wess-Zumino term C_3 is the pull-back of a 3-form potential defined in curved superspace and having the field strength

$$\begin{aligned} H_4 \quad &= dC_3 = \tfrac{1}{2}E^b \wedge E^a \wedge E^{\dot{\alpha}} \wedge E^{\dot{\beta}}\tilde{\sigma}_{ab\,\dot{\alpha}\dot{\beta}}\mathcal{Z} + c.c. \\ &+ \tfrac{1}{12}E^c \wedge E^b \wedge E^a \wedge \epsilon_{abcd}E^{\dot{\beta}}\sigma^d_{\alpha\dot{\beta}}\mathcal{D}^{\alpha}\mathcal{Z} + c.c. \\ &- \tfrac{i}{192}E^d \wedge E^c \wedge E^b \wedge E^a\epsilon_{abcd}(\mathcal{D}\mathcal{D} - 3\bar{R})\mathcal{Z} + c.c. \; . \end{aligned}$$

- Here $R = (\bar{R})^*$ and $G_a = (G_a)^*$ are main superfields of minimal SUGRA.

- This superspace 4-form

$$\begin{aligned} H_4 \quad &= dC_3 = \tfrac{1}{2} E^b \wedge E^a \wedge E^{\dot\alpha} \wedge E^{\dot\beta} \tilde\sigma_{ab\;\dot\alpha\dot\beta} \mathcal{Z} + c.c. \\ &+ \tfrac{1}{12} E^c \wedge E^b \wedge E^a \wedge \epsilon_{abcd} E^{\dot\beta} \sigma^d_{\alpha\dot\beta} \mathcal{D}^\alpha \mathcal{Z} + c.c. \\ &- \tfrac{i}{192} E^d \wedge E^c \wedge E^b \wedge E^a \epsilon_{abcd} (\mathcal{D}\mathcal{D} - 3\bar R)\mathcal{Z} + c.c. \; . \end{aligned}$$

  is closed, $dH_4 = 0$, if the supervielbein obey the SUGRA constraints and $\bar{\mathcal{D}}_{\dot\alpha}\mathcal{Z} = 0$. (Remember that $dd = 0$).
- However, the requirement that it is exact, i.e. that there exists a 3-form C_3 such that $H_4 = dC_3$, requires the chiral superfield $\mathcal{Z}$ to be special, namely to be constructed in terms of real superfield prepotential $\mathcal{P} = \mathcal{P}^*$,

$$\mathcal{Z} = -\frac{1}{4}\left(\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha} - R\right)\mathcal{P} \; .$$

- The component content of $\mathcal{Z}$ is different from that of the usual chiral superfield $\Phi = -\frac{1}{4}\left(\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha} - R\right)\mathbb{K}$ with complex prepotential $\mathbb{K} \neq \mathbb{K}^*$:
- The auxiliary F-component of that superfield is given by a complex linear combination of real scalar and a divergence of a real vector instead of two real scalars (scalar and pseudoscalar) in the case of Φ.

- Hence the name of single three form supermultiplet for the field content of $\mathcal{Z} = -\frac{1}{4}\left(\bar{\mathcal{D}}_{\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}} - R\right)\mathcal{P}$ with an arbitrary real superfield $\mathcal{P}$.
- Now, the real 3-form potential C_3, such that $dC_3 = H_4(\mathcal{Z})$, is expressed in terms of the same real superfield $\mathcal{P}$ by

$$\begin{aligned} C_3 \;\; = \;\; & -iE^a \wedge E^\alpha \wedge \bar{E}^{\dot{\alpha}} \sigma_{a\alpha\dot{\alpha}} \mathcal{P} - \frac{1}{4}\left(E^b \wedge E^a \wedge E^\alpha \sigma_{ab\;\alpha}{}^{\beta} \mathcal{D}_\beta \mathcal{P} + c.c.\right) + \\ & + \frac{1}{48} E^c \wedge E^b \wedge E^a \epsilon_{abcd}\left(\tilde{\sigma}^{d\dot{\alpha}\alpha}[\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}}]\mathcal{P} + 2G^d \mathcal{P}\right) . \end{aligned}$$

- Of course, this is the gauge fixed form of the 3-form potential.
- However, there exists a residual gauge invariance with respect to additive transformations of $\mathcal{P}$ with real linear superfield $\mathbb{L}$,

$$\delta\mathcal{P} = \mathbb{L}\,, \qquad \left(\bar{\mathcal{D}}_{\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}} - R\right)\mathbb{L} = 0\,, \qquad \left(\mathcal{D}^\alpha \mathcal{D}_\alpha - \bar{R}\right)\mathbb{L} = 0$$

leaving $H_4(\mathcal{Z}) = dC_3$ invariant.

- These transformations of the prepotential

$$\delta\mathcal{P} = \mathbb{L}\,, \qquad \left(\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha} - R\right)\mathbb{L} = 0\,, \qquad \left(\mathcal{D}^{\alpha}\mathcal{D}_{\alpha} - \bar{R}\right)\mathbb{L} = 0$$

result in the gauge transformations of the superspace 3-form

$$\delta C_3 = d\alpha_2$$

by closed 3–form $d\alpha_2$ constructed from the real linear superfield $\mathbb{L}$ as follows

$$\begin{aligned} d\alpha_2 \quad = \quad & -iE^a \wedge E^{\alpha} \wedge \bar{E}^{\dot\alpha}\sigma_{a\alpha\dot\alpha}\mathbb{L} - \frac{1}{4}\left(E^b \wedge E^a \wedge E^{\alpha}\sigma_{ab\,\alpha}{}^{\beta}\mathcal{D}_{\beta}\mathbb{L} + c.c.\right) + \\ & + \frac{1}{48}E^c \wedge E^b \wedge E^a \epsilon_{abcd}\left(\tilde\sigma^{d\dot\alpha\alpha}[\mathcal{D}_{\alpha}, \bar{\mathcal{D}}_{\dot\alpha}]\mathbb{L} + 2G^d\mathbb{L}\right)\,. \end{aligned}$$

- Clearly, $d\alpha_2 = C_3|_{\mathcal{P}\mapsto\mathbb{L}}$.

κ–symmetry

- The *closed* supermembrane action is invariant under local fermionic κ–symmetry transformations of the coordinate functions

$$i_\kappa E^a := \delta_\kappa z^M E_M^a = 0 , \qquad i_\kappa E^\alpha := \delta_\kappa z^M E_M^\alpha = \kappa^\alpha , \qquad i_\kappa E^{\dot\alpha} := \delta_\kappa z^M E_M^{\dot\alpha} = \bar\kappa^{\dot\alpha}$$

the fermionic parameters of which obey the conditions

$$\kappa_\alpha = -i \frac{\mathcal{Z}}{|\mathcal{Z}|} \Gamma_{\alpha\dot\alpha} \bar\kappa^{\dot\alpha} , \qquad \bar\kappa_{\dot\alpha} = -i \frac{\bar{\mathcal{Z}}}{|\mathcal{Z}|} \kappa^\alpha \Gamma_{\alpha\dot\alpha} ,$$

where

$$\Gamma_{\alpha\dot\alpha} = \frac{i}{3!\sqrt{h}} \sigma^a_{\alpha\dot\alpha} \epsilon_{abcd} \epsilon^{mnk} E_m^b E_n^c E_k^d ,$$

is imaginary, $(\Gamma_{\alpha\dot\alpha})^* = -\Gamma_{\alpha\dot\alpha}$, and obeys $\Gamma_{\alpha\dot\alpha}\Gamma^{\dot\alpha\beta} = \delta_\alpha{}^\beta$.

- This is important because the local κ-symmetry of the action guaranties that the ground state of the p–brane is BPS state preserving a part of supersymmetry, namely 1/2 of the SUSY in the present case.

SUGRA interacting with closed membrane

- As we have already stated, the action

$$S_{p=2} = \int_{W^3} d^3\xi \sqrt{|h|}\, |\mathcal{Z}| + \int_{W^3} C_3$$

can describe the supermembrane moving in the background of a three-form supergravity.

- In this case the above special chiral superfield $\mathcal{Z}$ should be treated as conformal compensator of a 3-form supergravity and the action for interacting system reads

$$\boxed{S = S_{sugra} + S_{p=2}}\,, \qquad \text{where}$$

$$S_{sugra} = -\frac{3}{4\kappa^2} \int d^8 z\, E\, (\mathcal{Z}\bar{\mathcal{Z}})^{\frac{1}{3}} - \frac{m}{2\kappa^2} \left(\int d^6\zeta_L\, \mathcal{E}\, \mathcal{Z} + c.c. \right) .$$

- (The chiral integration measure is defined by $-\frac{1}{4}\int d^6\zeta_L\, \mathcal{E}\, (\bar{\mathcal{D}}\bar{\mathcal{D}} - R)\, \mathbb{Y} = \frac{1}{2}\int d^8 z E \mathbb{Y}$ for any superfield $\mathbb{Y}$).

