CONFORMAL BLOCKS WITH HEAVY BACKGROUND OPERATORS

(large-c, AdS/CFT, geodesic networks)

Konstantin Alkalaev

Lebedev Physical Institute
and
Institute for Theoretical and Mathematical Physics

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Motivation

- The large central charge approximation in AdS_3/CFT_2 can be viewed as the block/length correspondence with two background operators

Outline

- CFT: conformal blocks and large central charge
- AdS: how to build the dual geometry
- AdS/CFT: 4-point block with three background operators as geodesic length
- More than three background operators

Heavy and light operators

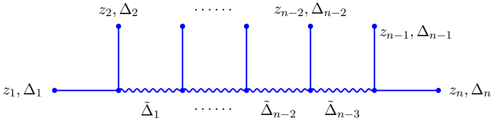
n-point correlation functions of $V_{\Delta_i,\bar{\Delta}_i}(z_i,\bar{z}_i)$, $i=1,...,n$ are given by

$$\langle V_{\Delta_1,\bar{\Delta}_1}(z_1,\bar{z}_1)\dots V_{\Delta_n,\bar{\Delta}_n}(z_n,\bar{z}_n)\rangle \sim \sum_{\{\tilde{\Delta}\}} C...C\ \mathcal{F}\ \bar{\mathcal{F}}$$

Conformal blocks

$$\mathcal{F}(z_1,...,z_n|\Delta_1,...,\Delta_n;\tilde{\Delta}_1,...,\tilde{\Delta}_{n-3}|c)$$

are conveniently depicted as (in a particular OPE channel)



Different large-c limits of the conformal blocks depend on the behavior of Δ_i and $\tilde{\Delta}_i$:

- $\Delta, \tilde{\Delta} = \mathcal{O}(c^1)$: heavy operators
- $\Delta, \tilde{\Delta} = \mathcal{O}(c^0)$: light operators

Three types of blocks:

- Global conformal block — all operators are light
- Classical conformal block — all operators are heavy
- Heavy-light blocks interpolate between these two extreme regimes

Classical conformal block

Let all conformal dimensions grow linearly with the central charge

$$\Delta_i = \mathcal{O}(c^1) \qquad \text{and} \qquad \tilde{\Delta}_j = \mathcal{O}(c^1)$$

The Laurent series around $c = \infty$ reads

$$\mathcal{F}(\Delta, \tilde{\Delta}, z|c) = \sum_{n\in\mathbb{N}} \frac{f_n(\epsilon, \tilde{\epsilon}|z)}{c^n} \qquad \text{where finite parameters} \qquad \epsilon_i = \frac{\Delta_i}{c} \quad \text{and} \quad \tilde{\epsilon}_j = \frac{\tilde{\Delta}_j}{c}$$

are *classical conformal dimensions*, and $f_n(\epsilon, \tilde{\epsilon}|z)$ are formal power series in the complex coordinates z with expansion coefficients being rational functions in ϵ and $\tilde{\epsilon}$.

Exponentiation hypothesis. At large c the principle part goes to zero. Less obvious is the fact that the regular part exponentiates (Zamolodchikov 1986). It follows that the large-c Virasoro block is asymptotically equivalent to

$$\mathcal{F}(\Delta, \tilde{\Delta}, z|c) \sim \exp\left[\, c\, f(\epsilon, \tilde{\epsilon}|z)\right] \qquad \text{at} \qquad c \gg 1$$

Function $f(\epsilon, \tilde{\epsilon}|q)$ is the *classical* conformal block.

Comments:

- Exponentiation is relevant for AdS/CFT within the GKP-W prescription
- The classical block is still hard to find

The problem: three heavy background operators

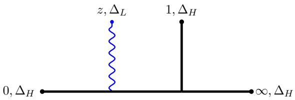
We consider the s-channel conformal block of the 4-point correlation function with three background operators and one perturbative operator,

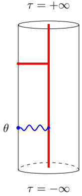
$$\text{HHHL type}: \qquad \langle \mathcal{O}_L(z,\bar{z})\mathcal{O}_H(0)\mathcal{O}_H(1)\mathcal{O}_H(\infty)\rangle\ ,$$

where $(z,\bar{z}) \in \mathbb{C}$, and the conformal dimensions are such that

$$\frac{\Delta_{L,H}}{c} = \text{fixed at } c \gg 1 \quad \text{and} \quad \frac{\Delta_L}{\Delta_H} \ll 1$$

- The large-c (i.e. classical) 4-point conformal block in the first order in Δ_L/Δ_H.
- The zeroth order: the 3-point function of the background operators $\mathcal{O}_H$ creates the AdS_3 space with three conical defects (3-conical space).
- $\mathcal{O}_L(z,\bar{z})$ is the geodesic line stretched from the conformal boundary to a distinguished point in the bulk. The geodesic length calculates the large-c conformal block of HHHL correlation function in the first order of the perturbative expansion.





