Quantum deformations of D=3 Lorentz symmetries

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Contents

The search for quantum gravity is linked with studies of noncommutative space-times and quantum deformations of space-time symmetries. The considerations of simple dynamical models in quantized gravitational background indicate that the presence of quantum gravity effects generates noncommutativity of $D = 4$ space-time coordinates, and as well the Lie-algebraic space-time symmetries (e.g. Euclidean, Lorentz, Kleinian, quaternionic and their ingomogeneous versions) are modified into respective quantum symmetries, described by noncocommutative Hopf algebras, named quantum deformations. Therefore, studing all aspects of the quantum deformations in details is an important issue in the search of quantum gravity models.

For classifications, constructions and applications of quantum Hopf deformations of an universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$, Lie bialgebras $(\mathfrak{g}, \delta)$ play an essential role. Here the *cobracket* δ is a linear skew-symmetric map $\mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ with the relations consisted with the Lie bracket in $\mathfrak{g}$:

$$\begin{aligned} &\delta([x,y]) = [\delta(x),\Delta_0(y)] + [\Delta_0(x),\delta(y)], \\ &(\delta\otimes\mathrm{id})\delta(x) + \mathrm{cycle} = 0, \end{aligned} \tag{1}$$

where $\Delta_0(\cdot)$ is a trivial (non-deformed) coproduct

$$\Delta_0(x) = x\otimes 1 + 1\otimes x, \tag{2}$$

for any $x, y \in \mathfrak{g}$. The first relation in (1) is a condition of the 1-cocycle and the second one is the co-Jacobi identity. The Lie bialgebra $(\mathfrak{g},\delta)$ is a correct infinitesimalization of the quantum Hopf deformation of $U(\mathfrak{g})$ and the operation δ is an infinitesimal part of difference between a coproduct Δ and an oposite coproduct $\tilde{\Delta}$ in the Hopf algebra, $\delta(x) = h^{-1}(\Delta - \tilde{\Delta}) \mod h$, where h is a deformation parameter. Any two Lie bialgebras $(\mathfrak{g},\delta)$ and $(\mathfrak{g},\delta')$ are isomorphic (equivalent) if they are connected by a $\mathfrak{g}$-automorphism φ satisfying the condition

$$\delta(x) = (\varphi\otimes\varphi)\delta'(\varphi^{-1}(x)) \tag{3}$$

for any $x \in \mathfrak{g}$.

Of our special interest here are the quasitriangle Lie bialgebras $(\mathfrak{g}, \delta_{(r)}) := (\mathfrak{g}, \delta, r)$, where the cobracket $\delta_{(r)}$ is given by the classical r-matrix $r \in \mathfrak{g} \wedge \mathfrak{g}$ as follows:

$$\delta_{(r)}(x) = [r, \Delta_0(x)]. \tag{4}$$

It is easy to see from (2) and (3) that *two quasitriangular Lie bialgebras $(\mathfrak{g}, \delta_{(r)})$ and $(\mathfrak{g}, \delta_{(r')})$ are isomorphic iff the classical r-matrices r and r' are isomorphic, i.e. $(\varphi \otimes \varphi)r' = r$.*

Therefore for a classification of all nonequivalent quasitriangular Lie bialgebras $(\mathfrak{g}, \delta_{(r)})$ of the given Lie algebra $\mathfrak{g}$ we need to find all nonequivalent (nonisomorphic) classical r-matrices. Because nonequivalent quasitriangular Lie bialgebras uniquely determine non-equivalent quasitriangular quantum deformations (Hopf algebras) of $U(\mathfrak{g})$ therefore the classification of all nonequivalent quasitriangular Hopf algebras is reduced to the classification of all nonequivalent classical r-matrices.

Let $\mathfrak{g}^* := (\mathfrak{g}, *)$ be a real form of a semisimple complex Lie algebra $\mathfrak{g}$, where $*$ is an antilinear involutive antiautomorphism of $\mathfrak{g}$, then *the bialgebra* $(\mathfrak{g}^*, \delta_{(r)})$ *is real iff the classical r-matrix r is $*$-anti-real ($*$-anti-Hermitian).*[1] Indeed, the condition of $*$-reality for the bialgebra $(\mathfrak{g}^*, \delta)$ means that

$$\delta(x)^{*\otimes *} = \delta(x^*). \tag{5}$$

Applying this condition to the relations (4) we abtain that

$$r^{*\otimes *} = -r, \tag{6}$$

i.e. the r-matrix r is $*$-anti-Hermitian.

[1] All bialgebras over the semisimple complex and real Lie algebras are quasitriangular, due to Whitehead lemma (see e.g. N. Jacobson, *"Lie algebras", Dover Publications, Inc., New York* (1979)).

In this paper we investigate the quantum deformations of $D = 3$ Lorentz symmetry. Firstly we obtain the complete classifications of the nonequivalent (nonisomorphic) classical r-matrices for complex Lie algebra $\mathfrak{sl}(2;\mathbb{C})$ and its real forms $\mathfrak{su}(2)$, $\mathfrak{su}(1,1)$ and $\mathfrak{sl}(2;\mathbb{R})$ with the help of explicite formulas for the automorphisms of these Lie algebras in terms of the Cartan-Weyl bases. In the case of $\mathfrak{sl}(2;\mathbb{C})$ there are two noniquivalent classical r-matrices - standard and Jordanian ones. For $\mathfrak{su}(2)$ algebra there is only the standard nonequivalent r-matrix. These results are well known. For the $\mathfrak{su}(1,1)$ case we obtained three noneqvivalent r-matrices - standard, quasi-standard and quasi-Jordanian ones. In the case of $\mathfrak{sl}(2;\mathbb{R})$ we find also three noneqvivalent r-matrices - standard, quasi-standard and Jordanian ones. Then using isomorphisms $\mathfrak{o}(2,1) \simeq \mathfrak{su}(1,1) \simeq \mathfrak{sl}(2;\mathbb{R})$ we express these r-matrices in terms of the Cartesian basis of the $D = 3$ Lorentz algebra $\mathfrak{o}(2,1)$ and we see that two systems with three r-matrices for $\mathfrak{su}(1,1)$ and $\mathfrak{sl}(2;\mathbb{R})$ algebras coincides. Thus we obtain that the isomorphic Lie algebras $\mathfrak{su}(1,1)$ and $\mathfrak{sl}(2;\mathbb{R})$ have the isomorphic systems of their quasitriangle Lie bialgebras. In the case of $\mathfrak{o}(2,1)$ we obtain that the $D = 3$ Lorentz algebra has two standard q-deformations and one Jordanian. These Hopf deformations are presented in explicite form in terms of the quantum Cartan-Weyl generators for the quantized universal enveloping algebras of $\mathfrak{su}(1,1)$ and $\mathfrak{sl}(2;\mathbb{R})$ and also in the terms of the quantum Cartesian generators.

