Francesco Toppan

CBPF, Rio de Janeiro, Brazil

Superconformal and Topological Conformal Mechanics

Talk at SQS'2017
Dubna, August 2017

Based on:

I - I.E. Cunha, N. L. Holanda & F.T., arXiv:1610.07205

II- F.T. & M. Valenzuela, arXiv:1705.04004

Conformal Mechanics (some basic references)

F. Calogero (1969) - sl_2**-invariance,** $\frac{1}{x^2}$ **potential.**

de Alfaro-Fubini-Furlan (1976) - **oscillator damping term (discrete, grounded from below spectrum, ground state)**.

Conformal Mechanics in the new Millennium (motivations):

Holography: $AdS_2 - CFT_1$

test particle close to RN BH horizon (Britto-Pacumio et al. 1999).

2012 State of the art review: Fedoruk-Ivanov-Lechtenfeld

Based on $(d, \mathcal{N}, \mathcal{N} - d)$, $\mathcal{N} = 1, 2, 4, 8$ worldline supermultiplets:

Existence of Critical scaling dimensions in $\mathcal{N} = 4$ and $\mathcal{N} = 8$ Superconformal Mechanics:
Z. Kuznetsova & F.T., JMP 53 (2012) 043513.

Classification of critical scaling dimensions for $\mathcal{N} = 7, \mathcal{N} = 8$:
S. Khodaee & F.T., JMP 53 (2012) 103518.

Recognition of the existence of extra allowed potentials, Papadopoulos CQG 2013

Two classes of superconformal mechanics, parabolic versus trigonometric/hyperbolic:
N. L. Holanda & F.T., JMP 55 (2014) 061703.

Quantization of superconformal mechanics (both cases, parabolic & trigonometric)

Part I of the talk:

Quantization of world-line superconformal actions ($1D$ sigma-models), based on
I. E. Cunha, N. L. Holanda & F.T., arXiv:1610.07205

Part II of the talk:

Symmetries of Matrix PDEs, based on
F.T. & M. Valenzuela, arXiv:1705.04004

$sl(2)$ algebra:

$$
\begin{aligned}
[D,H] &= H,\\
[D,K] &= -K,\\
[H,K] &= 2D.
\end{aligned}
$$

par vs hyp/tri D-module reps (Papadopoulos & Holanda-F.T.)

$$
\begin{aligned}
H &= \partial_t,\\
D &= t\partial_t + \lambda,\\
K &= t^2\partial_t + 2\lambda t.
\end{aligned}
$$

$$
\begin{aligned}
H &= e^{-\mu\tau}(\frac{1}{\mu}\partial_\tau - \lambda),\\
D &= \frac{1}{\mu}\partial_\tau,\\
K &= e^{\mu\tau}(\frac{1}{\mu}\partial_\tau + \lambda).
\end{aligned}
$$

Comments:

par $\leftrightarrow$ *hyp*/*tri* connection via similarity transformations and $t \mapsto t = \ln \tau$ change of the time coordinate.

λ is the scaling dimension: for $sl(2)$ is <u>non-critical</u>.

μ is a dimensional parameter (i.e.: it allows extra potential terms),
μ real: trigonometric case,
μ imaginary: hyperbolic case.

Clifford algebras encode worldline supersymmetry:

$\mathcal{N}$ operators Q_i such that

$$\begin{aligned} \{Q_i, Q_j\} &= 2\delta_{ij} H, \\ [H, Q_i] &= 0 \end{aligned}$$

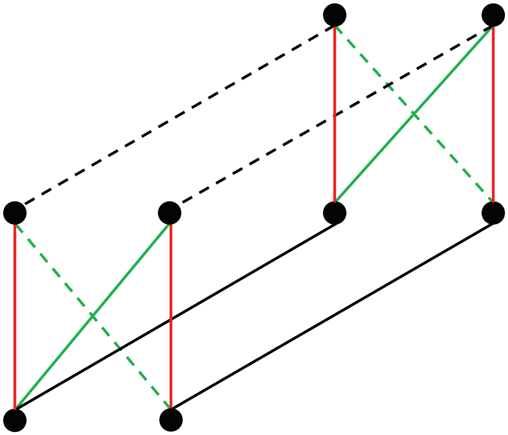
"usually" $H = i\partial_t$.

$\mathcal{N} = 1$ example

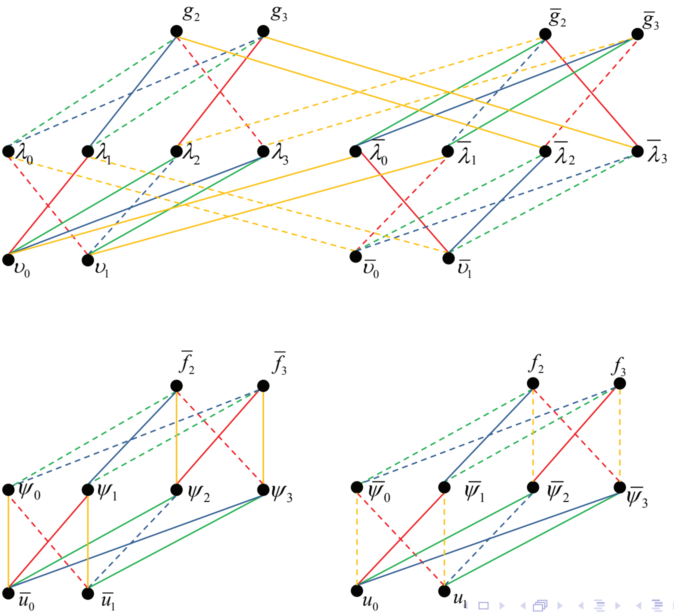
$$Q = \begin{pmatrix} 0 & 1 \\ i\partial_t & 0 \end{pmatrix},$$