Monodromy method

Let us consider $(n+1)$-point correlation functions with one degenerate operator. We have

$$\text{BPZ}: \qquad \left[c\frac{\partial^2}{\partial y^2}+\sum_{i=1}^{n}\left(\frac{\Delta_i}{(y-z_i)^2}+\frac{1}{y-z_i}\frac{\partial}{\partial z_i}\right)\right]\langle\Psi(y)V_1(z_1)\cdots V_n(z_n)\rangle=0$$

- In the classical limit $c\to\infty$ the $(n+1)$-point auxiliary correlation function behaves as

$$\mathcal{F}(y,z|\Delta_m,\tilde{\Delta}_k)\Big|_{c\to\infty}\to\psi(y|z)\exp\left[-\frac{c}{6}f(z|\epsilon_i,\tilde{\epsilon}_j)\right]$$

where $f(z_i|\epsilon_i,\tilde{\epsilon}_j)$ is the classical block and $\psi(y|z)$ is governed by the Fuchsian equation

$$\frac{d^2\psi(y|z)}{dy^2}+T(y|z)\psi(y|z)=0 \qquad \text{where} \qquad T(y|z)=\sum_{i=1}^{n}\left(\frac{\epsilon_i}{(y-z_i)^2}+\frac{c_i}{y-z_i}\right)$$

- Here $T(z)$ is the stress-energy tensor and c_i are the accessory parameters

$$\boxed{c_i(z)=\frac{\partial f(z)}{\partial z_i} \qquad i=1,...,n}$$

The monodromy properties of the correlation functions:

n algebraic relations for n accessory parameters.

- NP hard
- Low n points are solvable (using approximations)

Heavy-light perturbation expansion

Consider the HHHL type function. The conformal dimensions are organized as follows

$$\Delta_2/\Delta_{1,3,4} \ll 1 \quad \text{and} \quad \Delta_1 \sim \Delta_3 \sim \Delta_4 \ ,$$

The Fuchsian equation can be explicitly solved by expanding all functions in Δ_2 as

$$\psi(y,z) = \psi^{(0)}(y,z) + \psi^{(1)}(y,z) + \ldots \,, \qquad T(y,z) = T^{(0)}(y,z) + T^{(1)}(y,z) + \ldots \,,$$
$$f(z|\epsilon,\tilde{\epsilon}) = f^{(0)}(z|\epsilon,\tilde{\epsilon}) + f^{(1)}(z|\epsilon,\tilde{\epsilon}) + \ldots \,, \qquad c_2(z|\epsilon,\tilde{\epsilon}) = c_2^{(0)}(z|\epsilon,\tilde{\epsilon}) + c_2^{(1)}(z|\epsilon,\tilde{\epsilon}) + \ldots \,.$$

A few comments are in order.

- The term $f^{(0)} = 0$ because the conformal block for the 3-point HHH function equals 1.
- The zeroth order accessory parameter is also zero, $c_2^{(0)} = 0$.

The Fuchsian equation in the lowest orders takes the form

$$\left[\frac{d^2}{dy^2} + T^{(0)}(y)\right]\psi^{(0)}(y,z) = 0 \ , \qquad \left[\frac{d^2}{dy^2} + T^{(0)}(y)\right]\psi^{(1)}(y,z) = -T^{(1)}(y,z)\psi^{(0)}(y,z)$$

where

$$T^{(0)}(y) = \frac{\epsilon_1}{y^2} + \frac{\epsilon_3}{(1-y)^2} + \frac{\epsilon_1+\epsilon_3-\epsilon_4}{y(1-y)} \ , \quad T^{(1)}(y,z) = c_2\,\frac{(1-z)z}{y(1-y)(y-z)} + \frac{\epsilon_2}{(y-z)^2} + \frac{\epsilon_2}{y(1-y)}$$

Note that $T^{(1)}(y,z)$ is indeed the first order correction because $c_2 = \mathcal{O}(\epsilon_2)$.