It should be noted that the full list of the noniquivalent classical r-matrices for $\mathfrak{sl}(2;\mathbb{R})$ and $\mathfrak{o}(2,1)$ Lie algebras has been obtained early by different methods (A.G. Reyman, 1996; X. Gomez, 2000) , however the complete list of the nonequivalent Hopf quantizations for these Lie algebras has not been presented in the literature. Furthermore, there was put forward an incorrect hypothesis that the isomorphic Lie algebra $\mathfrak{su}(1,1)$ and $\mathfrak{sl}(2;\mathbb{R})$ do not have any isomorphic quasitriangular Lie bialgebras (see R. Borcherds, M. Haiman, N. Reshetikhin, V. Serganova, *"Berkeley Lecture on Lie Groups and Quantum Groups", ed. Anton Geraschenko and Theo Johnson-Freyd.* http://math.berkeley.edu/ theojf/LieQuantumGroups.pdf; see N. Reshetikhin, part II *Quantum groups,* Sect. 9.4.4.).
The isomorphic Lie algebras $\mathfrak{o}(2,1)$, $\mathfrak{sl}(2;\mathbb{R})$, $\mathfrak{su}(1,1)$ and their quantum deformations play very important role in physics as well as in mathematical considerations, so the structure of these deformations should be understood with full clarity. It should be noted also that the importance of $\mathfrak{o}(2,1)$ and its deformations follows also from the unique role of the $\mathfrak{o}(2,1)$ algebra as the lowest-dimensional rank one noncompact simple Lie algebra, endowed only with unitary infinite-dimensional representations.
Moreover using the obtained results we can describe all quantizations for all bialgebras of the complex $D = 4$ Euclidean algebra $\mathfrak{o}(4;\mathbb{C})$ and its real forms: Euclidian $\mathfrak{o}(4)$, quartenionic $\mathfrak{o}^{\star}(4)$, Kleinian $\mathfrak{o}(2,2)$ symmetries because of the the direct sums:

$$\mathfrak{o}(4;\mathbb{C}) = \mathfrak{o}(3;\mathbb{C}) \oplus \bar{\mathfrak{o}}(3;\mathbb{C}), \tag{7}$$
$$\mathfrak{o}(4) = \mathfrak{o}(3) \oplus \bar{\mathfrak{o}}(3), \tag{8}$$
$$\mathfrak{o}^{\star}(4) = \mathfrak{o}(3) \oplus \bar{\mathfrak{o}}(2,1). \tag{9}$$
$$\mathfrak{o}(2,2) = \mathfrak{o}(2,1) \oplus \bar{\mathfrak{o}}(2,1). \tag{10}$$

We first remind different most popular bases of the complex $D = 3$ Euclidean Lie algebra $\mathfrak{o}(3;\mathbb{C})$: *metric, Cartesian* and *Cartan-Weyl* bases.
The *metric* basis contains in its commutation relations an explicite metric, namely, the complex $D = 3$ Euclidean Lie algebra $\mathfrak{o}(3;\mathbb{C})$ is generated by three Euclidean basis elements $L_{ij} = -L_{ji} \in \mathfrak{o}(3;\mathbb{C})$ $(i,j = 1,2,3)$ satisfying the relations

$$[L_{ij},\, L_{kl}] \;=\; g_{jk}\, L_{il} - g_{jl}\, L_{ik} + g_{il}\, L_{jk} - g_{ik}\, L_{jl}\,, \tag{11}$$

where g_{ij} is the Euclidean metric: $g_{ij} = \mathrm{diag}\,(1,1,1)$. The Euclidean algebra $\mathfrak{o}(3;\mathbb{C})$, as a linear space, is a linear envelope of the basis $\{L_{ij}\}$ over $\mathbb{C}$.
The *Cartesian* (or *physical*) basis of $\mathfrak{o}(3;\mathbb{C})$ is related with the generators L_{ij} as follows

$$I_i \;:=\; -\frac{1}{2}\varepsilon_{ijk}L_{jk} \quad (i,j,k = 1,2,3). \tag{12}$$

From (11) and (12) we get

$$[I_i,\, I_j] \;=\; \varepsilon_{ijk}I_k. \tag{13}$$

If we consider a Lie algebra over $\mathbb{R}$ with the commutation relations (13) then we get the compact real form $\mathfrak{o}(3) := \mathfrak{o}(3;\mathbb{R})$ with the anti-Hermitian basis

$$I_i'^{*} \;=\; -I_i' \quad (i = 1,2,3) \quad \text{for} \;\; \mathfrak{o}(3). \tag{14}$$

The real form $\mathfrak{o}(2,1)$ is given by the formulas:

$$I_i''^{\dagger} \;=\; (-1)^{i-1}I_i'' \quad (i = 1,2,3) \quad \text{for} \;\; \mathfrak{o}(2,1). \tag{15}$$

The Cartesian generators for $\mathfrak{o}(3)$ and $\mathfrak{o}(2,1)$ we denote by I_i' and I_i'' correspondingly and they satisfy the same defining relations (12).

For the description of quantum deformations and in particular for the classification of classical r-matrices of the complex Euclidean algebra $\mathfrak{o}(3;\mathbb{C})$ and its real forms $\mathfrak{o}(3)$ and $\mathfrak{o}(2,1)$ it is convenient to use the *Cartan–Weyl* (CW) bases of the isomorphic complex Lie algebra $\mathfrak{sl}(2;\mathbb{C})$ and its real forms $\mathfrak{su}(2)$, $\mathfrak{sl}(1,1)$ and $\mathfrak{sl}(2,\mathbb{R})$. In the case of $\mathfrak{o}(3)$ the $\mathfrak{su}(2)$ Cartan–Weyl basis can be chosen as follows

$$\begin{array}{llll} H' &:= \imath I'_3, & E'_{\pm} &:= \imath I'_1 \mp I'_2, \\ [H', E'_{\pm}] &= \pm E'_{\pm}, & [E'_{+}, E'_{-}] &= 2H', \\ H'^{*} &= H', & E'^{*}_{\pm} &= E'_{\mp}, \end{array} \qquad (16)$$

where the conjugation $(^{*})$ is the same as for the $\mathfrak{so}(3)$ Cartesian basis $I'^{*}_i = -I'_i$ $(i = 1, 2, 3)$.

For the real form $\mathfrak{o}(2,1)$ we will use two CW bases of $\mathfrak{sl}(2;\mathbb{C})$ real forms: $\mathfrak{sl}(1,1)$ and $\mathfrak{sl}(2,\mathbb{R})$. Such bases are given by

$$\begin{array}{llll} H'' &:= \imath I''_2, & E''_{\pm} &:= \imath I''_1 \pm I''_3, \\ [H'', E''_{\pm}] &= \pm E_{\pm}, & [E''_{+}, E''_{-}] &= 2H'' \end{array} \quad \text{for } \ \mathfrak{su}(1,1), \qquad (17)$$

$$\begin{array}{llll} H''' &:= \imath I''_3, & E'''_{\pm} &:= \imath I''_1 \mp I''_2, \\ [H''', E'''_{\pm}] &= \pm E'''_{\pm}, & [E'''_{+}, E'''_{-}] &= 2H''' \end{array} \quad \text{for } \ \mathfrak{sl}(2,\mathbb{R}). \qquad (18)$$

Both bases $\{E''_{\pm}, H''\}$ and $\{E'''_{\pm}, H'''\}$ have the same commutation relations but they have different reality properties, namely

$$H''^{\dagger} = H'', \qquad E''^{\dagger}_{\pm} = -E''_{\mp} \quad \text{for } \mathfrak{su}(1,1), \tag{19}$$

$$H'''^{\dagger} = -H''', \qquad E'''^{\dagger}_{\pm} = -E'''_{\pm} \quad \text{for } \mathfrak{sl}(2;\mathbb{R}), \tag{20}$$

where the conjugation (†) is the same as for the $\mathfrak{so}(2,1)$ Cartesian basis $I''^{\dagger}_i = (-1)^{i-1} I''_i$ $(i = 1, 2, 3)$.
It should be noted that in the case of $\mathfrak{su}(1,1)$ the Cartan generator H'' is compact while for the case $\mathfrak{su}(2,\mathbb{R})$ the generator H''' is noncompact.
The relations between the $\mathfrak{su}(1,1)$ and $\mathfrak{su}(2,\mathbb{R})$ bases look as follows

$$\begin{aligned} H'' &= -\frac{\imath}{2}(E'''_{+} - E'''_{-}), \\ E''_{\pm} &= \mp\imath H''' + \frac{1}{2}(E'''_{+} + E'''_{-}). \end{aligned} \tag{21}$$

In this section we obtain complete classification bialgebras (classical r-matrices) for the complex $D=3$ Euclidean Lie algebra $\mathfrak{o}(3;\mathbb{C})$ and its real forms $\mathfrak{o}(3)$ and $\mathfrak{o}(2,1)$ using the isomorphisms: $\mathfrak{o}(3;\mathbb{C})\simeq\mathfrak{sl}(2;\mathbb{C})$, $\mathfrak{o}(2)\simeq\mathfrak{su}(2)$, $\mathfrak{o}(2,1)\simeq\mathfrak{su}(1,1)\simeq\mathfrak{sl}(2;\mathbb{C})$. In particular, we explicitly find out an isomorphism between $\mathfrak{su}(1,1)$ and $\mathfrak{sl}(2;\mathbb{R})$ bialgebras and fix on the basis $\mathfrak{o}(2,1)$ bialgebras in such forms which are convenient for quantizations.