Worldine supermultiplets

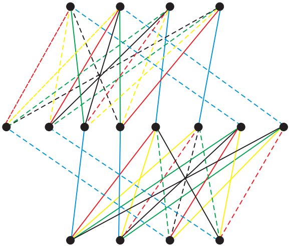
$\mathcal{N} = 3$ $(2, 4, 2)$:

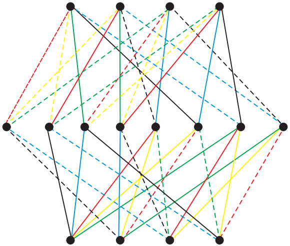


Equivalent $\mathcal{N} = 4$ $(4, 8, 4)$ Worldine supermultiplets:



Inequivalent $\mathcal{N} = 4$ $(4, 8, 4)$ Worldine supermultiplets:





Properties of finite SCA's:

Even sector $\mathcal{G}_{even}$: $sl(2) \oplus R$ (R is the R-symmetry).

Odd sector $\mathcal{G}_{odd}$: $2\mathcal{N}$ generators.

The dilatation operator D induces the grading

$$\mathcal{G} = \mathcal{G}_{-1} \oplus \mathcal{G}_{-\frac{1}{2}} \oplus \mathcal{G}_0 \oplus \mathcal{G}_{\frac{1}{2}} \oplus \mathcal{G}_1 .$$

The sector $\mathcal{G}_1$ ($\mathcal{G}_{-1}$) containes a unique generator given by H (K). The $\mathcal{G}_0$ sector is given by the union of D and the R-symmetry subalgebra ($\mathcal{G}_0 = \{D\} \bigcup \{R\}$).
The odd sectors $\mathcal{G}_{\frac{1}{2}}$ and $\mathcal{G}_{-\frac{1}{2}}$ are spanned by the supercharges Q_i's and their superconformal partners $\widetilde{Q}_i$'s, respectively.
The invariance under the global supercharges Q_i's and the generator K implies the invariance under the full superconformal algebra $\mathcal{G}$.

Most relevant cases: $\mathcal{N} = 4, 7, 8$ finite SCA's

$\mathcal{N} = 4$: simple SCA's are $A(1,1)$ and the exceptional superalgebras $D(2,1;\alpha)$, for $\alpha \in \mathbb{C}\backslash\{0,-1\}$.

Superalgebra isomorphism for α's connected via an S_3 group transformation:

$$\begin{array}{rclrclrcl} \alpha^{(1)} &=& \alpha, & \alpha^{(3)} &=& -(1+\alpha), & \alpha^{(5)} &=& -\frac{1+\alpha}{\alpha}, \\ \alpha^{(2)} &=& \frac{1}{\alpha}, & \alpha^{(4)} &=& -\frac{1}{(1+\alpha)}, & \alpha^{(6)} &=& -\frac{\alpha}{(1+\alpha)}. \end{array}$$

$A(1,1)$ can be regarded as a degenerate superalgebra recovered from $D(2,1;\alpha)$ at the special values $\alpha = 0, -1$.
For α real ($\alpha \in \mathbb{R}$) a fundamental domain under the action of the S_3 group can be chosen to be the closed interval

$$\alpha \in [0,1].$$

Over $\mathbb{C}$, there are four finite $\mathcal{N} = 8$ SCA's and one finite $\mathcal{N} = 7$ SCA.

The finite $\mathcal{N} = 8$ superconformal algebras are:

i) the $A(3,1) = sl(4|2)$ superalgebra, possessing 19 even generators and bosonic sector given by $sl(2) \oplus sl(4) \oplus u(1)$,
ii) the $D(4,1) = osp(8,2)$ superalgebra, possessing 31 even generators and bosonic sector given by $sl(2) \oplus so(8)$,
iii) the $D(2,2) = osp(4|4)$ superalgebra, possessing 16 even generators and bosonic sector given by $sl(2) \oplus so(3) \oplus sp(4)$,
iv) the $F(4)$ exceptional superalgebra, possessing 24 even generators and bosonic sector given by $sl(2) \oplus so(7)$.

The finite $\mathcal{N} = 7$ superconformal algebra is the exceptional superalgebra $G(3)$, possessing 17 even generators and bosonic sector given by $sl(2) \oplus g_2$.