First-order solution ($\Delta_3 = \Delta_4$)

0-th order. The Fuchsian equation can be reduced to the hypergeometric equation solved by

$$\psi_{\pm}^{(0)}(y) = (1-y)^{\frac{1+\alpha}{2}} y^{\frac{1\pm\beta}{2}} F_{\pm}(\alpha,\beta|y)\,,$$

where the hypergeometric functions are given by

$$F_{\pm}(\alpha,\beta|y) = {}_2F_1\left(\frac{1\pm\beta}{2}, \frac{1\pm\beta}{2}+\alpha, 1\pm\beta, y\right)\,,$$

and $\alpha = \sqrt{1-24\Delta_4/c}$, $\beta = \sqrt{1-24\Delta_1/c}$, and $0 < \alpha, \beta < 1$
1-st order. Using the method of variation of parameters we find the first order correction,

$$\psi_{\pm}^{(1)}(y,z) = \psi_{+}^{(0)}(y)\int dy\,\frac{\psi_{-}^{(0)}T^{(1)}(y,z)\psi_{\pm}^{(0)}}{W} - \psi_{-}^{(0)}(y)\int dy\,\frac{\psi_{+}^{(0)}T^{(1)}(y,z)\psi_{\pm}^{(0)}}{W}\,,$$

where the Wronskian is given by $W = \dfrac{\sin\pi\beta}{\pi}$. Thus, the first-order solution reads as

$$\psi_{\pm}(y,z) = \psi_{\pm}^{(0)}(y,z) + \psi_{\pm}^{(1)}(y,z)$$

It is parameterized by the background dimensions (α,β) and depends on the indeterminate accessory parameter c_2.

CFT result

- The accessory parameter

$$c_2 = \epsilon_2 \left[\frac{1+\alpha}{1-z} - \frac{1}{z} - \frac{d \log(F_+ F_-)}{dz} \right]$$

- The 4-point HHHL classical block

$$f(z|\alpha, \beta, \epsilon_2) = -\epsilon_2 \Big(\log(1-z)^{1+\alpha} + \log z + \log F_+(\alpha, \beta|z) + \log F_-(\alpha, \beta|z) \Big)$$

AdS dual : Bañados metric

In the AdS_3/CFT_2 correspondence, the locally AdS_3 geometry created by heavy insertions of the boundary CFT can be described in the Bañados form (Bañados, 1998)

$$ds^2 = R^2 \left(-H dz^2 - \bar{H} d\bar{z}^2 + \frac{u^2}{4} H \bar{H} \, dz d\bar{z} + \frac{du^2 + dz d\bar{z}}{u^2} \right) ,$$

with $u \in [0, \infty)$ and $z, \bar{z} \in \mathbb{C}$ being local coordinates, the radius is R. Arbitrary (anti)holomorphic functions $H = H(z)$ and $\bar{H} = \bar{H}(\bar{z})$ can be interpreted as components of the holographic CFT_2 energy-momentum tensor

$$T(z) = \frac{c}{6} H(z) ,$$

where the central charge is $c = 3R/2G_N$. Under $z \to w(z)$ it transforms in the standard fashion as

$$T(z) = \left(w'\right)^2 T(w) + \frac{c}{12} \{w, z\} , \qquad \text{where} \quad \{w, z\} = \frac{w'''}{w'} - \frac{3}{2} \left(\frac{w''}{w'} \right)^2 ,$$

where the prime denotes differentiation with respect to z.

AdS dual: Poincare metric

Let us find a map

$$z \to w(z) \qquad \text{such that} \qquad H(w(z)) = 0$$

Away from singularities it would correspond to pure AdS_3 in the Poincare coordinates (Asplund et al. 2014). This can be achieved provided that

$$H(z) = \frac{1}{2}\{w, z\} \ .$$

The solution to the above equation can be represented as the ratio of two independent solutions to the auxiliary Fuchsian equation

$$\psi'' + H\psi = 0$$

This is the Schwarz map

$$w(z) = \frac{A\,\psi_1(z) + B\,\psi_2(z)}{C\,\psi_1(z) + D\,\psi_2(z)} \ , \qquad AD - BC \neq 0 \ ,$$

where $\psi_{1,2}$ are two independent Fuchsian solutions, and $A, B, C, D \in \mathbb{C}$ parameterize the Möbius transformation of $\psi_1(z)/\psi_2(z)$.