Doble 3-form SUGRA interacting with closed membrane

- Actually, this action $S = S_{sugra} + S_{p=2}$ with

$$S_{sugra} = -\frac{3}{4\kappa^2}\int d^8z\, E\,(\mathcal{Z}\bar{\mathcal{Z}})^{\frac{1}{3}} - \frac{m}{2\kappa^2}\left(\int d^6\zeta_L\, \mathcal{E}\, \mathcal{Z} + c.c.\right) ,$$
$$S_{p=2} = \int_{W^3} d^3\xi \sqrt{|h|}\, |\mathcal{Z}| + \int_{W^3} C_3 ,$$

  and $\mathcal{Z} = -\frac{1}{4}\left(\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha} - R\right)\mathcal{P}$ describes the interaction of supermembrane with two versions of 3-forms supergravity:
- The system of *supermembrane and single 3-form supergravity* [Ovrut+Waldram 1996; Bandos+Meliveo 2012] if $\mathcal{P} = (\mathcal{P})^*$ is an independent, unconstrained real superfield;
- The system of *supermembrane and double 3-form supergravity* [Kuzenko+Tartaglino-Mazzucchelli, 2017] if $\mathcal{P}$ is constructed from the complex linear superfield

$$\mathcal{P} = \mathrm{Im}\,\Sigma := \frac{i}{2}(\bar\Sigma - \Sigma), \quad (\mathcal{D}^\alpha\mathcal{D}_\alpha - \bar{R})\Sigma = 0, \quad (\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha} - R)\bar\Sigma = 0 .$$

Supermembrane+SUGRA+3-form matter multilet

- In the case of simplest interacting system of supergravity, supermembrane and a 3-form matter multiplet the interaction action is

$$S = S_{sugra+matter} + S_{p=2} \ ,$$

with formally the same $S_{p=2} = \int\limits_{W^3} d^3\xi \sqrt{|h|}\, |\mathcal{Z}| + \int\limits_{W^3} C_3$ and

$$S_{sugra+matter} = -\frac{3}{4\kappa^2} \int d^8 z\, E\, \Omega(\mathcal{Z}, \bar{\mathcal{Z}}) - \frac{1}{2\kappa^2} \left(\int d^6 \zeta_L\, \mathcal{E}\, \mathcal{W}(\mathcal{Z}) + c.c. \right) \ .$$

- Here $\mathcal{W}(\mathcal{Z})$ is superpotential, $\Omega(\mathcal{Z}, \bar{\mathcal{Z}}) = e^{-\frac{\kappa^2}{3} K(\mathcal{Z}, \bar{\mathcal{Z}})}$.
- This action is invariant under the super-Weyl transformations

$$E^a \quad \mapsto \tilde{E}^a = e^{\Upsilon + \bar{\Upsilon}} E^a \ , \qquad E^\alpha \mapsto \tilde{E}^\alpha = e^{2\bar{\Upsilon} - \Upsilon} \left(E^\alpha - \tfrac{i}{2} E^a \bar{\mathcal{D}}_{\dot{\alpha}} \bar{\Upsilon} \tilde{\sigma}_a^{\dot{\alpha}\alpha} \right) \ , \ldots$$

supplemented by the Kähler trasformations of the Kähler potential.

$$K(\mathcal{Z}, \bar{\mathcal{Z}}) \mapsto K(\mathcal{Z}, \bar{\mathcal{Z}}) + 6\Upsilon(\mathcal{Z}) + 6\bar{\Upsilon}(\bar{\mathcal{Z}}) \ , \qquad \mathcal{W}(\mathcal{Z}) \mapsto \mathcal{W}(\mathcal{Z}) e^{-6\Upsilon(\mathcal{Z})} \ .$$

Supermembrane+3-form SUGRA as a particular case

- Hence any *nonvanishing* superpotential can be gauged to a constant m,

$$K(\mathcal{Z},\bar{\mathcal{Z}}) \mapsto \mathcal{K} = K(\mathcal{Z},\bar{\mathcal{Z}}) + \frac{2}{\kappa^2}\ln|\mathcal{W}(\mathcal{Z})| - \frac{2}{\kappa^2}\ln|m|, \qquad \mathcal{W}(\mathcal{Z}) \mapsto m$$

- In the case of $\mathcal{W}(\mathcal{Z}) = m\mathcal{Z}$ and $K(\mathcal{Z},\bar{\mathcal{Z}}) = -\frac{2}{\kappa^2}\ln|\mathcal{Z}|$, the above transformation removes the chiral superfield $\mathcal{Z}$ from the action.
- This indicates that, as stated, such an interacting action describes 3-form supergravity+supermembrane system (without matter) in the super-Weyl invariant formulation.

κ-symmetry of open supermembrane and superstring at its boundary

- When the supermembrane is not closed, $\partial W^3 = W^2 \neq \emptyset$, its action is not invariant under the κ-symmetry,

$$\begin{aligned} \delta_\kappa S_{p=2} &= \int_{W^2=\partial W^3} i_\kappa C_3 \; , \\ i_\kappa C_3 &= -iE^a \wedge E^\alpha \sigma_{a\alpha\dot\alpha} \bar\kappa^{\dot\alpha} \mathcal{P} - iE^a \wedge \bar E^{\dot\alpha} \kappa^\alpha \sigma_{a\alpha\dot\alpha} \mathcal{P} - \\ & -\frac{1}{4} E^b \wedge E^a \left(\kappa^\alpha \sigma_{ab\,\alpha}{}^\beta \mathcal{D}_\beta \mathcal{P} - \tilde\sigma_{ab}{}^{\dot\beta}{}_{\dot\alpha} \bar\kappa^{\dot\alpha} \bar{\mathcal{D}}_{\dot\beta} \mathcal{P} \right) . \end{aligned}$$

  Neither the open supermembrane action is invariant under the gauge transformations

$$\delta_{gauge} S_{p=2} = \int_{W^2=\partial W^3} \alpha_2 \equiv \int_{W^3} d\alpha_2 \; ,$$

  where $d\alpha_2 = -iE^a \wedge E^\alpha \wedge \bar E^{\dot\alpha} \sigma_{a\alpha\dot\alpha} \mathbb{L} + \ldots$ has been defined before.

3-form gauge symmetry and superstring at the boundary of open supermembrane

- To compensate these nonvanishing variations, it is necessary to put at the boundary of supermembrane a superstring.
- For the gauge symmetry the mechanism of compensation refers to the Wess–Zumino term of the superstring action which is given by integral over the worldsheet of a 2-form potential B_2,

$$- \int_{W^2} B_2 = - \int_{W^3} dB_2 \equiv - \int_{W^3} H_3 \, .$$

- The sum of the Wess-Zumino terms of string and membrane

$$\int_{W^3} C_3 - \int_{W^2} B_2 = \int_{W^3} (C_3 - dB_2)$$

will be invariant under the 3-form gauge transformations if 2-form potential transforms under these as a Stückelberg field,

$$\delta C_3 = d\alpha_2 \; , \qquad \delta B_2 = \alpha_2 \, .$$

open supermembrane and superstring at its boundary

- This is possible if B_2 is the pull-back of the superspace 2-form with the field strength expressed by

$$H_3 = dB_2 = C_3|_{\mathcal{P}\mapsto L} = -iE^a \wedge E^\alpha \wedge \bar{E}^{\dot\alpha}\sigma_{a\alpha\dot\alpha}L - \dots$$

in terms of the real tensor multiplet L

$$\left(\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha} - R\right)L = 0\,, \qquad \left(\mathcal{D}^\alpha\mathcal{D}_\alpha - \bar{R}\right)L = 0$$

which is transformed as Stückelberg superfield under the gauge symmetry acting on the prepotential,

$$\delta\mathcal{P} = \mathbb{L}\,, \qquad \delta L = \mathbb{L}\,.$$

open supermembrane and superstring at its boundary

- In the absence of open supermembrane, the action of closed superstring contains also a Nambu-Goto term

$$\frac{1}{2}\int_{W^2} d^2\sigma\sqrt{-\gamma}\,|L|$$

in which $\gamma = \det \gamma_{ij}$ is the determinant of the metric induced on W^2,

$$\gamma_{ij} = E_i^a \eta_{ab} E_j^b \ , \qquad E_i^a = \partial_i z^M(\sigma) E_M^a(z(\sigma)) \ .$$

- When the string is situated at the boundary of an open membrane, this should be modified to $\boxed{\frac{1}{2}\int_{W^2} d^2\sigma\sqrt{-\gamma}\,|\mathcal{P}-L|}$ which respects the gauge symmetry.
- Thus the action for superstring at the end of supermembrane is

$$S_{p=1} \quad = \quad \frac{1}{2}\int_{W^2} d^2\sigma\sqrt{-\gamma}\,|\mathcal{P}-L| - \int_{W^2} B_2,$$