Existence of critical scaling dimension λ's:

$\mathcal{N}=4$: $D(2,1;\alpha)$ reps are recovered from the $(k,4,4-k)$ supermultiplets, with a relation between α and the scaling dimension given by $\alpha=(2-k)\lambda$.

$\mathcal{N}=8$: for $k\neq 4$, all four $\mathcal{N}=8$ finite superconformal algebras are recovered, at the critical values $\lambda_k=\frac{1}{k-4}$, with the identifications:
$D(4,1)$ for $k=0,8$, $F(4)$ for $k=1,7$,
$A(3,1)$ for $k=2,6$ and $D(2,2)$ for $k=3,5$.

$\mathcal{N}=7$: the global supermultiplet $(1,7,7,1)$ induces, at $\lambda=-\frac{1}{4}$, a D-module representation of the exceptional superalgebra $G(3)$.

The $D(2,1;\alpha)$-invariant actions are

parabolic case:

$$\begin{aligned}\mathcal{L} &= A(\dot{\varphi}^2 + \psi_I \dot{\psi}_I + g_i^2) + A_\varphi(\psi_0 \psi_i g_i + \frac{1}{2}\epsilon^{ijk}\psi_i\psi_j g_k) + \\ &\quad \frac{1}{6}A_{\varphi\varphi}\epsilon^{ijk}\psi_0\psi_i\psi_j\psi_k, \\ \text{with} &\quad A = C\varphi^{-\frac{1+2\alpha}{\alpha}}\end{aligned}$$

hyperbolic case:

$$\begin{aligned}\mathcal{L} &= A(\dot{\varphi}^2 + \mu\psi_I \dot{\psi}_I + \mu^2 g_i^2) + \mu^2 A_\varphi(\psi_0 \psi_i g_i + \frac{1}{2}\epsilon^{ijk}\psi_i\psi_j g_k) + \\ &\quad \frac{1}{6}\mu^2 A_{\varphi\varphi}\epsilon^{ijk}\psi_0\psi_i\psi_j\psi_k + \mu^2\alpha^2 A\varphi^2, \\ \text{with} &\quad A = C\varphi^{-\frac{1+2\alpha}{\alpha}}.\end{aligned}$$

Fields redefinitions:

(constant kinetic basis, non-linear realization of supersymmetry)

$$
\begin{aligned}
\overline{\phi} &= -2\alpha\phi^{-\frac{1}{2\alpha}}, \\
\overline{\psi}_{I} &= \phi^{-\frac{1+2\alpha}{2\alpha}}\psi_{I}, \\
\overline{g}_{i} &= \phi^{-\frac{1+2\alpha}{2\alpha}} g_{i}.
\end{aligned}
$$

Lagrangians for Superconformal Mechanics

($D(2,1;\alpha)$-invariance, from Holanda-F.T., JMP 2014)

$(1,4,3)$ supermultiplet with $Q^2=H$:

$$
\begin{array}{rcl}
\mathcal{L} &=& (\dot{\bar\phi}^2+\bar\psi_I\dot{\bar\psi}_I+{\bar g_i}^2)+\frac{2(1+2\alpha)}{\bar\phi}\left(\bar\psi_0\bar\psi_i\bar g_i+\frac{1}{2}\epsilon^{ijk}\bar\psi_i\bar\psi_j\bar g_k\right)+\\
&&\frac{2(1+2\alpha)(1+3\alpha)}{3\bar\phi^2}\epsilon^{ijk}\bar\psi_0\bar\psi_i\bar\psi_j\bar\psi_k.
\end{array}
$$

$(1,4,3)$ supermultiplet with $Q^2=Z\neq H$:

$$
\begin{array}{rcl}
\mathcal{L} &=& (\dot{\bar\phi}^2+\mu\bar\psi_I\dot{\bar\psi}_I+\mu^2{\bar g_i}^2)+\frac{2(1+2\alpha)\mu^2}{\bar\phi}\left(\bar\psi_0\bar\psi_i\bar g_i+\frac{1}{2}\epsilon^{ijk}\bar\psi_i\bar\psi_j\bar g_k\right)+\\
&&\frac{2(1+2\alpha)(1+3\alpha)\mu^2}{3\bar\phi^2}\epsilon^{ijk}\bar\psi_0\bar\psi_i\bar\psi_j\bar\psi_k+\frac{\mu^2}{4}\bar\phi^2.
\end{array}
$$

1D superconformal invariance does not imply supersymmetry: $osp(1|2)$-invariant example

The hyperbolic action is

$$\mathcal{S} \;=\; \int dt(\dot{\varphi}^2 - \psi\dot{\psi} + \varphi^2).$$

The five invariant operators (closing the $osp(1|2)$ algebra) are

$$\begin{aligned} Q^{\pm}\varphi &= e^{\pm t}\psi, & Q^{\pm}\psi &= e^{\pm t}(\dot{\varphi} \mp \varphi),\\ Z^{\pm}\varphi &= e^{\pm 2t}(\dot{\varphi} \mp \varphi), & Z^{\pm}\psi &= e^{\pm 2t}\dot{\psi},\\ H\varphi &= \dot{\varphi}, & H\psi &= \dot{\psi}. \end{aligned}$$

One shoulde note that $Z^{\pm} = (Q^{\pm})^2$.