In the large-c regime the function H can be identified with the *classical* energy-momentum tensor arising in the zeroth-order Fuchsian equation of the monodromy method, i.e.,

$$H(z|\mathbf{z}) \equiv T^{(0)}(z|\mathbf{z}) \ ,$$

where the set of singular points $\mathbf{z}$ is the locations of the background operators.

AdS dual: Roberts solution

The boundary map $z \to w(z)$ can be extended to the whole three-dimensional space, $w = w(z, \bar{z}, u)$, $\bar{w} = \bar{w}(z, \bar{z}, u)$, and $v = v(z, \bar{z}, u)$, such that the resulting metric describes the Poincare patch

$$d\tilde{s}^2 = \frac{dv^2 + dw d\bar{w}}{v^2}$$

The explicit coordinate transformation reads (Roberts, 2012)

$$w(z, \bar{z}, u) = w(z) - \frac{2u^2 w'(z)^2 \bar{w}''(\bar{z})}{4w'(z)\bar{w}'(\bar{z}) + u^2 w''(z)\bar{w}''(\bar{z})} \qquad v(z, \bar{z}, u) = u \, \frac{4\left(w'(z)\bar{w}'(\bar{z})\right)^{3/2}}{4w'(z)\bar{w}'(\bar{z}) + u^2 w''(z)\bar{w}''(\bar{z})}$$

The length of a geodesic line in the Poincare coordinates:

$$\mathcal{L}_{AdS} = R \log \frac{(w_1 - w_2)(\bar{w}_1 - \bar{w}_2)}{v_1 v_2}$$

In the Euclidean case the Poincare patch covers the whole global AdS_3 space

$$d\hat{s}^2 = \frac{d\tau^2 + d\rho^2 + \sin^2 \rho d\phi^2}{\cos^2 \rho}$$

through the coordinate change

$$w = e^{\theta} \sin \rho, \qquad v = e^{\frac{\theta + \bar{\theta}}{2}} \cos \rho$$

where $\theta = \tau + i\phi$ and ρ are coordinates of the global AdS_3 (rigid cylinder). The conformal boundary is at $\rho = \pi/2$. There is a conformal map $\theta = \log w$ from the boundary $(w, \bar{w})$-plane to the boundary $(\theta, \bar{\theta})$-cylinder.

4-point HHHL block as geodesic length

- The classical energy-momentum tensor is given by

$$T^{(0)}(z) = \frac{\epsilon_1}{z^2} + \frac{\epsilon_3}{(z-1)^2} + \frac{\epsilon_1}{z(1-z)} \ ,$$

where ϵ_1 and $\epsilon_3 = \epsilon_4$ are classical dimensions of the heavy background operators at $(0, 1, \infty)$.

- The resulting space defined by the Bañados metric will be denoted as $AdS_3[3]$. There are three lines of coordinate singularities: $(z, \bar{z}, u) = (0, 0, u)$, $(z, \bar{z}, u) = (1, 1, u)$, $(z, \bar{z}, u) = (\infty, \infty, u)$ for any $u \in \mathbb{R}_+$.

- Choosing the Fuchsian solutions as $\psi_{1,2}(z) = \psi^{(0)}_{\pm}(z)$ we find the conformal mapping,

$$w(z) = z^{\beta} \ \frac{{}_2F_1\left(\frac{1+\beta}{2}, \frac{1+\beta}{2} + \alpha, 1+\beta, z\right)}{{}_2F_1\left(\frac{1-\beta}{2}, \frac{1-\beta}{2} + \alpha, 1-\beta, z\right)}$$

This is the Schwarz triangle function that maps the $(z, \bar{z})$-plane onto some curvilinear triangle on the $(w, \bar{w})$-plane.