κ-symmetry of open supermembrane and superstring at its boundary

- The sum of the actions of supermembrane and of the string at its boundary, $S_{p+2} + S_{p+1}$, is also invariant under the local fermionic κ–symmetry with parameters restricted, besides the membrane projection

  $$\kappa_\alpha = -i \frac{\mathcal{Z}}{|\mathcal{Z}|} \, \Gamma_{\alpha\dot{\alpha}} \bar{\kappa}^{\dot{\alpha}} \, ,$$

  also by

  $$\kappa_\alpha = \frac{\mathcal{P} - \mathbb{L}}{|\mathcal{P} - \mathbb{L}|} \, P_\alpha{}^\beta \kappa_\beta \, , \qquad \bar{\kappa}_{\dot{\alpha}} = \frac{\mathcal{P} - \mathbb{L}}{|\mathcal{P} - \mathbb{L}|} \, \bar{P}_{\dot{\alpha}}{}^{\dot{\beta}} \bar{\kappa}_{\dot{\beta}}$$

  $$\text{where} \qquad P_\beta{}^\alpha = \frac{1}{2\sqrt{-\gamma}} \epsilon^{ij} E_i^a E_j^b \sigma_{ab\,\beta}{}^\alpha , \qquad \bar{P}_{\dot{\alpha}}{}^{\dot{\beta}} = (P_\alpha{}^\beta)^*$$

  obey $P^2 = \mathbb{I}$, $\bar{P}^2 = \mathbb{I}$.
- The presence of the second projection condition indicates that superstring at the end of open supmembrane breaks one half of the one-half SUSY preserved by the supermembrane
- so that the ground state of the open supermembrane is at most 1/4 BPS.

- The superstring at the end of supermembrane also breaks the gauge symmetry characteristic for the three form potential.
- When the action is written with the use of the Stückelberg real linear superfield L, as above, this symmetry is formally maintained (realized dynamically) as $\mathcal{P} - L$ is invariant.
- We can fix the gauge under this symmetry by setting $L = 0$ and in this gauge the action of superstring at the end of supermembrane reduces to

$$S_{p=1}|_{L=0} \quad = \quad \frac{1}{2} \int_{W^2} d^2\sigma \sqrt{-\gamma}\, |\mathcal{P}| .$$

- The Wess–Zumino term of the superstring vanishes in this gauge.
- The κ-symmetry holds if its parameter is restricted also by the condition

$$\kappa_\alpha = \frac{\mathcal{P}}{|\mathcal{P}|}\, P_\alpha{}^\beta \kappa_\beta \,, \qquad \bar{\kappa}_{\dot{\alpha}} = \frac{\mathcal{P}}{|\mathcal{P}|}\, \bar{P}_{\dot{\alpha}}{}^{\dot{\beta}} \bar{\kappa}_{\dot{\beta}}$$

with the same $P_\alpha{}^\beta = (\bar{P}_{\dot{\alpha}}{}^{\dot{\beta}})^*$ as above.

- When restoring the membrane tension in the gauge fixed action, it takes the form $S_{p=1}|_{L=0} = \frac{T_2}{2} \int\limits_{W^2} d^2\sigma \sqrt{-\gamma}\, |\mathcal{P}|$
- which makes manifest that the effective tension of the string at the end of supermembrane is defined by the supermembrane tension: $T_1(\sigma) = T_2|\mathcal{P}(z(\sigma)|$.
- Thus in the gauge invariant form the interacting action $S = S_{sugra+matter} + S_{p=2} + S_{p=1}$ includes

$$S_{p=2} + S_{p=1} = T_2 \int d^3\xi \sqrt{|h|}\, |\mathcal{Z}| + T_2 \int\limits_{W^3} (C_3 - H_3) + \frac{T_2}{2} \int\limits_{W^2} d^2\sigma \sqrt{-\gamma}\, |\mathcal{P} - L| \,.$$

- As far as the breaking of the three form gauge symmetry is allowed, we can consider the supergravity plus matter part of the action including the mass term for the 3-form matter multiplet [Farrar+Gabadadze+Schwetz 97]

$$S_{sugra+matter} = -\frac{3}{4\kappa^2} \int d^8z\, E\, \Omega(\mathcal{Z}, \bar{\mathcal{Z}}) - \frac{1}{2\kappa^2} \left(\int d^6\zeta_L\, \mathcal{E}\, \mathcal{W}(\mathcal{Z}) + c.c. \right) -$$
$$-\mathfrak{m}^4 \int d^8z\, E\, \frac{(\mathcal{P} - L)^2}{(\mathcal{Z}\bar{\mathcal{Z}})^{1/3}} \,.$$

- The generalization for the case of interaction of open supermembrane and of the string at the end of open supermembrane with
- more complicated systems of numerous and nonlinearly interacting supermultiplets can be obtained from the above described case just
- by writing a particular expression for the composite real prepotential $\mathcal{P}$.
- In particular, setting [Bandos+Farakos+Martucho+Lanza+Sorokin 18]

$$\mathcal{P} \quad = q_I \mathcal{P}^I - p^I \tilde{\mathcal{P}}_I = q_I \mathcal{M}^{IJ} \mathrm{Im} \Sigma_J - p^I \mathrm{Im}(\bar{\mathcal{G}}_{IJ} \mathcal{M}^{JK} \Sigma_K)$$

with

$$\mathcal{P}^I \equiv \mathrm{Im}\left(\mathcal{M}^{IJ} \Sigma_J\right), \qquad \tilde{\mathcal{P}}_I \equiv \mathrm{Im}\left(\mathcal{G}_{IJ} \mathcal{M}^{JK} \Sigma_K\right)$$

and

$$\mathcal{G}_{IJ}(\mathcal{S}) := \partial_I \partial_J \mathcal{G}(\mathcal{S}) \qquad \mathcal{G}(w\mathcal{S}) = w^2 \mathcal{G}(\mathcal{S}), \qquad \mathcal{M}_{IJ} := \mathrm{Im}\, \mathcal{G}_{IJ}$$

we can obtain the action describing interaction of the open supermembrane with a set of nonlinear self-interacting 3-form multiplets described by special chiral superfields $\mathcal{S}^I = \frac{i}{2}(\bar{\mathcal{D}}^2 - R)\mathcal{P}^I$.

- The interacting action includes the sum of the action for supermembrane carrying the electric and magnetic charges q_I and p^I

$$S_{p=2}(q_I, p^I) = \frac{1}{2}\int d^3\xi\sqrt{|\hat{g}|}\,|q_I\mathcal{S}^I - p^I\mathcal{G}_I(\mathcal{S})| + q_I\int_{W^3}\mathcal{C}_3^I - p^I\int_{W^3}\tilde{\mathcal{C}}_{3I}$$

- and the action of superstring at the boundary of the worldvolume,

$$S_{p=1} = \frac{1}{2}\int_{W^2} d^2\sigma\sqrt{-\gamma}\,|q_I(\mathcal{P}^I - L^I) - p^I(\mathcal{P}_I - \tilde{L}_I)| - q_I\int_{W^2} B_2^I + p^I\int_{W^2}\tilde{B}_{2I}\ ,$$

which is invariant under the κ-symmetry provided the parameters obey

$$\kappa_\alpha = -i\frac{q_I\mathcal{S}^I - p^I\mathcal{G}_I(\mathcal{S})}{|q_I\mathcal{S}^I - p^I\mathcal{G}_I(\mathcal{S})|}\,\Gamma_{\alpha\dot{\alpha}}\bar{\kappa}^{\dot{\alpha}}$$

and

$$\kappa_\alpha = \frac{q_I(\mathcal{P}^I - L^I) - p^I(\mathcal{P}_I - \tilde{L}_I)}{|q_I(\mathcal{P}^I - L^I) - p^I(\mathcal{P}_I - \tilde{L}_I)|}\,P_\alpha{}^\beta\kappa_\beta\ ,$$

with the projectors defined above.

- The sum of the open supermembrane and superstring actions is also invariant under the gauge symmetry

$$\delta\Sigma_I = \tilde{\mathbb{L}}_I + \mathcal{G}_{IJ}\mathbb{L}^J\,, \qquad \delta L^I = \mathbb{L}^I\,, \qquad \delta\tilde{L}_I = \tilde{\mathbb{L}}_I\,.$$

- A quite general action for nonlinearly self interacting system of the supermembrane, 3-form matter multiplets and supergravity, of a kind which appear in string compactifications, includes also

$$S_{sugra+matter} = -\frac{3}{4\kappa^2}\int d^8z\, E\,\Omega(\mathcal{S}^I,\bar{\mathcal{S}}^I) - \frac{1}{2\kappa^2}\left(\int d^6\zeta_L\,\mathcal{E}\,\mathcal{W}(\mathcal{S}^I) + c.c.\right) -$$
$$-c_2\int d^8z\, E\,\frac{\mathcal{M}^{IJ}\left(\Sigma_I - \tilde{L}_I - \bar{\mathcal{G}}_{IK}L^K\right)\left(\bar{\Sigma}_J - \tilde{L}_J - \mathcal{G}_{IL}L^L\right)}{(\mathcal{S}^P\mathcal{M}_{PQ}\bar{\mathcal{S}}^Q)^{\frac{1}{3}}}\,.$$

- Such an action will be invariant under the $Sp(2n+2|\mathbb{Z})$ symmetry, characteristic for string compactifications, provided $\Omega(\mathcal{S}^I,\bar{\mathcal{S}}^I)$ and $\mathcal{W}(\mathcal{S}^I)$ are invariant and the supermembrane charges (p^I, q_I) are transformed as symplectic vector (see [Bandos+Farakos+Martucho+Lanza+Sorokin 18]).