No change of time variable $t \mapsto \tau(t)$ allows to represent either Z^{+} or Z^{-} as a time-derivative operator with respect to the new time τ.

Simplest example of "fermionization"

Wigner's dual picture of the harmonic oscillator from

"Do the Equations of Motion Determine the Quantum Mechanical Commutation Relations?" E.P. Wigner, Phys. Rev. 77 (1950) 711. (Wigner discovered $osp(1|2)$ as a byproduct).

$$H = \frac{1}{2}\{a, a^\dagger\}, \qquad [H, a^\pm] = \pm a^\pm,$$

(also $E^\pm = \{a^\pm, a^\pm\}$**).**

The Fock vacuum $a|vac>= 0$ is replaced by a lowest weight representation of $osp(1|2)$.

Where is the supersymmetry in the harmonic oscillator?

1) The action is invariant under $osp(1|2)$.

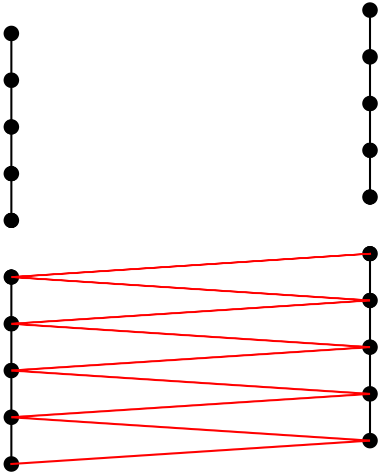
2) The supersymmetry is "weak" ($Q^2 = Z \neq H$).

3) The supersymmetry is spontaneously broken:
$(a + a^\dagger)|vac > \neq 0$.

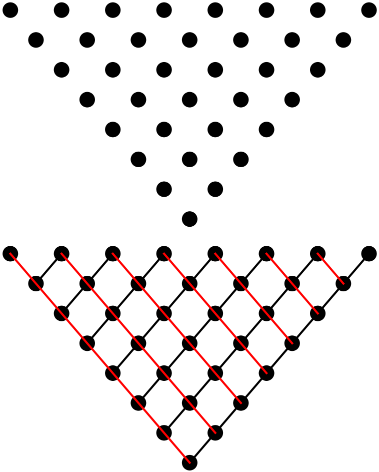
4) Supersymmetry as spectrum-generating algebra.

5) Fermion parity $\Leftrightarrow$ eigenfunction's parity.

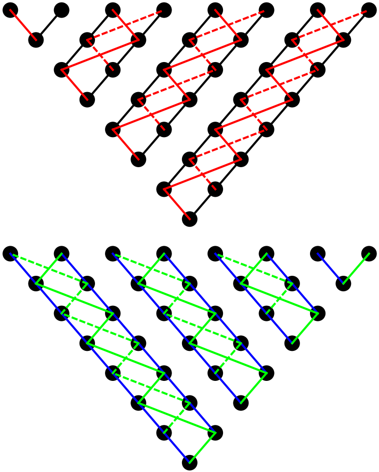
SPECTRUM GENERATING ALGEBRA OF THE HARMONIC OSCILLATOR :



TRIANGULAR NUMBERS (AKA 2D HARMONIC OSCILLATOR) :



SUPERSYMMETRIC CARPETS (N=2):



From Lagrangian to Hamiltonian framework (classical)

$osp(1|2)$ example

$$\mathcal{S} = \int dt\mathcal{L} = \int dt\frac{1}{2}(\dot{y}^2 + i\chi\dot{\chi}),$$

with χ a Grassmann variable.

The classical Noether charges ar:e

$$C_H = \frac{\dot{y}^2}{2}, \quad C_D = \frac{t\dot{y}^2}{2} - \frac{y\dot{y}}{2}, \quad C_K = \frac{t^2\dot{y}^2}{2} - ty\dot{y} + \frac{y^2}{2},$$
$$C_Q = \dot{y}\chi, \quad C_{\bar{Q}} = t\dot{y}\chi + y\chi.$$

The Hamiltonian formalism requires the conjugate momenta:

$$p = \frac{\partial\mathcal{L}}{\partial\dot{y}} = \dot{y}, \quad \pi = \frac{\partial\mathcal{L}}{\partial\dot{\chi}} = -\frac{i\chi}{2}.$$

The last step requires defining the Dirac brackets:

The conjugate momentum π to the Grassmann variable χ is not an invertible function of the velocity $\dot{\chi}$. The second equation is a second class constraint on the phase space:

$$u = \pi + \frac{i\chi}{2}.$$

The non-vanishing Dirac brackets are

$$\{y, p\}_D = 1, \qquad \{\chi, \chi\}_D = -i.$$

Canonical Quantization:

$$\{A, B\}_D \;\rightarrow\; \frac{1}{i\hbar}[A, B\}.$$

The non-vanishing (anti)commutators are

$$[\hat{y}, \hat{p}] = i\hbar, \qquad \{\hat{\chi}, \hat{\chi}\} = \hbar.$$

In the position-space representation the above operators are

$$\hat{y} = y, \qquad \hat{p} = -i\hbar\partial_y, \qquad \hat{\chi} = \sqrt{\frac{\hbar}{2}}.$$

Comment: the fermionic field χ, classically represented by a Grassmann variable, is a Clifford variable $\hat{\chi}$ in the quantum case.