Comments:

- # The conformal mapping is defined up to Möbius transformations. The Möbius group acts triply transitively and conformally.
- # By construction, the Schwarz function has three singular points $z = 0, 1, \infty$ identified with background operator locations. The angle in the point $w(0)$ is equal to $\pi\beta$, the second angle in $w(1)$ is equal to $-\pi\alpha$, and the third angle in $w(\infty)$ is equal to $\pi\alpha$. We have angle deficit/excess.

- Let us consider now the HHHL conformal block function in three boundary coordinate systems:

$$(z, \bar{z})\text{-plane}, \ (w, \bar{w})\text{-domain}, \ (\theta, \bar{\theta})\text{-cylinder}$$

Assuming that we do some coordinate change $x \to x(y)$ the transformation formula is

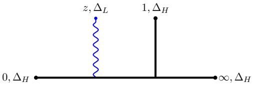
$$f(x|\alpha, \beta, \epsilon_2) = f(y(x)|\alpha, \beta, \epsilon_2) + \epsilon_2 \log y'(x)$$

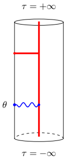
The block function in different coordinate systems is given by

$$f(w|\alpha, \beta, \epsilon_2) = -\epsilon_2 \log w \ , \qquad f(\theta|\alpha, \beta, \epsilon_2) = 0$$

- Let us consider AdS$_3$[3] in the Poincare coordinates and fix two points: the boundary insertion of the perturbative operator $(w, \bar{w}, \varepsilon)$ and the distinguished point in the bulk $(0, 0, 1)$, where the cut-off $\varepsilon \to 0$. The distinguished point belongs to the trivalent graph connecting the background heavy insertions: two at infinities, one in a finite region of the conformal boundary. The geodesic length is

$$\mathcal{L}_{AdS_3[3]}(w, \bar{w}) = R\left(\log w + \log \bar{w}\right) - R \log \varepsilon \ ,$$





The (holomorphic) block/length relation is given by

$$f(w|\alpha, \beta, \epsilon_2) \sim -\frac{\epsilon_2}{R} \ \mathcal{L}_{AdS_3[3]}(w)$$

More than three background operators

- We consider $\mathrm{H}^{n-k}\mathrm{L}^{k}$ type correlation functions.

- Let $\mathrm{AdS}_3[n-k]$ be a three-dimensional space with the Bañados metric defined by the classical tensor $T(z|\mathbf{z})$ with $n-k$ singular points.
- The boundary Schwarz mappings and the Poincare coordinates are build using the solutions of the associated Fuchsian equation,

$$\left[\frac{d^2}{dz^2}+T(z|\mathbf{z})\right]\psi(z)=0\ , \quad \text{where} \quad T(z|\mathbf{z})=\sum_{i=k+1}^{n}\frac{\epsilon_i}{(z-z_i)^2}+\frac{c_i}{z-z_i}\ ,$$

where $\mathbf{z}=(z_{k+1},...,z_n)$ are locations of the background operators with classical dimensions ϵ_i, the c_i are respective accessory parameters.
- The resulting space $\mathrm{AdS}_3[n-k]$ will have $n-k$ conical defects parameterized by background conformal dimensions as can be directly seen from the Schwarz map of the $(z,\bar{z})$-plane to some curvilinear polygon with $n-k$ vertices on the $(w,\bar{w})$-plane.
- Assuming that $\epsilon_j/\epsilon_i \ll 1$ for $j=1,...,k$ and $i=k+1,...,n$ we can use the heavy-light expansion and introduce type $H^{n-k}L^{k}$ perturbative conformal blocks $f_{(k,n-k)}(w)$. The energy-momentum tensor arising in the zeroth order is exactly $T(z|\mathbf{z})$.
- It is tempting to conjecture that type $H^{n-k}L^{k}$ conformal blocks are equal to the length of dual geodesic trees in $\mathrm{AdS}_3[n-k]$,

$$f_{(k,n-k)}(w|\epsilon)\sim-\frac{1}{R}\ \mathcal{L}_{AdS_3[n-k]}(w|\epsilon)\ ,$$

where the right-hand side is the weighted length of the dual geodesic tree, and w are locations of perturbative operators in the Poincare coordinates.

Conclusion

- Up to now, the case of HHL^{n-2} type functions is fully understood.
- We considered 4-point HHHL, the next non-trivial check is 5-point HHHLL, and then n-point $HHHL^{n-3}$.
- Towards H^{n-k}L^{k} type functions and their duals.