Outline

SYM

- The $\mathcal{N}=1$ SYM multiplet: $A_m(x)$ (gluon), $\lambda_\alpha(x)$, $\bar\lambda_{\dot\alpha}(x)$ (gluino)+ auxiliary field $D(x)$, all in the adjoint representation of $SU(N)$.

$$\begin{aligned}\mathcal{L}_{\mathrm{SYM}} \quad = \quad & -\frac{\mathrm{i}}{2g^2}\,\mathrm{Tr}\,\lambda\sigma^m\nabla_m\bar\lambda + \frac{\mathrm{i}}{2g^2}\,\mathrm{Tr}\,\nabla_m\lambda\sigma^m\bar\lambda - \frac{1}{4g^2}\,\mathrm{Tr}\,F_{mn}F^{mn} \\ & + \frac{1}{2g^2}\,\mathrm{Tr}\,D^2 + \frac{\vartheta}{32\pi^2}\,\mathrm{Tr}(\varepsilon_{mnpl}F^{mn}F^{pl} + 4\partial_m(\lambda\sigma^m\bar\lambda))\,,\end{aligned}$$

  where $F_2 = \mathrm{d}A + \mathrm{i}A\wedge A$, $\nabla_m = \partial_m - \mathrm{i}A_m$, g is the SYM coupling constant and ϑ is the angle of the topological term.
- In the superfield formalism, the SYM Lagrangian is an F-term

$$\mathcal{L}_{\mathrm{SYM}} = \frac{\tau}{8\pi}\int \mathrm{d}^2\theta\,\mathrm{Tr}\,\mathcal{W}^\alpha\mathcal{W}_\alpha + \mathrm{c.c.}\,,$$

  where $\tau = \frac{\mathrm{i}\vartheta}{2\pi} + \frac{2\pi}{g^2}$ and $\mathcal{W}_\alpha(x_L,\theta)$ is the chiral superfield

$$\mathcal{W}_\alpha = -\mathrm{i}\lambda_\alpha + \theta_\alpha D - \frac{\mathrm{i}}{2}F_{mn}\sigma^{mn}{}_\alpha{}^\beta\theta_\beta + \theta^2\sigma^m_{\alpha\dot\beta}\nabla_m\bar\lambda^{\dot\beta}.$$

SYM vacua

- The classical $U(1)$ R-symmetry of the SYM action ($\lambda \to \lambda e^{i\varphi}$) is broken by quantum anomaly down to Z_{2N}. The instanton effects create a gluino condensate [Witten 1982] whose values are [Shifman+Veinstein 1987]

$$< \lambda\lambda > \equiv < \mathrm{Tr}\, \lambda^{\alpha}\lambda_{\alpha} > \propto \Lambda^3 e^{\frac{2\pi i n}{N}}, \quad n = 0, 1, \ldots, N-1,$$

  where Λ^3 is a SYM dynamical scale
- and the parameter n labels N degenerate supersymmetric vacua of the SYM theory related by Z_N symmetry.
- Thus the gluino condensate further breaks the Z_{2N} R-symmetry down to Z_2 ($\lambda \mapsto -\lambda$).

VY model

- The gluino condensate and the N vacua of $SU(N)$ SYM are effectively described by the Veneziano-Yankielowicz Lagrangian [VY 1982] for the chiral superfield

$$S = \mathrm{Tr}\, \mathcal{W}^{\alpha}\mathcal{W}_{\alpha}\ .$$

- This chiral superfield S accommodates a gaugino bi-linear (the *gluino-ball*) $s = -\,\mathrm{Tr}\,\lambda^\alpha\lambda_\alpha$ and its superpartners:

$$\begin{aligned} S &= \mathrm{Tr}\,\mathcal{W}^\alpha\mathcal{W}_\alpha = s + \sqrt{2}\theta^\alpha\chi_\alpha + \theta^2 F, \\ s &= -\,\mathrm{Tr}\,\lambda^\alpha\lambda_\alpha\,, \qquad \chi_\alpha = \sqrt{2}\,\mathrm{Tr}\left(\frac{1}{2}F_{mn}\sigma_\alpha^{mn\beta}\lambda_\beta - \mathrm{i}\lambda_\alpha D\right), \\ F &= \mathrm{Tr}\left(-2\mathrm{i}\lambda\sigma^m\nabla_m\bar\lambda - \frac{1}{2}F_{mn}F^{mn} + D^2 - \frac{\mathrm{i}}{4}\varepsilon_{mnpl}F^{mn}F^{pl}\right). \end{aligned}$$

- In the VY theory s, χ and F were regarded as elementary colorless fields.
- But [Burgess+Derendinger+Quevedo+Quiros 1995] noticed that S is a *special* chiral superfield which describes single 3-form multiplet:
- Its F-component $\boxed{F = \hat{D} + \mathrm{i}\partial_m C^m = \hat{D} + \frac{\mathrm{i}}{3!}\epsilon^{mnkp}\partial_m C_{nkp}}$ contains the field strength of a (composite) 3-form, the $SU(N)$ Chern-Simons term

$$\begin{aligned} F_4 &= d^4x\,\mathrm{Im}\,F = -\,\mathrm{Tr}\,F_2\wedge F_2 - d^4x\,\partial_m(\mathrm{Tr}\,\lambda\sigma^m\bar\lambda) \\ &= -d\,\mathrm{Tr}\left(AdA + \frac{2\mathrm{i}}{3}A^3 + \frac{1}{3!}dx^k dx^n dx^m \epsilon_{mnkl}\,\mathrm{Tr}\,\lambda\sigma^l\bar\lambda\right) \equiv dC_3. \end{aligned}$$

- Thus this is a special chiral superfield of the type which can be coupled to a supermembrane. It can be written in the form

$$S = -\frac{1}{4}\bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}U,$$

where $U = U^*$ is a real superfield prepotential (we called it $\mathcal{P}$ in the general case) with the bosonic field content

$$\begin{aligned} U| &= u, \\ -\frac{1}{8}\bar{\sigma}^{\dot{\alpha}\alpha m}[D_\alpha, \bar{D}_{\dot{\alpha}}]U| &= C^m \;=:\; \frac{1}{3!}\epsilon^{mnkl}C_{nkl}, \\ \frac{1}{4}D^2U| &= -\bar{s} = \mathrm{Tr}\,\bar{\lambda}\bar{\lambda}, \\ \frac{1}{16}D^2\bar{D}^2U| &= \hat{D} + \mathrm{i}\partial^m C_m \equiv F\,. \end{aligned}$$

- As $S = -\frac{1}{4}\bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}U$ is invariant under the gauge transformations

$$U' = U + L\,, \qquad \bar{D}^2 L = 0 = D^2 L\,,$$

- the leading bosonic component of U is a pure gauge.

- The original Veneziano-Yankielowicz Lagrangian is

$$\mathcal{L}_{\mathrm{VY}} = \frac{1}{\rho}\int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}(S\bar{S})^{\frac{1}{3}} + \int \mathrm{d}^2\theta\, W(S) + \mathrm{c.c.}\,,$$

in which the VY superpotential is uniquely fixed by anomalous superconformal Ward identities of the SYM theory:

$$W(S) = \frac{N}{16\pi^2} S\left(\ln\frac{S}{\Lambda^3} - 1\right)\,, \qquad W_S := \partial_S W(S) = \frac{N}{16\pi^2}\ln\frac{S}{\Lambda^3}\,.$$

- The Kähler potential in the first term is chosen

$$K(S,\bar{S}) = \frac{1}{\rho}(S\bar{S})^{\frac{1}{3}}$$

due to the mass dimension 3 of the superfield S.

- Here ρ is a dimensionless (a priori arbitrary) positive constant.
- In general, the kinetic part of the Lagrangian is not fixed by anomalous symmetries and can also include higher order terms.

- In [arXiv:1905.02743 [hep-th]] we have shown that by treating U as the independent superfield and modifying the Lagrangian with an appropriate boundary term allows one to consistently eliminate the auxiliary fields by solving their equations of motion and to get additional contributions to the effective scalar potential of (quantized) numerical integration parameters similar to those introduced in [Kovner+Shifman+Smilga 97].
- This also solves the second issue with the VY Lagrangian whose superpotential is not single-valued: because of the presence of the logarithmic term, it gets shifted by the (identical) phase transformation

$$S(x,\theta) \to S'(x,\theta e^{\pi \mathrm{i}}) = e^{2\pi \mathrm{i}} S(x,\theta)\,, \qquad W(S) \to W(S) + \frac{\mathrm{i}N}{8\pi} S\,.$$

The addition of the boundary term

$$\mathcal{L} = \mathcal{L}_{\mathrm{VY}} + \mathcal{L}_{\mathrm{bd}}$$

compensates the shift in the superpotential and makes the whole Lagrangian single-valued.