By the definition any classical r-matrix of arbitrary complex or real Lie algebra $\mathfrak{g}$, $r\in\mathfrak{g}\wedge\mathfrak{g}$, satisfy the classical Yang-Baxter equation (CYBE):

$$[[r,\,r]] \;=\; \tilde{\Omega}\,. \tag{22}$$

Here $[[\cdot,\cdot]]$ is the Schouten bracket which for any monomial skew-symmetric two-tensors $r_1=x\wedge y$ and $r_2=u\wedge v$ $(x,y,u,v\in\mathfrak{g})$ is given by

$$\begin{aligned}[][[x\wedge y,\,u\wedge v]] \;&:=\; x\wedge\big([y,u]\wedge v+u\wedge[y,v]\big)\\ &\quad\; -y\wedge\big([x,u]\wedge v+u\wedge[x,v]\big)\\ &=\; [[u\wedge v,\,x\wedge y]]\end{aligned} \tag{23}$$

and $\tilde{\Omega}$ is the $\mathfrak{g}$-invariant element, $\tilde{\Omega}\in(\overset{3}{\wedge}\,\mathfrak{g})_{\mathfrak{g}}$, that in the case of $\mathfrak{g}:=\mathfrak{sl}(2;\mathbb{C})$ looks as follows:

$$\tilde{\Omega} \;=\; \gamma\,\Omega(\mathfrak{sl}(2;\mathbb{C})) \;=\; \gamma\,E_+\wedge H\wedge E_-\,, \tag{24}$$

where $\gamma\in\mathbb{C}$.

First we will shown that *any two-tensor of* $\mathfrak{sl}(2;\mathbb{C})\wedge\mathfrak{sl}(2;\mathbb{C})$ *is a classical* $\mathfrak{sl}(2;\mathbb{C})$ *r-matrix*. Indeed, let

$$r \;:=\; \beta_+ r_+ + \beta_0 r_0 + \beta_- r_- \quad (\beta_+, \beta_0, \beta_- \in \mathbb{C}) \tag{25}$$

be arbitrary element of $\mathfrak{sl}(2;\mathbb{C})\wedge\mathfrak{sl}(2;\mathbb{C})$, where

$$r_+ \;:=\; E_+\wedge H, \quad r_0 \;:=\; E_+\wedge E_-, \quad r_- \;:=\; H\wedge E_- \tag{26}$$

are basis elements of $\mathfrak{sl}(2;\mathbb{C})\wedge\mathfrak{sl}(2;\mathbb{C})$. Because all basis elemets (26) are classical r-matrices, and, moreover, the Schouten brackets of the elemets $r_\pm$ with r_0 are also equal to zero, $[[r_\pm, r_0]] = 0$, we have

$$\begin{aligned}[][[r,r]] &= 2\beta_+\beta_-[[r_+,r_-]] + \beta_0^2[[r_0,r_0]] \\ &= -4(\beta_0^2+\beta_+\beta_-)E_+\wedge H\wedge E_- \;\equiv\; \gamma\Omega.\end{aligned} \tag{27}$$

Thus arbitrary element (25) is a classical r-matrix, and if its coefficients satisfy the condition $\gamma := \beta_0^2 + \beta_+\beta_- = 0$ then it satisfies the homogeneous CYBE, and if $\gamma := \beta_0^2 + \beta_+\beta_- \neq 0$ then it satisfies the non-homogeneous CYBE. We shall call the parameter γ the γ-characteristic of the classical r-matrix (25). It is evident that the γ-characteristic of the classical r-matrix r is invariant under the $\mathfrak{sl}(2;\mathbb{C})$-automorphisms, i.e. any two r-matrices r and r', which are connected by a $\mathfrak{sl}(2;\mathbb{C})$-automorphism, have the same γ-characteristic, $\gamma = \gamma'$. We can show also that any two $\mathfrak{sl}(2;\mathbb{C})$ r-matrices r and r' with the same γ-characteristic can be connected by a $\mathfrak{sl}(2;\mathbb{C})$-automorphism.

There are two types of explicit $\mathfrak{sl}(2;\mathbb{C})$-automorphisms. First type connecting the classical r-matrices with zero γ-characteristic is given by the formulas :

$$
\begin{aligned}
\varphi_0(E_+) &= \chi(\tilde{\beta}_+ E_+ - 2\tilde{\beta}_0 H + \tilde{\beta}_- E_-),\\
\varphi_0(E_-) &= \chi^{-1}(\tilde{\beta}_- E_+ - 2\kappa\tilde{\beta}_0 H + \tilde{\beta}_+ E_-),\\
\varphi_0(H) &= \tilde{\beta}_0 E_+ + (\kappa\tilde{\beta}_+ + \tilde{\beta}_-) H + \kappa\tilde{\beta}_0 E_-,
\end{aligned}
\tag{28}
$$

where χ is a non-zero rescaling parameter (including $\chi=1$), κ takes two values $+1$ or -1, and the parameters $\tilde{\beta}_i$ $(i=+,0,-)$ satisfy the conditions:

$$
\gamma \;:=\; \tilde{\beta}_0^2 + \tilde{\beta}_+\tilde{\beta}_- \;=\; 0, \qquad \kappa\tilde{\beta}_+ - \tilde{\beta}_- = 1. \tag{29}
$$

Let us consider two general r-matrices with zero γ-characteristics:

$$
\begin{aligned}
r &:= \beta_+ E_+ \wedge H + \beta_0 E_+ \wedge E_- + \beta_- H \wedge E_-,\\
r' &:= \beta'_+ E_+ \wedge H + \beta'_0 E_+ \wedge E_- + \beta'_- H \wedge E_-,
\end{aligned}
\tag{30}
$$

where $\gamma = \beta_0^2 + \beta_+\beta_- = 0$ and $\gamma = \beta_0'^2 + \beta'_+\beta'_- = 0$. Moreover, we suppose that the parameters $\beta_\pm$ and $\beta'_\pm$ satisfy the additional relations:

$$
\kappa\beta_+ - \beta_- = \chi\beta'_+ - \chi^{-1}\kappa\beta'_- \neq 0, \tag{31}
$$

where the parameters κ and χ are the same as in (28).

One can check that the following formula is valid:

$$r \;=\; (\varphi_0\otimes\varphi_0)r', \tag{32}$$

i.e.

$$\begin{aligned}&\beta_+E_+\wedge H+\beta_0E_+\wedge E_-+\beta_-H\wedge E_-\\&=\beta'_+\varphi_0(E_+)\wedge\varphi_0(H)+\beta'_0\varphi_0(E_+)\wedge\varphi_0(E_-)+\beta'_-\varphi_0(H)\wedge\varphi_0(E_-),\end{aligned} \tag{33}$$

where φ_0 is the $\mathfrak{sl}(2;\mathbb{C})$-automorphism (28) with the following parameters:

$$\begin{aligned}\tilde\beta_0 &= \frac{\beta_0(\chi\beta'_+ + \chi^{-1}\kappa\beta'_-)-\beta'_0(\kappa\beta_+ + \beta_-)}{(\kappa\beta_+ - \beta_-)(\chi\beta'_+ - \chi^{-1}\kappa\beta'_-)}\;,\\ \tilde\beta_+ &= \frac{\kappa(\kappa\beta_+ + \beta_-)(\chi\beta'_+ + \chi^{-1}\kappa\beta'_-)+4\beta_0\beta'_0}{2(\kappa\beta_+ - \beta_-)(\chi\beta'_+ - \chi^{-1}\kappa\beta'_-)}+\frac{\kappa}{2}\;,\\ \tilde\beta_- &= \frac{(\kappa\beta_+ + \beta_-)(\chi\beta'_+ + \chi^{-1}\kappa\beta'_-)+4\kappa\beta_0\beta'_0}{2(\kappa\beta_+ - \beta_-)(\chi\beta'_+ - \chi^{-1}\kappa\beta'_-)}-\frac{1}{2}\;.\end{aligned} \tag{34}$$