The DFF Quantum $D(2,1;\alpha)$ $(1,4,3)$ Model:

It is obtained from the $\mathfrak{h}_1 \oplus C_4$ superalgebra.
The Hamiltonian is the Cartan operator

$$\hat{D} = (\frac{\hat{p}^2}{2} + \frac{\hat{y}^2}{8} + \frac{(1+2\alpha)^2}{8\hat{y}^2})\mathbb{I}_4 + \frac{(1+2\alpha)}{4\hat{y}^2}\mathcal{F}_4.$$

($\mathcal{F}_4$ is the Fermion Parity Operator.
Explicitly,

$$\hat{D} = \left(\begin{array}{c|c} (\frac{\hat{p}^2}{2} + \frac{4\alpha^2+8\alpha+3}{8\hat{y}^2} + \frac{\hat{y}^2}{8})\mathbb{I}_2 & 0 \\ \hline 0 & (\frac{\hat{p}^2}{2} + \frac{4\alpha^2-1}{8\hat{y}^2} + \frac{\hat{y}^2}{8})\mathbb{I}_2 \end{array}\right).$$

Both upper (bosonic) and lower (fermionic) diagonal blocks of $\hat{D}$ contain a Calogero Hamiltonian with the DFF term

$$\hat{\mathcal{H}}_{DFF} = \frac{1}{2}\hat{p}^2 + \frac{g^2}{\hat{y}^2} + \frac{\hat{y}^2}{8}.$$

The inequality $g^2 > -\frac{1}{8}$ guarantees the existence of physically acceptable solutions.

The discrete spectrum is

$$E_n = \frac{1}{2}(n + \nu + 1).$$

The parameter ν entering the Casimir energy $\frac{1}{2}(\nu + 1)$ is

$$\nu = \frac{1}{2}(1 + 8g^2)^{\frac{1}{2}}.$$

$\alpha \neq 0, -1$ ensures that both g_b^2 and g_f^2 are greater than $-\frac{1}{8}$.
The scaling dimension α can be regarded as an external control parameter of the theory, so that the vacuum energy can be interpreted as a Casimir energy.
The Casimir energy of the $(1, 4, 3)$ $D(2, 1; \alpha)$ (un)deformed oscillator admits a very nice expression in terms of α:

$$E_{vac} = \frac{1}{4}(1 + |2\alpha + 1|).$$

Part II: Analysis from PDE's symmetries

$\Omega = \Omega^\dagger$ is a hermitian (matrix) PDE, a first-order differential operator Σ is a symmetry operator if it satisfies the equation

$$[\Sigma, \Omega] \;=\; \Phi_\Sigma \cdot \Omega,$$

for a given matrix-valued function Φ_Σ.

2×2-matrix differential operator containing Calogero potentials:

$$\Omega \;=\; (e_{11} + e_{22})(i\partial_t + \frac{1}{2}\partial_x^2) - v_1(x)e_{11} - v_2(x)e_{22}.$$

$4 + 4$ diagonal symmetry generators and non-diagonal symmetry operators $[\Sigma_{up}, \Omega] = 0, [\Sigma_{down}, \Omega] = 0$ for

$$v_i(x) \;=\; \frac{a_i}{x^2} + b_i x^2 + c_i,$$

$a_1 + a_2 - a_1^2 - a_2^2 + 2a_1 a_2 = 0,\; b_2 = b_1$ and, without loss of generality via similarity transformations, $c_1 = c_2 = 0$.

After setting $\nu = 2(a_2 - a_1)$,

$$a_1 = \frac{1}{8}\nu(\nu-2), \qquad a_2 = \frac{1}{8}\nu(\nu+2).$$

we have

$$\Omega_\epsilon = (e_{11}+e_{22})(i\partial_t + \frac{1}{2}\partial_x^2 - \frac{1}{2}\epsilon x^2 - \frac{\nu^2}{8x^2}) + (e_{11}-e_{22})\frac{\nu}{4x^2},$$

with $\epsilon = 0, 1$.
The operator K,

$$K = e_{11} - e_{22},$$

plays different roles, depending on the context:
- **either the Fermion Parity Operator (SQM) or**
- **Klein Operator (entering deformed oscillators).**

Triplet of operators, $\Omega_{\pm 1}, \Omega_0$.

They carry an $sl(2)$ representation generated by $z'_{\pm}\mathbb{I}_2, z_0\mathbb{I}_2$:

$$[z'_{-}\mathbb{I}_2, \Omega_{+1}] = \Omega_0 = 2it\Omega_{+1}, \qquad [z'_{-}\mathbb{I}_2, \Omega_0] = \Omega_{-1} = -2t^2\Omega_{+1}.$$

$\Omega_{\pm 1}, \Omega_0$ close an $sl(2)$ algebra:

$$[\Omega_0, \Omega_{\pm 1}] = \pm 2\Omega_{\pm 1}, \qquad [\Omega_{+1}, \Omega_{-1}] = -2\Omega_0.$$

Under redefinition of the time coordinate and a similarity transformation Ω_0 is mapped into $\Omega_{\epsilon=1}$.