- The total space-time derivative term in question has the following form

$$\mathcal{L}_{\rm bd} \;=\; -\frac{1}{8}\left(\int \mathrm{d}^2\theta \bar{D}^2 - \int \mathrm{d}^2\bar{\theta} D^2\right)\left[\left(\frac{1}{12\rho}\bar{D}^2\frac{\bar{S}^{\frac{1}{3}}}{S^{\frac{2}{3}}} + \frac{1}{16\pi^2}\ln\frac{\Lambda^{3N}}{S^N}\right)U\right] + \text{c.c.}$$

- For a general class of models involving three-form chiral supermultiplets the boundary terms of this kind were derived in [Farakos+Lanza+Martucci+Sorokin 2017].
- Setting fermions to zero we obtain the following expressions for the bulk and boundary terms of the Lagrangian

$$\mathcal{L}^{\rm bos}_{\rm VY} = K_{s\bar{s}}\left(-\partial_m s\partial^m\bar{s} + (\partial_m C^m)^2 + \hat{D}^2\right) + \left(W_s\left(\hat{D} + \mathrm{i}\partial_m C^m\right) + \text{c.c.}\right)$$

$$\text{and} \qquad \mathcal{L}^{\rm bos}_{\rm bd} = -2\partial_m\left(C^m K_{s\bar{s}}\partial_n C^n\right) - \mathrm{i}\partial_m\left(C^m(W_s - \bar{W}_{\bar{s}})\right) .$$

- What we would like to do now is to add the action of supermembrane coupled to the VY fields.

Supermembrane action

- The most general action describing the coupling a supermembrane to a single special chiral three-form superfield $S = -\frac{1}{4}\bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha}U$ in flat $\mathcal{N}=1$, $D=4$ superspace reads

$$S_{\text{membrane}} = -\frac{1}{4\pi}\int_{\mathcal{M}_3} d^3\xi\sqrt{-\det h_{ij}}\,|kS+c| - \frac{k}{4\pi}\int_{\mathcal{M}_3}\mathcal{C}_3 - \left(\frac{\bar{c}}{4\pi}\int_{\mathcal{M}_3}\mathcal{C}_3^0 + c.c.\right),$$

  where $c = k_1 + ik_2$, and k, k_1 and k_2 are real constant charges characterizing the membrane coupling to a real three-form gauge superfield $\mathcal{C}_3$ and a complex super three-form $\mathcal{C}_3^0$,
- In the Nambu-Goto part of action $S(x,\theta,\bar\theta)$ is evaluated on the membrane worldvolume $z^M = z^M(\xi)$ parametrized by ξ^i ($i = 0,1,2$),

$$h_{ij}(\xi) \equiv \eta_{ab}E_i^a(\xi)E_j^b(\xi), \quad \text{with} \quad E_i^a(\xi) \equiv \partial_i z^M(\xi)E_M^a(z(\xi)),$$

  is the induced metric on the membrane worldvolume and

$$E^a(\xi) \equiv dz^M(\xi)E_M^a(z(\xi)) = dx^a(\xi) + i\theta\sigma^a d\bar\theta(\xi) - id\theta\sigma^a\bar\theta(\xi)\,.$$

WZ term

- In the second and third Wess–Zumino terms of the action contains

$$
\begin{aligned}
\mathcal{C}_3 = & \, \mathrm{i} E^a \wedge \mathrm{d}\theta^\alpha \wedge \mathrm{d}\bar{\theta}^{\dot{\alpha}} \sigma_{a\alpha\dot{\alpha}} U \\
& - \frac{1}{4} E^b \wedge E^a \wedge \mathrm{d}\theta^\alpha \sigma_{ab\,\alpha}{}^\beta D_\beta U - \frac{1}{4} E^b \wedge E^a \wedge \mathrm{d}\bar{\theta}^{\dot{\alpha}} \bar{\sigma}_{ab}{}^{\dot{\beta}}{}_{\dot{\alpha}} \bar{D}_{\dot{\beta}} U \\
& - \frac{1}{48} E^c \wedge E^b \wedge E^a \epsilon_{abcd}\, \bar{\sigma}^{d\dot{\alpha}\alpha} [D_\alpha, \bar{D}_{\dot{\alpha}}] U
\end{aligned}
$$

and

$$
\mathcal{C}_3^0 = \mathrm{i} E^a \wedge \mathrm{d}\theta^\alpha \wedge \mathrm{d}\bar{\theta}^{\dot{\alpha}} \sigma_{a\alpha\dot{\alpha}}\, \theta^2 - \frac{1}{2} E^b \wedge E^a \wedge \mathrm{d}\theta^\alpha \sigma_{ab\,\alpha}{}^\beta \theta_\beta
$$

with

$$
\mathcal{H}_4^0 = -\frac{1}{2} E^b \wedge E^a \wedge \mathrm{d}\theta^\alpha \sigma_{ab\,\alpha}{}^\beta \mathrm{d}\theta_\beta
$$

- which are also calculated at the worldvolume.

κ–symmetry

- By construction, the supermembrane action is invariant under the worldvolume diffeomorphisms $\xi^i \to f^i(\xi)$ and under the κ-symmetry transformations

$$\delta\theta^\alpha = \kappa^\alpha(\xi), \quad \delta\bar{\theta}^{\dot{\alpha}} = \bar{\kappa}^{\dot{\alpha}}(\xi), \quad \delta x^m = \mathrm{i}\kappa\sigma^m\bar{\theta} - \mathrm{i}\theta\sigma^m\bar{\kappa},$$

such that $\delta_\kappa z^M E_M^a = 0$,

- where the local fermionic parameter $\kappa^\alpha(\xi)$ and its c.c. $\bar{\kappa}^{\dot{\alpha}}(\xi)$ satisfy

$$\kappa_\alpha = -\mathrm{i}\frac{kS+c}{|kS+c|}\Gamma_{\alpha\dot{\alpha}}\bar{\kappa}^{\dot{\alpha}} \quad \Leftrightarrow \quad \bar{\kappa}_{\dot{\alpha}} = -\mathrm{i}\frac{k\bar{S}+\bar{c}}{|kS+c|}\Gamma_{\alpha\dot{\alpha}}\kappa^\alpha,$$

with

$$\Gamma_{\alpha\dot{\alpha}} \equiv \frac{\mathrm{i}\epsilon^{ijk}}{3!\sqrt{-\det h}}\epsilon_{abcd}E_i^b E_j^c E_k^d\, \sigma^a_{\alpha\dot{\alpha}}, \qquad \Gamma_{\alpha\dot{\alpha}}\Gamma^{\dot{\alpha}\beta} = \delta_\alpha^\beta.$$

- The κ-symmetry corresponds to half of the bulk supersymmetry preserved by a BPS state, the ground state of the extended object, while another half of supersymmetry is spontaneously broken.

Supermembrane is described by Goldstone fields

- W/v reparametrization inv. can be used to fix $x^i(\xi) = \xi^i = (\xi^0, \xi^1, \xi^2)$.
- κ-symm. can be used to kill 1/2 of the components of $\theta(\xi)$ and $\bar{\theta}(\xi)$.
- Hence the propagating fields on the membrane worldvolume are a scalar $\varphi(\xi) = x^3(\xi)$ and two of four fermionic fields $\theta(\xi)$ and $\bar{\theta}(\xi)$.
- These fields form an $\mathcal{N} = 1$, $d = 3$ Goldstone supermultiplet associated with a half of $\mathcal{N} = 1$, $D = 4$ supersymmetry spontaneously broken by the presence of the membrane.
- The membrane action describes its coupling to the special chiral superfield S.
- An additional contribution of S to the membrane 'central' charge can be easily calculated (using the technique of [de Azcarraga et al 1989]) when $\theta(\xi) = 0 = \bar{\theta}(\xi)$ with the following result

$$\{Q_\alpha, Q_\beta\} = \int dx^m \wedge dx^n \, \sigma_{mn\alpha\beta} \frac{\bar{c} + k\bar{s}}{4\pi}$$

Therefore the membrane ground state preserving half of the bulk supersymmetry saturates the BPS bound with this central charge.

Outline

- The inclusion of the membrane in the VY theory is necessary to induce and take care of the discontinuity of the VY superpotential and the corresponding cusp of the VY potential.