It is easy to check that as expected the formulas (34) satisfy the conditions (29). Let us assume in (33) and (34) that the parameters β'_0 and β'_- are equal to zero. Then the general classical r-matrix r, satisfying the homogeneous CYBE, is reduced to usual Jordanian form by the authomorphism φ_0 with the parameters:

$$\tilde\beta_0 = \frac{\beta_0}{\kappa\beta_+ - \beta_-}\;,\quad \tilde\beta_\pm = \frac{\beta_\pm}{\kappa\beta_+ - \beta_-}\;. \tag{35}$$

Second type of $\mathfrak{sl}(2;\mathbb{C})$-automorphism connecting the classical r-matrices with non-zero γ-characteristic is given as follows

$$\begin{aligned}
\varphi_1(E_+) &= \frac{\chi}{2}\Big((\tilde{\beta}_0+1)\,E_+ + 2\tilde{\beta}_- H - \frac{\tilde{\beta}_-^2}{\tilde{\beta}_0+1}\,E_-\Big), \\
\varphi_1(E_-) &= \frac{\chi^{-1}}{2}\Big(\frac{-\tilde{\beta}_+^2}{\tilde{\beta}_0+1}\,E_+ + 2\tilde{\beta}_+ H + (\tilde{\beta}_0+1)E_-\Big), \qquad (36)\\
\varphi_1(H) &= \frac{1}{2}(-\tilde{\beta}_+ E_+ + 2\tilde{\beta}_0 H - \tilde{\beta}_- E_-),
\end{aligned}$$

where χ is a non-zero rescaling parameter, and $\tilde{\beta}_0^2 + \tilde{\beta}_+\tilde{\beta}_- = 1$.
Let us consider two general r-matrices with non-zero γ-characteristics:

$$\begin{aligned}
r &:= \beta_+ E_+ \wedge H + \beta_0 E_+ \wedge E_- + \beta_- H \wedge E_-, \qquad (37)\\
r' &:= \beta'_+ E_+ \wedge H + \beta'_0 E_+ \wedge E_- + \beta'_- H \wedge E_-,
\end{aligned}$$

where the parameters $\beta_\pm$, β_0 and $\beta'_\pm$, β'_0 can be equal to zero provided that $\gamma = \beta_0^2 + \beta_+\beta_- = \gamma' = (\beta'_0)^2 + \beta'_+\beta'_- \neq 0$, i.e. both r-matrices r and r' have the same non-zero γ-characteristic $\gamma = \gamma' \neq 0$.

One can check the following relation:

$$r \;=\; (\varphi_1\otimes\varphi_1)r', \tag{38}$$

i.e.

$$\begin{aligned}&\beta_+E_+\wedge H+\beta_0E_+\wedge E_- +\beta_-H\wedge E_-\\ &=\beta'_+\varphi_1(E_+)\wedge\varphi_1(H)+\beta'_0\varphi_1(E_+)\wedge\varphi_0(E_-)+\beta'_-\varphi_1(H)\wedge\varphi_1(E_-),\end{aligned} \tag{39}$$

where φ_1 is the $\mathfrak{sl}(2;\mathbb{C})$-automorphism (36) with the parameters:

$$\begin{aligned}\tilde{\beta}_0 &= \frac{(\beta_0+\beta'_0)^2-(\beta_+-\chi\beta'_+)(\beta_- -\chi^{-1}\beta'_-)}{(\beta_0+\beta'_0)^2+(\beta_+-\chi\beta'_+)(\beta_- -\chi^{-1}\beta'_-)}\,,\\ \tilde{\beta}_\pm &= \frac{2(\beta_0+\beta'_0)(\beta_\pm-\chi^{\pm1}\beta'_\pm)}{(\beta_0+\beta'_0)^2+(\beta_+-\chi\beta'_+)(\beta_- -\chi^{-1}\beta'_-)}\,.\end{aligned} \tag{40}$$

It is easy to check that the formulas (40) satisfy the condition $\tilde{\beta}_0^2+\tilde{\beta}_+\tilde{\beta}_-=1$. If we assume in (39) and (40) that the parameters $\beta'_\pm$ are equal to zero then the general classical r-matrix r, satisfying the non-homogeneous CYBE, is reduced to the usual standard form by the automorphism (36) with the following parameters:

$$\tilde{\beta}_0 = \frac{\beta_0}{\beta'_0}\,,\quad \tilde{\beta}_\pm = \frac{\beta_\pm}{\beta'_0}\,. \tag{41}$$

Finally for $\mathfrak{sl}(2,\mathbb{C})$ we obtain the well-known result:
For the complex Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ there exists up to $\mathfrak{sl}(2,\mathbb{C})$ automorphisms two solutions of CYBE, namely Jordanian r_J and standard r_{st}:

$$r_J \;=\; \beta E_+\wedge H,\quad [[r_J,r_J]] \;=\; 0, \tag{42}$$

$$r_{st} \;=\; \beta' E_+\wedge E_-,\quad [[r_{st},r_{st}]] \;=\; \beta'^2\Omega, \tag{43}$$

where the complex parameter β in (42) can be removed by the rescaling automorphism: $\varphi(E_+)=\beta^{-1}E_+$, $\varphi(E_-)=\beta E_-$, $\varphi(H)=H$; in (43) the parameter $\beta'=e^{\imath\phi}|\beta'|$ for $|\phi|\leq\frac{\pi}{2}$ is effective.
The general non-reduced expression of the r-matrix r is convenient for the application of reality conditions:

$$r^{\circledast} \;:=\; \beta_+^{*}E_+^{\circledast}\wedge H^{\circledast}+\beta_0^{*}E_+^{\circledast}\wedge E_-^{\circledast}+\beta_-^{*}H^{\circledast}\wedge E_-^{\circledast} \;=\; -r, \tag{44}$$

where $\circledast$ is the conjugation associated with corresponding real form ($\circledast=*,\dagger$), and β_i^{*} ($i=+,0,-$) means the complex conjugation of the number β_i. It should be noted that for any classical r-matrix r, $r^{\circledast}$ is again a classical r-matrix. Moreover, if r-matrix is *anti-real* (anti-Hermitian), i.e. it satisfies the condition (44), then its γ-characteristic is real. Indeed, applying the conjugation $\circledast$ to CYBE we have for the left-side: $[[r,r]]^{\circledast}=-[[r^{\circledast},r^{\circledast}]]=-[[r,r]]$ and for the right-side: $(\gamma\Omega)^{\circledast}=-\gamma^{*}\Omega$ for all real forms $\mathfrak{su}(2)$, $\mathfrak{su}(1,1)$, $\mathfrak{su}(2;\mathbb{R})$. It follows that the parameter γ is real, $\gamma^{*}=\gamma$.