$\epsilon = 0$ example:

$$\begin{aligned}
\Sigma_1 &= e_{12}(\partial_x + \frac{\nu}{2x}), \\
\Sigma_2 &= e_{12}(t\partial_x + t\frac{\nu}{2x} - ix), \\
\Sigma_3 &= e_{21}(\partial_x - \frac{\nu}{2x}), \\
\Sigma_4 &= e_{21}(t\partial_x - t\frac{\nu}{2x} - ix).
\end{aligned}$$

Basis of 4 off-diagonal hermitian operators:

$$\begin{aligned}
Q_1 &= \frac{i}{\sqrt{2}}(\Sigma_1 + \Sigma_3), \\
Q_2 &= iKQ_1 = \frac{1}{\sqrt{2}}(\Sigma_3 - \Sigma_1), \\
\widetilde{Q}_1 &= \frac{i}{\sqrt{2}}(\Sigma_2 + \Sigma_4), \\
\widetilde{Q}_2 &= iK\widetilde{Q}_1 = \frac{1}{\sqrt{2}}(\Sigma_4 - \Sigma_2).
\end{aligned}$$

Several symmetry (super)algebras:

$\mathcal{N} = 2$ supersymmetry:

$$\{Q_i, Q_j\} = 2\delta_{ij}\mathbf{H}.$$

$osp(2|2)$ superalgebra:

$$\begin{aligned}
\{Q_i, Q_j\} &= 2\delta_{ij}\mathbf{H},\\
\{\widetilde{Q}_i, \widetilde{Q}_j\} &= 2\delta_{ij}\mathbf{K},\\
\{Q_i, \widetilde{Q}_j\} &= \delta_{ij}\mathbf{D} + \epsilon_{ij}\mathbf{J}.
\end{aligned}$$

Deformed Heisenberg algebra induced by

$$[Q_1, \widetilde{Q}_1] = [Q_2, \widetilde{Q}_2] = \frac{i}{2}(\mathbb{I}_2 + \nu K).$$

Connection with higher spin:

From

$$\mathcal{Q}^1_\alpha := \mathcal{Q}_\alpha, \qquad \mathcal{Q}^2_\alpha := iK\mathcal{Q}_\alpha,$$

Associative algebra $Aq(2;\nu)$ (introduced by Vasiliev 1989) of Weyl ordered (hermitian) monomials

$$\mathcal{Q}_{\alpha(n)} = \mathcal{Q}_{12\cdots n} := \sum_{\bar{1},2;\sigma} \frac{1}{n!}\, \mathcal{Q}_{\sigma(1)}\mathcal{Q}_{\sigma(2)}\cdots\mathcal{Q}_{\sigma(n)}\,, \qquad n = 0, 1, \ldots,$$

As a vector space $Aq(2;\nu)$ can be endowed with two different types of brackets:

i) either ordinary brackets realized by commutators (**deformed Schrödinger algebra**) or

ii) $\mathbb{Z}_2$-graded brackets realized by (anti)commutators. It is the **Vasiliev's higher spin superalgebra**

$$q(2;\nu) := \{Aq(2;\nu)\,|\,[\mathfrak{a},\mathfrak{b}\} \in Aq(2;\nu)\,,\,\forall\; \mathfrak{a},\,\mathfrak{b}\}.$$

Covariant form of $osp(2|2) \subset q(2;\nu)$

$osp(2|2)$ admits a covariant description in terms of three types of indices: vector indices $\mu, \lambda, \ldots = 0,1,2$ labeling a three-dimensional Minkowski space, the (Majorana) spinorial indices $\alpha, \beta, \ldots = 1,2$ labeling the associated real 2-component spinors and the scalar indices $A, B, \ldots = 1,2$ labeling an internal space.

If we set $\mathcal{Q}^1_\alpha := \mathcal{Q}_\alpha, \quad \mathcal{Q}^2_\alpha := iK\mathcal{Q}_\alpha$. then

$$\{\mathcal{Q}^A_\alpha, \mathcal{Q}^B_\beta\} \;=\; \delta^{AB}(C\gamma_\mu)_{\alpha\beta}J^\mu + \epsilon^{AB}C_{\alpha\beta}R$$

is a generalized supersymmetry.

The closure of the $osp(2|2)$ superalgebra is guaranteed by the non-vanishing commutators

$$\begin{aligned} [J_\mu, J_\nu] &= 4i\epsilon_{\mu\nu\lambda}J^\lambda, \\ [J_\mu, \mathcal{Q}^A_\alpha] &= 2i(\gamma_\mu)_\alpha{}^\beta \mathcal{Q}^A_\beta, \\ [R, \mathcal{Q}^A_\alpha] &= -2iS^A{}_B\mathcal{Q}^B_\alpha. \end{aligned}$$

In the last equation the matrix $S^A{}_B$ is given by $S = e_{12} - e_{21}$.