- We consider a purely bosonic domain wall solution preserving 1/2 of the $\mathcal{N} = 1$ supersymmetry
- and assume that the membrane which sources the VY domain walls is static: stretches along x^0, x^1 and x^2 and sits at $x^3 = 0$ and $\theta = 0$.
- The action describing the coupling of the scalar sector of the VY effective theory to the static membrane has the following form

$$S = \int d^4x \, (\mathcal{L}^{\rm bos}_{\rm VY\ bulk} + \mathcal{L}^{\rm bos}_{\rm VY\ bd}) - \frac{1}{4\pi} \int d^3\xi \left(|ks + c| + kC^3 \right) ,$$

where

$$\mathcal{L}^{\rm bos}_{\rm VY\ bulk} = K_{s\bar{s}} \left(-\partial_m s \partial^m \bar{s} + (\partial_m C^m)^2 + \hat{D}^2 \right) + \left(W_s \left(\hat{D} + \mathrm{i}\partial_m C^m \right) + \text{c.c.} \right) ,$$

$$\mathcal{L}^{\rm bos}_{\rm VY\ bd} = -2\partial_m \left(C^m K_{s\bar{s}} \partial_n C^n \right) - \mathrm{i}\partial_m \left(C^m (W_s - \bar{W}_{\bar{s}}) \right) .$$

- Varying this action we find that the membrane provides source for the scalar s and 3-form field equations ($F = \hat{D} + \mathrm{i}\partial_m C^m$)

$$\Box s\, K_{s\bar{s}} + \partial_m s \partial^m s\, K_{ss\bar{s}} + F\bar{F} K_{s\bar{s}\bar{s}} + \bar{F}\bar{W}_{\bar{s}\bar{s}} = \frac{k}{8\pi}\delta(x^3)\frac{ks+c}{|ks+c|}\,,$$

$$\partial_m(K_{s\bar{s}}\partial_n C^n - \mathrm{Im}\, W_s) = -\frac{k}{8\pi}\delta^3_m \delta(x^3).$$

while the equations for auxiliary scalar $\hat{D}$ remains 'free'

$$\partial_{S\bar{S}} K \hat{D} + \mathrm{Re}\, \partial_S W = 0\,.$$

- The solution of last two eqs expresses the auxiliary field F as

$$F = \hat{D} + \mathrm{i}\partial_m C^m = -\frac{16\pi^2 \bar{W}_{\bar{s}} + \mathrm{i}(2\pi n + 2\pi k \Theta(x^3))}{16\pi^2 K_{s\bar{s}}}.$$

where $\Theta(x^3)$ is the step function, $\Theta(x^3) = \begin{cases} 1, & \text{if } x^3 > 0\,, \\ 0, & \text{if } x^3 < 0 \end{cases}.$

- This prompts us to introduce the discontinuous superpotential

$$\hat{W}(s) \equiv W(s) - \frac{\mathrm{i}}{8\pi}(n + k\Theta(x^3))\, s$$

which "jumps" at the position of the membrane.

- Thus its local minima describe two SYM vacua, one on the left of the membrane labeled by n and another one on the right labelled by $n+k$.
- We should also take into account the equation of motion of the membrane field $x^3(\xi)$, which for $\partial_i x^3 = 0$ reduces to

$$(\partial_3 |ks + c| + k\partial_m C^m)|_{x^3=0} = 0.$$

- We are interested in 1/2 supersymmetric BPS domain wall configurations interpolating between two vacua at $x^3 \to -\infty$ and $x^3 \to +\infty$ separated by the membrane. With VY superpotential $\partial_s W(s) = \frac{N}{16\pi^2} \ln \frac{s}{\Lambda^3}$ and these are described by

$$< s >_{-\infty} = \Lambda^3 e^{\frac{2\pi \mathrm{i} n}{N}} \quad \text{and} \quad < s >_{+\infty} = \Lambda^3 e^{\frac{2\pi \mathrm{i}(n+k)}{N}}\ .$$

- We chose the ansatz $s = s(x^3)$ [Abraham+Townsend 90, Dvali+Shifman 96, Shifman+Yung 09]:
- and study the conditions of preserved SUSY

$$\delta\chi_\alpha = \mathrm{i}\sigma^3_{\alpha\dot\alpha}\bar\epsilon^{\dot\alpha}\dot s + \epsilon_\alpha F = 0\,.$$

- taking into account that 1/2 of the SUSY is broken (spontaneously) by the membrane.
- The preserved part is defined by the κ-symmetry of the supermembrane action and its parameter obeys

$$\epsilon_\alpha = e^{\mathrm{i}\alpha}\sigma^3_{\alpha\dot\alpha}\bar\epsilon^{\dot\alpha}, \qquad e^{\mathrm{i}\alpha} := \frac{ks+c}{|ks+c|}\Big|_{x^3=0}.$$

- With this condition $\delta\chi_\alpha = 0$ is satisfied if the following BPS equation holds

$$\dot s = \mathrm{i}e^{\mathrm{i}\alpha}F = -\mathrm{i}e^{\mathrm{i}\alpha}\frac{\overline{\hat W_{\bar s}}}{K_{s\bar s}},$$

- It is not difficult to check that this relation solves the field equation.

- If we make a particular choice that on the membrane $ks(0)+c$, $ks(0)$ and c have the same phase α, the equations imply that $\forall x^3$

$$\frac{\mathrm{d}}{\mathrm{d}x^3}\mathrm{Re}(\hat{W}e^{-\mathrm{i}\alpha}) = 0\,, \qquad \Rightarrow \qquad \mathrm{Re}(\hat{W}e^{-\mathrm{i}\alpha}) = \mathtt{const}\,.$$

- We are now ready to compute the energy density (namely, the **tension**) of the domain wall configuration sourced by the membrane which is determined by the on-shell value of the action of the interacting system

$$S_{\text{on-shell}} \equiv -\int \mathrm{d}^3\xi\ T_{\mathrm{DW}}\,, \quad d^3\xi := \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2\,.$$

- Substituting the solution of the auxiliary field eqs, assuming that $s = s(x^3)$ and taking into account the form of the boundary term we get

$$S = \int \mathrm{d}^3\xi\,\mathrm{d}x^3 \left(-K_{s\bar{s}}\dot{s}\dot{\bar{s}} - \frac{1}{K_{s\bar{s}}}\hat{W}_s\bar{\hat{W}}_{\bar{s}}\right) - \int \mathrm{d}^3\xi\,\mathrm{d}x^3\ \delta(x^3)\,T_{\mathrm{M}}\,,$$

where $\boxed{T_{\mathrm{M}} = \frac{|ks+c|}{4\pi}}$ is the membrane tension.

- This action can be more elegantly written in the *BPS*-form

$$S = \int \mathrm{d}^3\xi \, \mathrm{d}x^3 \Big[- K_{s\bar{s}} \left(\dot{s} \pm \mathrm{i} e^{\mathrm{i}\beta} \bar{\hat{W}}_{\bar{s}} / K_{s\bar{s}} \right) \left(\dot{\bar{s}} \mp \mathrm{i} e^{-\mathrm{i}\beta} \hat{W}_s / K_{s\bar{s}} \right)$$
$$\mp \mathrm{i} \left(\dot{s} \, \hat{W}_s e^{-\mathrm{i}\beta} - \dot{\bar{s}} \, \bar{\hat{W}}_{\bar{s}} e^{\mathrm{i}\beta} \right) \Big] - \int \mathrm{d}^3\xi \, \mathrm{d}x^3 \; \delta(x^3) \, T_{\mathrm{M}}$$

  where β is an arbitrary phase.
- If we take $\beta = \alpha$, then, for the upper sign, the first term vanishes due to the BPS eq and we get

$$S = \int \mathrm{d}^3\xi \, dx^3 \, 2\mathrm{Im} \left(\dot{s} \, \hat{W}_s e^{-\mathrm{i}\alpha} \right) - \int \mathrm{d}^3\xi \, \mathrm{d}x^3 \; \delta(z) \, T_{\mathrm{M}} \, .$$

- Now the integration along the transverse direction may be easily performed by noticing that, due to $\hat{W}(s) \equiv W(s) - \frac{\mathrm{i}}{8\pi}(n + k\Theta(x^3))\, s$,

$$\dot{s} \, \hat{W}_s = \frac{\mathrm{d}}{\mathrm{d}x^3} \hat{W} + \frac{\mathrm{i}k}{8\pi} s \, \delta(x^3).$$

- In such a way we arrive at

$$
\begin{aligned}
S &= -\int \mathrm{d}^3\xi \left(T_{\mathrm{M}} - \frac{1}{4\pi}\mathrm{Re}\,(ks(0)e^{-\mathrm{i}\alpha}) \right) \\
&\quad -\int \mathrm{d}^3\xi\, 2\,\mathrm{Im}\,[(\hat{W}_{+\infty} - \hat{W}_{-\infty})e^{-\mathrm{i}(\alpha-\pi)}]\,.
\end{aligned}
$$