I. *The compact real form* $\mathfrak{su}(2)$ $(H'^{*} = H',\ E'^{*}_{\pm} = E'_{\mp})$.
In this case it follows from (44) that

$$\beta_0^{*} \;=\; \beta_0, \quad \beta_{\pm}^{*} \;=\; \beta_{\mp}. \tag{45}$$

If in (25) $\gamma = \beta_0^2 + \beta_+\beta_- = 0$ then $\beta_0\beta_0^{*} + \beta_+\beta_+^{*} = 0$ and it follows that $\beta_0 = \beta_+ = \beta_- = 0$, i.e. *any classical r-matrix, which satisfies the homogeneous CYBE and the* $\mathfrak{su}(2)$ *reality condition, is equal zero.*
If in (44) $\gamma = \beta_0^2 + \beta_+\beta_- \neq 0$ we have three possible $\mathfrak{su}(2)$ real classical r-matrices:

$$\begin{aligned} r_1 &:= \beta_0 E'_+ \wedge E'_-\,, \\ r_2 &:= \beta_+ E'_+ \wedge H' + \beta_+^{*} H' \wedge E'_-\,, \\ r_3 &:= \beta'_+ E'_+ \wedge H' + \beta'_0 E'_+ \wedge E'_- + \beta'^{*}_+ H' \wedge E'_-\,, \end{aligned} \tag{46}$$

where β_0 and β'_0 are real numbers and we use the conditions (45). The r-matrices r_i $(i = 1, 2, 3)$ satisfy the non-homogeneous CYBE

$$[[r_i, r_i]] \;=\; \gamma_i \Omega, \tag{47}$$

where all γ_i $(i = 1, 2, 3)$ are positive: $\gamma_1 = \beta_0^2 > 0$, $\gamma_2 = \beta_+\beta_+^{*} > 0$,
$\gamma_3 = \beta'^2_0 + \beta'_+\beta'^{*}_+ > 0$.

Let the classical r-matrices (37) be $\mathfrak{su}(2)$-antireal, i.e. their parameters satisfy the reality conditions (45). It follows that the functions (40) for $\chi = e^{\imath\phi}$ have the same congugation properties, i.e. $\tilde{\beta}_0^* = \tilde{\beta}_0$, $\tilde{\beta}_\pm^* = \tilde{\beta}_\mp$, and we obtain that the automorphism φ_1 with such parameters is $\mathfrak{su}(2)$-real, i.e.:

$$
\begin{aligned}
\varphi_1^*(E'_\pm) &= \varphi_1(E'^*_\pm) = \varphi_1(E'_\mp), \\
\varphi_1^*(H') &= \varphi_1(H'^*) = \varphi_1(H').
\end{aligned} \tag{48}
$$

We see that the r-matrices r_2 and r_3 in (46) can be reduced to the standard r-matrix $r_{st} := r_1$ using φ_1.
It is easy to see that the standard r-matrix $r_{st} = r_1$ in (46) effectively depends only on positive values of the parameter $\alpha := \beta_0$. Indeed, we see that

$$
\alpha\varphi(E'_+) \wedge \varphi(E'_-) = -\alpha E'_+ \wedge E'_-, \tag{49}
$$

where φ is the simple $\mathfrak{su}(2)$ automorphism: $\varphi(E'_\pm) = E'_\mp$, $\varphi(H') = -H'$, i.e. any negative value of parameter α in r_{st} can be replaced by the positive one.
We obtain the following result:
For the compact real form $\mathfrak{su}(2)$ there exists up to the $\mathfrak{su}(2)$ automorphisms only one solution of CYBE and this solution is the usual standard classical r-matrix r_{st}:

$$
r_{st} := \alpha E'_+ \wedge E'_-, \quad [[r_{st}, r_{st}]] = \gamma\Omega', \tag{50}
$$

where the effective parameter α is a positive number, and $\gamma = \alpha^2$.

II. *The non-compact real form* $\mathfrak{su}(1,1)$ $(H''^{\dagger} = H''$, $E''^{\dagger}_{\pm} = -E''_{\mp})$.

In this case it follows from the reality condition (44) that

$$\beta_0^* = \beta_0, \quad \beta_{\pm}^* = -\beta_{\mp}. \tag{51}$$

If $\beta_0^2 + \beta_+\beta_- = 0$ in the genaral formula for $r$then $\beta_0\beta_0^* - \beta_+\beta_+^* = 0$, i.e. $\beta_{\pm} = \pm e^{\pm\imath\phi}|\beta_0|$, and we have the following ϕ-family of $\mathfrak{su}(1,1)$ homogeneous CYBE solutions:

$$r_{\phi} := \beta_0\Big(e^{i\phi}\frac{|\beta_0|}{\beta_0}E''_+\wedge H'' + E''_+\wedge E''_- - e^{-i\phi}\frac{|\beta_0|}{\beta_0}H''\wedge E''_-\Big), \tag{52}$$

where β_0 is real. By using the $\mathfrak{su}(1,1)$-real rescaling automorphism $\varphi(E''_{\pm}) = \Big(-\imath e^{i\phi}\frac{|\beta_0|}{\beta_0}\Big)^{\pm 1}E''_{\pm}$, $\varphi(H'') = H''$ we can reduce the ϕ-family (52) to $r_{qJ} := \beta_0(\imath(E''_+ - E''_-)\wedge H'' + E''_+\wedge E''_-)$:

$$\begin{aligned} r_{\phi} &= \beta_0\Big(e^{\imath\phi}\frac{|\beta_0|}{\beta_0}E''_+\wedge H'' + E''_+\wedge E''_- - e^{-\imath\phi}\frac{|\beta_0|}{\beta_0}H''\wedge E''_-\Big) \\ &= \beta_0\Big(\imath(\varphi(E''_+) - \varphi(E''_-))\wedge\varphi(H'') + \varphi(E''_+)\wedge\varphi(E''_-)\Big). \end{aligned} \tag{53}$$

We shall call a $\mathfrak{su}(1,1)$-real r-matrix "quasi-Jordanian" if it can not be reduced to Jordanian form by a $\mathfrak{su}(1,1)$-real automorphism, but after complexification of $\mathfrak{su}(1,1)$ it can be reduced to Jordanian form by an appropriate complex $\mathfrak{sl}(2,\mathbb{C})$-authomorphism. Thus all r-matrices in the ϕ-family (52) are quasi-Jordanian and they are connected with each other by the $\mathfrak{su}(1,1)$-real rescaling automorphism. We take r_{qJ} as an representative of the ϕ-family. It is easy to see that the quasi-Jordanian r-matrix r_{qJ} effectively depends only on positive values of the parameter β_0, indeed,

$$\begin{aligned} r_{qJ} &= \beta_0(\imath E''_+\wedge H'' + E''_+\wedge E''_- + \imath H''\wedge E''_-) \\ &= -\beta_0\big(\imath\varphi(E''_+)\wedge\varphi(H'') + \varphi(E''_+)\wedge\varphi(E''_-) + \imath\varphi(H'')\wedge\varphi(E''_-)\big), \end{aligned} \tag{54}$$

where φ is the simple $\mathfrak{su}(1,1)$ automorphism $\varphi(E''_{\pm}) = E''_{\mp}$, $\varphi(H'') = -H''$, i.e. any negative value of parameter β_0 in r_{qJ} can be changed into a positive one.

In the case $\beta_0^2 + \beta_+\beta_- \neq 0$ in the general formula for r we have four versions of $\mathfrak{su}(1,1)$-real classical r-matrices. Two of them are characterized by positive value of γ_i, $(i = 1,2)$:

$$
\begin{aligned}
r_1 &:= \beta_0 E''_+ \wedge E''_-, \\
r_2 &:= \beta'_+ E''_+ \wedge H'' + \beta'_0 E''_+ \wedge E''_- - \beta'^*_+ H'' \wedge E''_-, \\
[[r_i, r_i]] &:= \gamma_i \Omega \quad (i = 1,2),
\end{aligned} \tag{55}
$$

where β_0 and β'_0 are real (see (51)), and $\gamma_1 = \beta_0^2 > 0$, $\gamma_2 = \beta'_0\beta'^*_0 - \beta'_+\beta'^*_+ > 0$. The remaining two are with negative values of γ_i, $(i = 3,4)$:

$$
\begin{aligned}
r_3 &:= \beta''_+ E''_+ \wedge H'' - \beta''^*_+ H'' \wedge E''_-, \\
r_4 &:= \beta'''_+ E''_+ \wedge H'' + \beta'''_0 E''_+ \wedge E''_- - \beta'''^*_+ H'' \wedge E''_-, \\
[[r_i, r_i]] &:= \gamma_i \Omega'' \quad (i = 3,4),
\end{aligned} \tag{56}
$$

where β'''_0 is real (see (51)), and $\gamma_3 = -\beta''_+\beta''^*_+ < 0$, $\gamma_4 = \beta'''_0\beta'''^*_0 - \beta'''_+\beta'''^*_+ < 0$. It terns out that there exist the automorphisms $\varphi_1(\cdot)$ which are $\mathfrak{su}(1,1)$-real $\varphi_1^\dagger(E''_\pm) = \varphi_1(E''^\dagger_\pm) = -\varphi_1(E''_\mp)$, $\varphi_1^\dagger(H'') = \varphi_1(H''^\dagger) = \varphi_1(H'')$ and which allow to reduce the r-matrix r_2 to the standard r-matrix $r_{st} := r_1$ for $\gamma_1 = \gamma_2 > 0$ and the r-matrix r_4 to the r-matrix r_3 for $\gamma_3 = \gamma_4 < 0$. By analogy to the notation of quasi-Jordanian r-matrix we shall call the r-matrices r_3 and r_4 as quasi-standard ones and take $r_{qst} := \alpha(E''_+ + E''_-) \wedge H''$ as their representative.