Topological conformal mechanics

Part III of the talk:

L. Baulieu & F.T., Nucl. Phys. B 2017

Witten's type Topological Quantum Field Theory (1988)

g_{rs} background metric independence.

Stress-energy tensor $T_{rs} = \frac{\delta S}{\delta g_{rs}}$ independent of local metric fluctuations.

Concrete implementation: Witten's TFT.

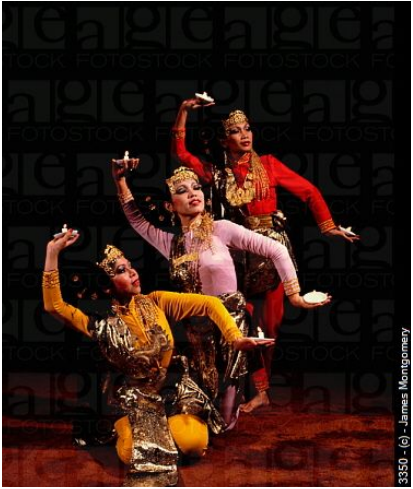
$$\begin{aligned} T_{rs} &= \delta G_{rs}, \\ \delta^2 &= 0 \end{aligned}$$

δ is a nilpotent operator.

Physical condition (cohomology):

$$\delta|\Psi>= 0, \qquad |\Psi>\neq \delta|something> .$$

Spinorial properties encoded in classical system
Example: Balinese Candle Dance



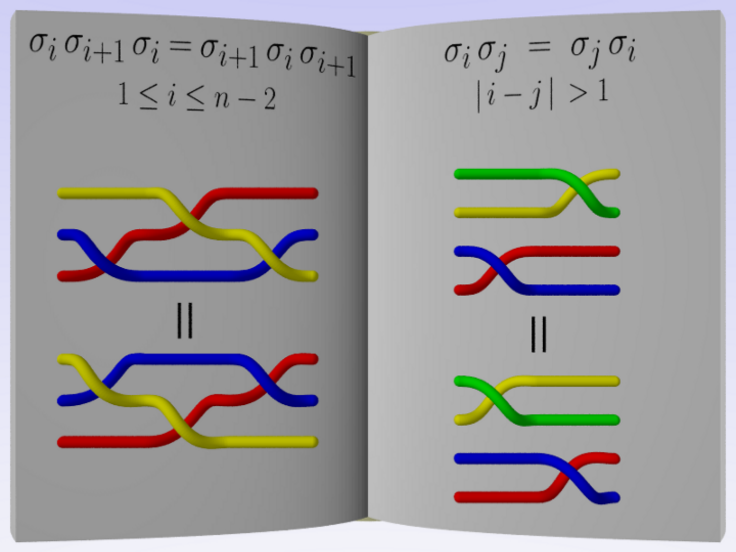
(the Balinese dancer is a classical, not a quantum-mechanical system!)

Kitaev:

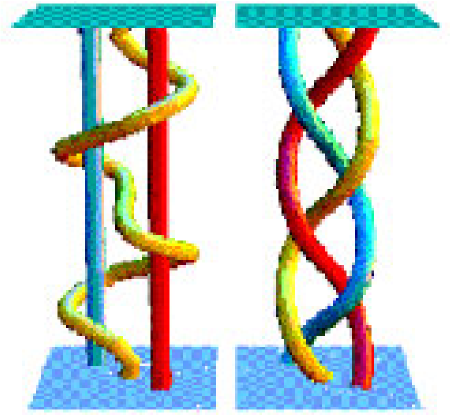
emergent Majorana fermions embodied in Topological Quantum Computers

(offering topological protection from decoherence)

Braids

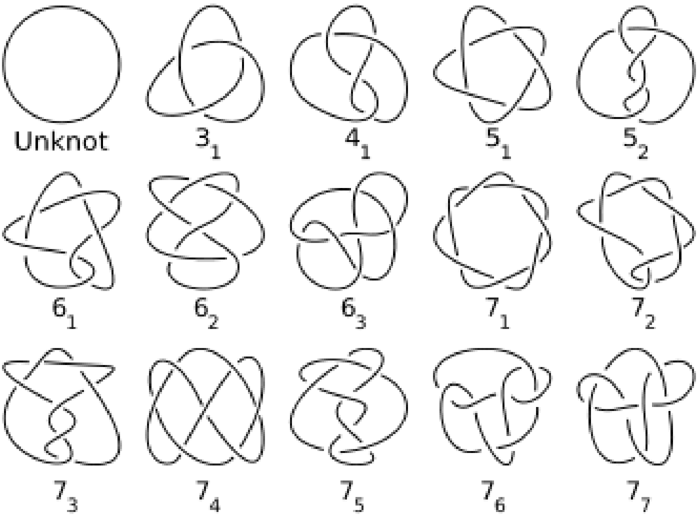


$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$
$1 \leq i \leq n-2$
$\sigma_i \sigma_j = \sigma_j \sigma_i$
$|i-j| > 1$



and

Knots:



Connection with Clifford algebras

$$\begin{aligned} \gamma_i\gamma_j + \gamma_j\gamma_i &= 2\eta_{ij}, \\ \eta_{ij} &= diag(+\ldots, -\ldots). \end{aligned}$$

$Cl(p,q)$**:** p $+$**'s,** q $-$**'s.**

"almost" Pauli matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

They are $Cl(2,1)$ split-quaternions

Braid reps from Majorana fermions (Clifford algebras):

From

$$\begin{aligned} b_k b_{k+1} b_k &= b_{k+1} b_k b_{k+1}, \\ b_i b_j &= b_j b_i, \quad |i-j| > 1, \end{aligned}$$

and

$$\begin{aligned} \gamma_i^\dagger &= \gamma_i, \\ \{\gamma_i, \gamma_j\} &= 2\delta_{ij}, \end{aligned}$$

then $b_k = \frac{1}{\sqrt{2}}(1 + \gamma_{k+1}\gamma_k)$, **with** $b_k^8 = \mathbb{I}$.

Ordinary supersymmetry $\Rightarrow$ supersymmetric σ-models

Pseudo-supersymmetry $\Rightarrow$ topological σ-models

Pseudo-supersymmetry:

$$\{Q_i, Q_j\} \;=\; 2\eta_{ij}H.$$

From $\gamma_1^2 = -\gamma_2^2 = \mathbb{I}$**,** $\gamma_1\gamma_2 + \gamma_2\gamma_1 = 0$
one construct the nilpotent operators $\gamma_\pm$**:**

$$\gamma_\pm = \frac{1}{2}(\gamma_1 \pm \gamma_2), \qquad \gamma_\pm^2 = 0.$$

Question

Which topological mechanics should be used?

1) ordinary supersymmetry applied to the vacuum ($\Rightarrow 2^{nd}$-order differential equations):

$$Q^2 = H, \qquad H|vac>= 0,$$

2) or cohomological mechanics ($\Rightarrow 1^{st}$-order differential equations):

$$Q^2 = 0.$$

Is it supersymmetry realized in Nature (maybe at LHC)?

Exact supersymmetry cannot be realized

1) From observation: we do not see same mass superparticles
$\Rightarrow$ supersymmetry must be (spontaneously) broken (which scale?).

2) From anthropic considerations: exact supersymmetry cannot be observed: Ordinary chemistry is not supported.
$\Rightarrow$ in a supersymmetric world we are simply not there.

Intriguing possibility:

maybe some supersymmetry is already there and we are observing it without realizing that we are seeing it.

“Beauty is in the eye of the beholder”

from Molly Bawn (1878) by Margaret Wolfe Hungerford née Hamilton:

Confront the complementarity principle:

is light a wave or a particle?

It depends from the observer!

Is supersymmetry in the eye of the beholder?

Keyword: fermionization (rather than bosonization)

Coleman's observation:

sine-Gordon model and massive Thirring models are equivalent.

$$\underset{(Bosonization)}{Fermionic \quad System \quad A \quad \Rightarrow \quad Bosonic \quad System \quad B},$$

$$\underset{(Fermionization)}{Fermionic \quad System \quad A \quad \Leftarrow \quad Bosonic \quad System \quad B},$$

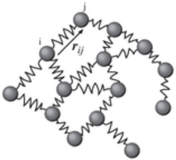
$$A \;\Leftrightarrow\; B$$

(Bosonization/Fermionization works fine in $d = 1, 2$ dimensions.)

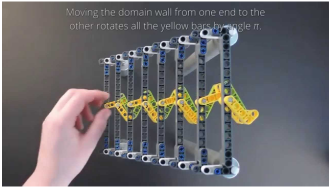
If supersymmetry is not found (yet) at LHC, could be found in some other place?

A trip to LEGOLAND:

2012 Kane-Lubensky:
modeling topological insulators through mechanical models
(e.g.: $1D$ polyacetylene with springs)



2014 Vitelli:
no-energy cost zero-modes $\Rightarrow$ rods can replace strings
metamaterials (lego)



Lagrangians for Topological Conformal Mechanics

(from Baulieu-Holanda-F.T., JMP 2015)

$(2,2,0)$ supermultiplet:

$$\mathcal{L} = \dot{y}\dot{\bar{y}} + \dot{\chi}\bar{\chi} + \frac{\epsilon}{4}y\bar{y} + (1+2\lambda)\frac{\chi\bar{\chi}\dot{\bar{y}}}{\bar{y}}.$$

$(1,2,1)$ supermultiplet:

$$\mathcal{L} = a^2 + a\dot{y} + \dot{\chi}\bar{\chi} + \frac{\epsilon}{2}ya + \frac{\epsilon}{2}(1+2\lambda)\bar{\chi}\chi + 2(1+2\lambda)\frac{\bar{\chi}\chi a}{y}.$$

- either $\epsilon = 0$ (parabolic) or $\epsilon = 1$ (hyperbolic) case.
- $2\lambda + 1$ is a coupling constant.
- The actions are invariant under $sl(2|1)$.

Thanks a lot for the attention!