- In view of $\mathrm{Re}\,(\hat{W}e^{-\mathrm{i}\alpha}) = \mathtt{const}$ and requiring the non-positive definiteness of the second term of S, we find that the phase of $\hat{W}_{+\infty} - \hat{W}_{-\infty}$ coincides with $\alpha - \frac{\pi}{2} \pmod{2\pi}$.
- Then, as on the membrane $\arg(ks(0)) = \arg c = \alpha$, we see that

$$
S = -\int \mathrm{d}^3\xi\, \frac{|c|}{4\pi} - \int \mathrm{d}^3\xi\, 2\,|(\hat{W}_{+\infty} - \hat{W}_{-\infty})|
$$

and hence

$$
T = 2\,|\hat{W}_{+\infty} - \hat{W}_{-\infty}| + \frac{|c|}{4\pi}.
$$

- In this expression

$$T = 2\,|\hat{W}_{+\infty} - \hat{W}_{-\infty}| + \frac{|c|}{4\pi}$$

- the first term $T_{\rm DW} = 2\,|\hat{W}_{+\infty} - \hat{W}_{-\infty}|$ is the tension of the domain walls saturating the BPS bound [Abraham+Townsend 90, Dvali+Shifman 96, Shifman+Yung 09]
- and the second term is the contribution of the free membrane of tension

$$T_0 = \frac{|c|}{4\pi}.$$

- Coming back to the original expression (with later observations),

$$S = -\int d^3\xi \left(T_{\rm M} - \frac{1}{4\pi}{\rm Re}\,(ks(0)e^{-{\rm i}\alpha}) \right) - \int d^3\xi\, 2\,|(\hat{W}_{+\infty} - \hat{W}_{-\infty}|\,,$$

we see that for $T_0 = 0$, the contribution of the membrane tension T_M cancels exactly the 'jump' $|ks(0)|/4\pi$ of the superpotential, and

$$T = 2\,|\hat{W}_{+\infty} - \hat{W}_{-\infty}| = T_{DW}.$$

- If the membrane were not present, i.e. $T_M = 0$, and the superpotential is discontinuous at $x^3 = 0$ then we would get the tension

$$T = T_{\mathrm{DW}} - \frac{|ks(0)|}{4\pi}$$

whose value is less than that of the BPS bound and can even be not positive definite.

- This discrepancy was found in [KKS98=Kogan+Kovner+Shifman PRD 98]. KKS suggested that at the cusp of the VY potential there should leave an object, associated with integrated heavy modes of the theory whose tension compensates the above negative contribution and restores the BPS value of the domain wall tension.
- As we have shown, this object is the dynamical (super)membrane, which sources the domain wall solutions.

- We have thus shown that the tension of the BPS domain-wall+membrane configurations interpolating between two supersymmetric vacua in the VY effective theory coincides with the value of the tension of the BPS domain walls in $\mathcal{N}=1$ SYM [Dvali+ Shifman 96], i.e.

$$T_{\rm DW}^{\rm SYM} = \frac{N\Lambda^3}{8\pi^2}\left|e^{2\pi\mathrm{i}\frac{n+k}{N}} - e^{2\pi\mathrm{i}\frac{n}{N}}\right| = \frac{N\Lambda^3}{4\pi^2}\left|\sin\frac{\pi k}{N}\right| .$$

- Let us also remind that that the membrane's own tension is

$$T_M = \frac{|ks(0)+c|}{4\pi}.$$

- This result is in agreement with the calculation (see [de Azcarraga+Gauntlett+Izquierdo+Townsend 89, Cvetic+Griffies 92, Cvetic+Griffies+Rey 92] for $c=0$) of the total tensorial central charge of the $1\mathcal{N}$, $4D$ superalgebra generated by the domain-wall+membrane system

$$\{Q_\alpha, Q_\beta\} = \int \mathrm{d}x^m\wedge \mathrm{d}x^n\,\sigma_{mn\alpha\beta}\left[2\mathrm{i}(\overline{W}_{+\infty}-\overline{W}_{-\infty}) + \frac{\bar{c}}{4\pi}\right].$$

Outline

- We have presented the action of interacting system of open supermembrane, supergravity and matter and discussed its property.
- To preserve (a part of) κ–symmetry of closed membrane the action of open membrane have to include a boundary term which allows for interpretation as an action of string at the boundary of supermembrane.
- On the superstring worldsheet the κ–symmetry parameter is restricted by additional projection condition
- which implies that the supersymmetric configuration with open supermembrane can be at best 1/4 BPS.
- The gauge symmetry of the 3–form matter is broken by the string at the end of supermembrane and can be restored by introducing the Stükelberg real linear multiplet(s).
- Actually this restoration was quite useful to obtain the action of supertring at the boundary of supermembrane starting from its WZ term.
- In the gauge where this Stükelberg superfield is set to zero this superstring Wess-Zumino term vanishes and the superstring action reduces to the integral over the worldsheet of the pull–back of the characteristic real prepotential of the supergrvaity+matter which also defines the supermembrane tension.

- We believe that our results will be useful to construct the effective actions for phenomenologically interesting models of string theory compactifications with open branes and branes at the boundary of open branes.
- Actually, some of such actions with open brane constructions in the effective field theory have been written (independently) and studied in recent [Lanza+Marchesano+Martucci+Sorokin, arXiv:1907.11256].
- Another direction is to obtain equations of motion for 3-form matter and supergravity from our action and to search for their solution describing open supermembrane systems and supermembrane junctions.

- Here, as an application we considered the supermembrane interaction with SYM and VY effective theory.
- In particular, we have shown that that the inclusion of supermembrane is necessary to solve the old standing problem of the domain wall tension in VY theory.
- The membrane sources VY field equation, separates two distinct SYM vacua and provides the missing contribution to the tension of the BPS saturated domain-wall configurations.

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- THANK YOU FOR YOUR ATTENTION!
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Outline

- Let us consider

$$S_{\text{membrane}} = -\frac{1}{4\pi}\int_{\mathcal{M}_3} d^3\xi \sqrt{-\det h_{ij}}\,|kS+c| - \frac{k}{4\pi}\int_{\mathcal{M}_3} \mathcal{C}_3 - \left(\frac{\bar{c}}{4\pi}\int_{\mathcal{M}_3} \mathcal{C}_3^0 + c.c.\right)$$

  with $S = \mathrm{Tr}\,\mathcal{W}^\alpha \mathcal{W}_\alpha$. As this is nilpotent, the supermembrane tension $\boxed{T_M = \frac{1}{4\pi}|kS+c|}$ is well defined if $c \neq 0$.
- It is tempting to assume that this membrane action is associated with an effective field theory on the worldvolume of a BPS domain wall, including the explicit coupling to the SYM multiplet of its Goldstone sector.
- In the absence of the Goldstone field this is believed to be described by the action of an $\mathcal{N}=1$, $d=3$ $SU(N)$ Chern-Simons theory

$$S_{\text{static}} = -\frac{\mathrm{i}k}{4\pi}\int_{\mathcal{C}} d^3\xi\,\mathrm{Tr}\,\psi^\alpha\psi_\alpha + \frac{k}{4\pi}\int_{\mathcal{C}}\left[\mathrm{Tr}\left(AdA + \frac{2\mathrm{i}}{3}A^3\right)\right] - T_0\int_{\mathcal{C}} d^3\xi\,.$$

- For $T_0 = 0$ and $k = 1$ this action was obtained in [Bashmakov+Benini+Benvenuti+Bertolini 2018] by inserting an interface operator into the SYM action.

- An agrument in favour of the above assumption would be a derivation of the $\mathcal{N}=1$, $d=3$ $SU(N)$ Chern-Simons action

$$S_{\rm static} = -\frac{\mathrm{i}k}{4\pi}\int_{\mathcal{C}} \mathrm{d}^3\xi\, \mathrm{Tr}\,\psi^\alpha\psi_\alpha + \frac{k}{4\pi}\int_{\mathcal{C}}\left[\mathrm{Tr}\left(A\mathrm{d}A + \frac{2\mathrm{i}}{3}A^3\right)\right] - T_0\int_{\mathcal{C}}\mathrm{d}^3\xi\,,$$

from the supermembrane action

$$S_{\rm membrane} = -\tfrac{1}{4\pi}\int_{\mathcal{M}_3}\mathrm{d}^3\xi\sqrt{-\det h_{ij}}\,|kS+c| - \tfrac{k}{4\pi}\int_{\mathcal{M}_3}\mathcal{C}_3 - \left(\tfrac{\bar{c}}{4\pi}\int_{\mathcal{M}_3}\mathcal{C}_3^0 + c.c.\right)$$

with $S = \mathrm{Tr}\,\mathcal{W}^\alpha\mathcal{W}_\alpha$ in the case of static membrane configuration

$$x^3(\xi)=0, \qquad \theta^\alpha(\xi)=0\,, \qquad \bar{\theta}^{\dot{\alpha}}(\xi)=0\,.$$

- Let us, following [I.B.+Lanza+Sorokin 2019], sow how this can be done and what additional assumptions are needed to complete this goal.