Finally for $\mathfrak{su}(1,1)$ we obtain:
For the non-compact real form $\mathfrak{su}(1,1)$ there exists up to $\mathfrak{su}(1.1)$ automorphisms three solutions of CYBE, namely quasi-Jordanian r_{qJ}, standard r_{st} and quasi-standard r_{qst}:

$$r_{qJ} = \frac{\alpha}{2}\big(\imath(E''_{+} - E''_{-}) \wedge H'' + E''_{+} \wedge E''_{-}\big), \quad [[r_{qJ}, r_{qJ}]] = 0, \tag{57}$$

$$r_{st} = \alpha E''_{+} \wedge E''_{-}, \quad [[r_{st}, r_{st}]] = \alpha^{2}\Omega'', \tag{58}$$

$$r_{qst} = \alpha(E''_{+} + E''_{-}) \wedge H'', \quad [[r_{qst}, r_{qst}]] = -\alpha^{2}\Omega'', \tag{59}$$

where α effectively is a positive number.

III. *The non-compact real form* $\mathfrak{sl}(2;\mathbb{R})$ $({H'}^{\dagger} = -H',\ {E'_{\pm}}^{\dagger} = -E'_{\pm})$.
In this case from

$$r^{\divideontimes} \;:=\; \beta_{+}^{*}E_{+}^{\divideontimes}\wedge H^{\divideontimes} + \beta_{0}^{*}E_{+}^{\divideontimes}\wedge E_{-}^{\divideontimes} + \beta_{-}^{*}H^{\divideontimes}\wedge E_{-}^{\divideontimes} \;=\; -r, \tag{60}$$

we obtain

$$\beta_{0}^{*} \;=\; -\beta_{0}, \quad \beta_{\pm}^{*} \;=\; -\beta_{\pm}, \tag{61}$$

i.e. all parameters β_i $(i = +, 0, -)$ are purely imaginary. Consider the case $\beta_0^2 + \beta_+\beta_- = 0$. We have three $\mathfrak{su}(2;\mathbb{R})$ solutions of the homogeneous CYBE:

$$\begin{aligned} r'_1 \;&=\; \beta_{+}E'_{+}\wedge H', \quad r'_2 \;=\; \beta_{-}H'\wedge E'_{-}, \\ r'_3 \;&=\; \beta'_{+},E'_{+}\wedge H' + \beta'_{0}E'_{+}\wedge E'_{-} + \beta'_{-}H'\wedge E'_{-}, \end{aligned} \tag{62}$$

where all parameters β_i $(i = +, -)$, β'_i $(i = +, 0, -)$ are purely imaginary, and ${\beta'_0}^2 + \beta'_+\beta'_- = 0$. It easy to see that there exists a $\mathfrak{sl}(2;\mathbb{R})$-real automorphisms $\varphi_0(\cdot)$: $\varphi_0(E'''_{\pm})^{\dagger} = \varphi_0({E'''_{\pm}}^{\dagger}) = -\varphi_0(E'''_{\pm})$, $\varphi_0(H''')^{\dagger} = \varphi_0({H'''}^{\dagger}) = -\varphi_0(H''')$, that allow to reduce the r-matrices r'_2 and r'_3 in (62) to the Jordanian r-matrix $r'_J := r'_1$.

In the case $\beta_0^2 + \beta_+\beta_- \neq 0$ in (25) we have seven versions of $\mathfrak{sl}(2;\mathbb{R})$-real classical r-matrices. Five of them are with negative values of γ_i, $(i = 1, 2, \ldots, 5)$:

$$\begin{aligned}
r'_1 \;&:=\; \beta_0 E'''_+ \wedge E'''_-,\\
r'_2 \;&:=\; \beta_+ E'''_+ \wedge H''' + \beta_0 E'''_+ \wedge E'''_-,\\
r'_3 \;&:=\; \beta_0 E'''_+ \wedge E'''_- + \beta_- H''' \wedge E'''_-,\\
r'_4 \;&:=\; \beta'_+ E'''_+ \wedge H''' + \beta'_- H''' \wedge E'''_-,\\
r'''_5 \;&:=\; \beta''_+ E'''_+ \wedge H''' + \beta''_0 E'''_+ \wedge E'''_- + \beta''_- H''' \wedge E'''_-,\\
[[r'_i, r'_i]] \;&:=\; \gamma_i \Omega''' \quad (i = 1, 2, \ldots, 5),
\end{aligned} \tag{63}$$

where all parameters β are purely imaginary, and $\gamma_1 = \gamma_2 = \gamma_3 = \beta_0^2 < 0$, $\gamma_4 = \beta'_+\beta'_- < 0$, $\gamma_5 = \beta''_0 + \beta''_+\beta''_- < 0$; Ω''' is the $\mathfrak{sl}(2;\mathbb{R})$-invariant element $\Omega''' = \gamma(4E'''_- \wedge H''' \wedge E'''_+)$. The remaining two r-matrices r'_i $(i = 6, 7)$ have positive values of γ_i:

$$\begin{aligned}
r'_6 \;&:=\; \beta'''_+ E'''_+ \wedge H''' + \beta'''_- H' \wedge E'''_-,\\
r'_7 \;&:=\; \beta''''_+ E'''_+ \wedge H''' + \beta''''_0 E'''_+ \wedge E'''_- + \beta''''_- H''' \wedge E'''_-,\\
[[r'_i, r'_i]] \;&:=\; \gamma_i \Omega''' \quad (i = 6, 7),
\end{aligned} \tag{64}$$

where $\gamma_6 = \beta'''_+\beta'''_- > 0$ and $\gamma_7 = \beta''''^2_0 + \beta''''_+\beta''''_- > 0$. It easy to see that there exist $\mathfrak{sl}(2;\mathbb{R})$-real automorphisms $\varphi_1(\cdot)$ that allow to reduce all r-matrices r_i $(i = 2, \ldots, 5)$ with the negative γ-characteristics $\gamma_i < 0$ $(i = 1, \ldots, 5)$ in (63) to the standard formula $r'_{st} := r'_1$. In the case of the positive γ-characteristics $\gamma_i > 0$ $(i = 6, 7)$ the classical r-matrix r'_7 in (64) is reduced to the quasi-standard r-matrix $r'_{qst} := r'_6$.

Finally for $\mathfrak{sl}(2,\mathbb{R})$ we obtain the following result:
For the non-compact real form $\mathfrak{sl}(2,\mathbb{R})$ there exists up to $\mathfrak{sl}(2,\mathbb{R})$ automorphisms three solutions of CYBE, namely Jordanian r'_J, standard r'_{st} and quasi-standard r'_{qst}:

$$r'_J = \imath\alpha E'_+ \wedge H', \quad [[r'_J, r'_J]] = 0, \tag{65}$$

$$r'_{st} = \imath\alpha E'_+ \wedge E'_-, \quad [[r'_{st}, r'_{st}]] = -\alpha^2\Omega', \tag{66}$$

$$r'_{qst} = \imath\alpha(E'_+ + E'_-) \wedge H', \quad [[r'_{qst}, r'_{qst}]] = \alpha^2\Omega', \tag{67}$$

where the parameter α is a positive number.