- Let us consider the SYM fields λ, A_m and D inside S in S_{membrane} as dynamical
- and set the worldvolume Goldstone fields to zero

$$x^i = \xi^i\ , \qquad x^3(\xi) = 0, \qquad \theta^\alpha(\xi) = 0\ , \qquad \bar{\theta}^{\dot{\alpha}}(\xi) = 0\ .$$

- Then the supermembrane action becomes

$$S_{\text{static}} = -\frac{1}{4\pi}\int_{\mathcal{C}} \mathrm{d}^3\xi \left(|k\,\mathrm{Tr}\,\lambda\lambda - \ c| - k\,\mathrm{Tr}\,\lambda\sigma^3\bar{\lambda}\right) + \frac{k}{4\pi}\int_{\mathcal{C}}\left[\mathrm{Tr}\left(A\mathrm{d}A + \frac{2\mathrm{i}}{3}A^3\right)\right].$$

- Now we may consider the equations of motion of $\theta(\xi)$ and $\bar{\theta}(\xi)$ [Bandos+Meliveo 2010], in which we set all the W/V fields to zero.
- These impose the kappa-symmetry projection condition on the fermion $\chi_\alpha = \sqrt{2}\,\mathrm{Tr}\left(\frac{1}{2}\,F_{mn}\sigma_\alpha^{mn\beta}\lambda_\beta - \mathrm{i}\lambda_\alpha D\right)$,

$$\chi_\alpha = -\mathrm{i}\frac{ks+c}{|ks+c|}\Gamma_{\alpha\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = -\frac{k\,\mathrm{Tr}\,\lambda\lambda - c}{|k\,\mathrm{Tr}\,\lambda\lambda - c|}\sigma^3_{\alpha\dot{\beta}}\bar{\chi}^{\dot{\beta}}.$$

- From the very definition $\chi_\alpha = \sqrt{2}\,\mathrm{Tr}\left(\frac{1}{2}\,F_{mn}\sigma_\alpha^{mn\,\beta}\lambda_\beta - \mathrm{i}\lambda_\alpha D\right)$ the previous condition implies the following general constraint on λ

$$\frac{1}{2}\,\mathrm{Tr}\left[F_{ij}\sigma_\alpha^{ij\,\beta}\left(\lambda_\beta + \frac{k\,\mathrm{Tr}\,\lambda\lambda - c}{|k\,\mathrm{Tr}\,\lambda\lambda - c|}\sigma^3_{\alpha\dot\beta}\bar\lambda^{\dot\beta}\right)\right]$$
$$= \mathrm{Tr}\left[\left(\mathrm{i}D\delta_\alpha{}^\beta - F_{i3}\sigma_\alpha^{i3\,\beta}\right)\left(\lambda_\beta - \frac{k\,\mathrm{Tr}\,\lambda\lambda - c}{|k\,\mathrm{Tr}\,\lambda\lambda - c|}\sigma^3_{\alpha\dot\beta}\bar\lambda^{\dot\beta}\right)\right]\ .$$

- If we consider a particular solution such that λ is subject to the same projection condition as χ,

$$\lambda_\alpha = -\frac{k\,\mathrm{Tr}\,\lambda\lambda - c}{|k\,\mathrm{Tr}\,\lambda\lambda - c|}\sigma^3_{\alpha\dot\beta}\bar\lambda^{\dot\beta},$$

then the above equation implies that on the membrane worldvolume

$$F_{3i}|_{\mathcal{C}_3} = 0 = D|_{\mathcal{C}_3}.$$

- Furthermore, if λ satisfies $\lambda_\alpha = -\frac{k\,\mathrm{Tr}\,\lambda\lambda - c}{|k\,\mathrm{Tr}\,\lambda\lambda - c|}\sigma^3_{\alpha\dot\beta}\bar\lambda^{\dot\beta}$, then we have

$$\mathrm{Tr}\,\lambda\lambda = \mathrm{Tr}\,\lambda\sigma^3\bar\lambda\,e^{i\alpha}, \qquad \text{where} \qquad \alpha = \arg(-k\,\mathrm{Tr}\,\lambda\lambda + c).$$

- Upon some algebra, we also have

$$|k\,\mathrm{Tr}\,\lambda\lambda - c| = \pm|c| - k\,\mathrm{Tr}\,\lambda\sigma^3\bar{\lambda}\,.$$

Note that only the upper sign solution is consistent with $\lambda \to 0$.

- Hence, we pick this one and find

$$\arg c = \alpha + 2\pi n\,.$$

- so that $\alpha = \arg(-k\,\mathrm{Tr}\,\lambda\lambda + c)$ should be constant on the static membrane.
- solving $\lambda_\alpha = -\frac{k\,\mathrm{Tr}\,\lambda\lambda - c}{|k\,\mathrm{Tr}\,\lambda\lambda - c|}\sigma^3_{\alpha\dot{\beta}}\bar{\lambda}^{\dot{\beta}}$ by

$$\lambda_1 = \frac{1}{2}(\psi_1 + \mathrm{i}\psi_2), \qquad \lambda_2 = e^{i\alpha}\bar{\lambda}_1 = \frac{e^{i\alpha}}{2}(\psi_1 - \mathrm{i}\psi_2),$$

in terms of real $SL(2,\mathbb{R})$ spinor $\psi_\alpha = (\psi_1, \psi_2)$, we find

$$|k\,\mathrm{Tr}\,\lambda\lambda - c| = |c| - k\,\mathrm{Tr}\,\lambda\sigma^3\bar{\lambda} = |c| + \frac{\mathrm{i}k}{2}\,\mathrm{Tr}\,\psi^\alpha\psi_\alpha\,.$$

- Let us look at, the membrane bosonic equations for $x^3 = \theta = \bar{\theta} = 0$

$$\partial_{x^3}(|k\,\mathrm{Tr}\,\lambda\lambda - c| - k\,\mathrm{Tr}\,\lambda\sigma^3\bar{\lambda}) = k\varepsilon^{ijk3}\,\mathrm{Tr}\,F_{ij}F_{k3} + k\partial_i(\mathrm{Tr}\,\lambda\sigma^i\bar{\lambda}).$$

- its r.h.s. vanishes fue to $F_{3i}|_{\mathcal{C}_3} = 0 = D|_{\mathcal{C}_3}$ and projection conditions on λ; the latter also imply that two terms in the l.h.s. give the equal contribution so that we get

$$\partial_{x^3}(\mathrm{Tr}\,\lambda\sigma^3\bar{\lambda})|_{\mathcal{C}_3} = 0.$$

  This and $F_{3i}|_{\mathcal{C}_3} = 0 = D|_{\mathcal{C}_3}$ imply that the fields λ and A_i get localized on the membrane.
- Finally, substituting

$$|k\,\mathrm{Tr}\,\lambda\lambda - c| = |c| - k\,\mathrm{Tr}\,\lambda\sigma^3\bar{\lambda} = |c| + \frac{\mathrm{i}k}{2}\,\mathrm{Tr}\,\psi^\alpha\psi_\alpha\,.$$

  into the action for the supermembrane in the static gauge we arrive at the wanted expression for the $\mathcal{N} = 1$, $d = 3$ $SU(N)$ Chern-Simons action

$$S_{\mathrm{static}} = -\frac{\mathrm{i}k}{4\pi}\int_{\mathcal{C}} \mathrm{d}^3\xi\,\mathrm{Tr}\,\psi^\alpha\psi_\alpha + \frac{k}{4\pi}\int_{\mathcal{C}}\left[\mathrm{Tr}\left(A\mathrm{d}A + \frac{2\mathrm{i}}{3}A^3\right)\right] - T_0\int_{\mathcal{C}}\mathrm{d}^3\xi\,,$$

- Actually, our supermembrane originated action

$$S_{\text{static}} = -\frac{\mathrm{i}k}{4\pi}\int_{\mathcal{C}} \mathrm{d}^3\xi\, \mathrm{Tr}\, \psi^\alpha \psi_\alpha + \frac{k}{4\pi}\int_{\mathcal{C}} \left[\mathrm{Tr}\left(A\mathrm{d}A + \frac{2\mathrm{i}}{3}A^3\right)\right] - T_0 \int_{\mathcal{C}} \mathrm{d}^3\xi\,,$$

  differs from the $\mathcal{N}=1$, $d=3$ $SU(N)$ Chern-Simons action of level $-k$ by $-T_0\int_{\mathcal{C}}\mathrm{d}^3\xi$. But this constrant tension contribution completely decouples and can be removed by $T_0 \to 0$.
- The obtained action is level/rank dual to the Acharya-Vafa worldvolume theory of the $k=1$ domain wall [Acharya-Vafa 2001], but differs from the latter for $k>1$.
- Our action does not take into account additional worldvolume fields associated with relative fluctuations of a stack of k coincident D-branes in the stringy construction of Acharya and Vafa.
- Nevertheless, the account of the effects of the membrane of charge k (or of a stack of k parallel membranes of charge 1) allowed us [I.B.+Lanza+Sorokin 2019] to consistently derive the BPS domain wall tension and explicitly construct k-walls in the Veneziano-Yankielowicz effective theory.