(0) For the complex Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ there exists up to $\mathfrak{sl}(2,\mathbb{C})$ automorphisms two solutions of CYBE, namely Jordanian r_J and standard r_{st}:

$$
\begin{aligned}
r_J &= \beta E_+ \wedge H, \quad [[r_J, r_J]] = 0, && (68)\\
r_{st} &= \beta' E_+ \wedge E_-, \quad [[r_{st}, r_{st}]] = \beta'^2 \Omega, && (69)
\end{aligned}
$$

where the complex parameter β in (68) can be removed.
(I) For the compact real form $\mathfrak{su}(2)$ there exists up to the $\mathfrak{su}(2)$ automorphisms only one solution of CYBE and this solution is the usual standard classical r-matrix r'_{st}:

$$
r'_{st} := \alpha E'_+ \wedge E'_-, \quad [[r'_{st}, r'_{st}]] = \gamma \Omega', \qquad (70)
$$

where the effective parameter α is a positive number, and $\gamma = \alpha^2$.
(II) For the non-compact real form $\mathfrak{su}(1,1)$ there exists up to $\mathfrak{su}(1.1)$ automorphisms three solutions of CYBE, namely quasi-Jordanian r''_{qJ}, standard r'''_{st} and quasi-standard r'''_{qst}:

$$
\begin{aligned}
r''_{qJ} &= \frac{\alpha}{2}\big(\imath(E''_+ - E''_-) \wedge H'' + E''_+ \wedge E''_-\big), \quad [[r''_{qJ}, r''_{qJ}]] = 0, && (71)\\
r''_{st} &= \alpha E''_+ \wedge E''_-, \quad [[r''_{st}, r''_{st}]] = \alpha^2 \Omega'', && (72)\\
r''_{qst} &= \alpha(E''_+ + E''_-) \wedge H'', \quad [[r''_{qst}, r''_{qst}]] = -\alpha^2 \Omega'', && (73)
\end{aligned}
$$

where α effectively is a positive number.
(III) For the non-compact real form $\mathfrak{sl}(2,\mathbb{R})$ there exists up to $\mathfrak{sl}(2,\mathbb{R})$ automorphisms three solutions of CYBE, namely Jordanian r'''_J, standard r'''_{st} and quasi-standard r'''_{qst}:

$$
\begin{aligned}
r'''_J &= \imath\alpha E'''_+ \wedge H''', \quad [[r'''_J, r'''_J]] = 0, && (74)\\
r'''_{st} &= \imath\alpha E'''_+ \wedge E'''_-, \quad [[r'''_{st}, r'''_{st}]] = -\alpha^2 \Omega''', && (75)\\
r'''_{qst} &= \imath\alpha(E'''_+ + E'''_-) \wedge H''', \quad [[r'''_{qst}, r'''_{qst}]] = \alpha^2 \Omega''', && (76)
\end{aligned}
$$

where the parameter α is a positive number.

Using the formulas of connections between CW and the Cartesian bases we can express the triplets of the classical $\mathfrak{su}(1,1)$ and $\mathfrak{sl}(2;\mathbb{R})$ r-matrices in terms of the $\mathfrak{o}(2,1)$ Cartesian basis. We get the following results.

(i) The $\mathfrak{su}(1,1)$ case:

$$\begin{aligned} r''_{qJ} &= \frac{\alpha}{2}(\imath(E''_{+} - E''_{-}) \wedge H'' + E''_{+} \wedge E''_{-})) \\ &= -\alpha(\imath I''_1 - I''_2) \wedge I''_3, \quad [[r''_{qJ}, r''_{qJ}]] \;=\; 0, \end{aligned} \tag{77}$$

$$\begin{aligned} & r''_{st} \;=\; \alpha E''_{+} \wedge E''_{-} \;=\; -2\imath\alpha I''_1 \wedge I''_3, \\ & [[r''_{st}, r''_{st}]] \;=\; \alpha^2 \Omega'', \end{aligned} \tag{78}$$

$$\begin{aligned} & r''_{qst} \;=\; \alpha(E''_{+} + E''_{-}) \wedge H'' \;=\; -2\alpha I''_1 \wedge I''_2, \\ & [[r''_{qst}, r''_{qst}]] \;=\; -\alpha^2 \Omega'', \end{aligned} \tag{79}$$

(ii) The $\mathfrak{su}(2;\mathbb{R})$ case:

$$\begin{aligned} & r'''_{J} \;=\; \imath\alpha E'''_{+} \wedge H''' = -\alpha(\imath I''_1 - I''_2) \wedge I''_3), \\ & [[r'''_{J}, r'''_{J}]] \;=\; 0, \end{aligned} \tag{80}$$

$$\begin{aligned} & r'''_{st} \;=\; \imath\alpha E'''_{+} \wedge E'''_{-} \;=\; -2\alpha I''_1 \wedge I'''_2, \\ & [[r'''_{st}, r'''_{st}]] \;=\; -\alpha^2 \Omega'''. \end{aligned} \tag{81}$$

$$\begin{aligned} & r'''_{qst} \;=\; \imath\alpha(E'''_{+} + E'''_{-}) \wedge H''' \;=\; -2\imath\alpha I''_1 \wedge I''_3, \\ & [[r'''_{qst}, r'''_{qst}]] \;=\; \alpha^2 \Omega''', \end{aligned} \tag{82}$$

where $\Omega''' = \Omega'' = -8 I''_1 \wedge I''_2 \wedge I''_3$.

Comparing the r-matrix expressions (77)–(79) with (80)–(82) we obtain that

$$r''_{qJ} = r'''_{J} = -\alpha(\imath I''_1 - I''_2)\wedge I''_3), \tag{83}$$

$$r''_{st} = r'''_{qst} = -2\imath\alpha I''_1 \wedge I''_3, \tag{84}$$

$$r''_{qst} = r'''_{st} = -2\alpha I''_1 \wedge I''_2, \tag{85}$$

We see that the quasi-Jordanian r-matrix r''_{qJ} in the $\mathfrak{su}(1,1)$ basis is the same as the Jordanian r-matrix r'''_{J} in the $\mathfrak{sl}(2;\mathbb{R})$ basis, and the standard r-matrix r''_{st} in the $\mathfrak{su}(1,1)$ basis becomes the quasi-standard r-matrix r'''_{qst} in the $\mathfrak{sl}(2;\mathbb{R})$ basis. Conversely, the quasi-standard r-matrix r''_{qst} in the $\mathfrak{su}(1,1)$ basis is the same as the standard r-matrix r'''_{st} in the $\mathfrak{sl}(2;\mathbb{R})$ basis.
The relations (83)–(85) show that the $\mathfrak{su}(1,1)$ and $\mathfrak{sl}(2;\mathbb{R})$ bialgebras are isomorphic. This result finally resolves the doubts about isomorphisms of these two bialgebras.
Using the isomorphisms of the $\mathfrak{su}(1,1)$ and $\mathfrak{sl}(2;\mathbb{R})$ bialgebras we take as basic r-matrices for the $D = 3$ Lorentz algebra $\mathfrak{o}(2,1)$ the following ones:

$$r''_{st} = -2\imath\alpha I''_1 \wedge I''_3 = \alpha E''_{+} \wedge E''_{-}, \tag{86}$$

$$r'''_{st} = -2\alpha I''_1 \wedge I''_2 = \imath\alpha E'''_{+} \wedge E'''_{-}, \tag{87}$$

$$r'''_{J} = -\alpha(\imath I''_1 - I''_2)\wedge I''_3 = \imath\alpha E'''_{+} \wedge H'''. \tag{88}$$

The first two r-matrices r''_{st} and r'''_{st} with the effective positive parameter α correspond to the q-analogs of $\mathfrak{su}(1,1)$ and $\mathfrak{sl}(2;\mathbb{R})$ real algebras, the third r-matrix r'''_{J} presents the Jordanian twist deformation of $\mathfrak{sl}(2;\mathbb{R})$.

In this section we describe explicitly all quantizations of the complex $D = 3$ Euclidean $\mathfrak{o}(3;\mathbb{C})$ and its real forms: the compact Euclidean $\mathfrak{o}(3)$ and the non-compact Lorentz $\mathfrak{o}(2,1)$ symmetries. These quantizations are given in terms of the CW bases for these algebras, i.e. we use isimorphisms

$$\mathfrak{o}(3;\mathbb{C}) \simeq \mathfrak{sl}(2;\mathbb{C}), \quad \mathfrak{o}(3) \simeq \mathfrak{su}(2), \quad \mathfrak{o}(2,1) \simeq \mathfrak{su}(1,1) \simeq \mathfrak{sl}(2;\mathbb{R}). \tag{89}$$

We first consider the standard quantizations, i.e. q-analogs that correspond to the standard classical r-matrices:

$$r_{st} = -2z\, I_1 \wedge I_2 = \imath z\, E_+ \wedge E_- \quad \text{for } \mathfrak{sl}(2;\mathbb{C}), \tag{90}$$

$$r'_{st} = -2\imath\alpha I'_1 \wedge I'_2 = \alpha E'_+ \wedge E'_- \quad \text{for } \mathfrak{su}(2)\ ({H'}^{\dagger} = H',\ {E'_{\pm}}^{\dagger} = E'_{\mp}), \tag{91}$$

$$r''_{st} = -2\imath\alpha I''_1 \wedge I''_3 = \alpha E''_+ \wedge E''_- \quad \text{for } \mathfrak{su}(1,1)\ ({H''}^{\dagger} = H'',\ {E''_{\pm}}^{\dagger} = -E''_{\mp}), \tag{92}$$

$$r'''_{st} = -2\alpha I''_1 \wedge I''_2 = \imath\alpha E'''_+ \wedge E'''_- \quad \text{for } \mathfrak{sl}(2;\mathbb{R})\ ({H'''}^{\dagger} = -H''',\ {E'''_{\pm}}^{\dagger} = -E'''_{\pm}). \tag{93}$$

The q-analogs of the universal enveloping algebras $U(\mathfrak{g})$ for $\mathfrak{g}=\mathfrak{sl}(2;\mathbb{C}))$, $\mathfrak{su}(2)$, $\mathfrak{su}(1,1)$, $\mathfrak{sl}(2;\mathbb{R})$ are well known and they are given as follows. The quantum deformation (q-analog) of $U(\mathfrak{g})$ is an unital associative algebra $U_q(\mathfrak{g})$ with generators $X_{\pm}$, $q^{\pm X_0}$ for $\mathfrak{g}=\mathfrak{sl}(2;\mathbb{C}))$ and the defining relations:

$$\begin{aligned} q^{X_0}q^{-X_0} &= q^{-X_0}q^{X_0} \;=\; 1, \\ q^{X_0}X_{\pm} &= q^{\pm 1}X_{\pm}q^{X_0}, \\ [X_+, X_-] &= \frac{q^{2X_0}-q^{-2X_0}}{q-q^{-1}}, \end{aligned} \qquad (94)$$

where $q := e^{z}$. In the case of the real forms we replace $X_{\pm}\to X'_{\pm}$, $q^{X_0}\to q^{X'_0}$ for $\mathfrak{su}(2)$, and $X_{\pm}\to X''_{\pm}$, $q^{X_0}\to q^{X''_0}$ for $\mathfrak{su}(1,1)$, and $X_{\pm}\to X'''_{\pm}$, $q^{X_0}\to q^{X'''_0}$ for $\mathfrak{sl}(2;\mathbb{R})$ with the reality conditions:

$$\begin{aligned} U_q(\mathfrak{su}(2)):\quad & (X'_{\pm})^{\dagger} = X'_{\mp}, \quad (q^{X'_0})^{\dagger} = q^{X'_0}, \quad q := e^{\alpha}, \\ U_q(\mathfrak{su}(1,1)):\quad & (X''_{\pm})^{\dagger} = -X''_{\mp}, \quad (q^{X''_0})^{\dagger} = q^{X''_0}, \quad q := e^{\alpha}, \\ U_q(\mathfrak{sl}(2;\mathbb{R})):\quad & (X'''_{\pm})^{\dagger} = -X'''_{\pm}, \quad (q^{X'''_0})^{\dagger} = q^{X'''_0}, \quad q := e^{\imath\alpha}. \end{aligned} \qquad (95)$$

A Hopf structure on $U_q(\mathfrak{g})$ is defined with help of three additional operations: coproduct (comultiplication) Δ_q, antipode S_q and counit ϵ_q:

$$\begin{aligned} \Delta_q(q^{\pm X_0}) &= q^{\pm X_0}\otimes q^{\pm X_0}, \\ \Delta_q(X_{\pm}) &= X_{\pm}\otimes q^{X_0} + q^{-X_0}\otimes X_{\pm}, \\ S_q(q^{\pm X_0}) &= q^{\mp X_0}, \quad S_q(X_{\pm}) = -q^{\pm 1}X_{\pm}, \\ \epsilon_q(q^{\pm X_0}) &= 1, \quad \epsilon_q(X_{\pm}) = 0, \end{aligned} \qquad (96)$$

with the reality conditions[2] for the real form $\mathfrak{su}(2)$, $\mathfrak{su}(1,1)$, $\mathfrak{sl}(2;\mathbb{R})$:

$$\Delta_q^{\circledast}(X) = \Delta_q(X^{\circledast}), \quad S_q^{\circledast}(X) = S_q^{-1}(X^{\circledast}), \quad \epsilon_q^{\circledast}(X) = \epsilon_q(X^{\circledast}). \tag{97}$$

for any $X \in U_q(\mathfrak{g})$, where $\circledast$ is the conjugation associated with corresponding real form ($\circledast = *, \dagger$). The quantum algebra $U_q(\mathfrak{g})$ is endowed also with the *opposite* Hopf structure: opposite coproduct $\tilde{\Delta}_q$[3], corresponding antipode $\tilde{S}_q$ and counit $\tilde{\epsilon}_q$.
In the limit $q \to 1$ we obtain $(X_i, X_i', X_i'', X_i''') \to (E_i, E_i', E_i'', E_i''')$:

$$\delta_{(r)}(x) = \alpha^{-1}(\Delta - \tilde{\Delta}) = [r, \Delta_0(x)] \pmod{\alpha}. \tag{98}$$

Here r is a standard classical r-matrix of $\mathfrak{sl}(2;\mathbb{C})$), $\mathfrak{su}(2)$, $\mathfrak{su}(1,1)$ or $\mathfrak{sl}(2;\mathbb{R})$: $r \in \{r_{st}, r'_{st}, r''_{st}, r'''_{st}\}$. Thus our q-analogs $U_q(\mathfrak{g})$ are the correct quantizations of the bialgebras corresponding to these standard classical r-matrices.

[2] $\Delta_q^{\circledast}(X) := (\Delta_q(X))^{\circledast\otimes\circledast}$.

[3] The opposite (transformed) coproduct $\tilde{\Delta}_q(\cdot)$ is a coproduct with permuted components, i.e. $\tilde{\Delta}_q(\cdot) = \tau \circ \Delta_q(\cdot)$ where τ is the flip operator: $\tau \circ \sum X_{(1)} \otimes X_{(2)} = \sum X_{(2)} \otimes X_{(1)}$.

An invertible element $R_q := R_q(\mathfrak{g})$ which satisfies the relations:

$$\begin{aligned} R_q\Delta_q(X) &= \tilde{\Delta}_q(X)R_q, \quad \forall X \in U_q(\mathfrak{g}), \\ (\Delta_q \otimes \mathrm{id})R_q &= R_q^{13}R_q^{23}, \quad (\mathrm{id} \otimes \Delta_q)R_q = R_q^{12}R_q^{13} \end{aligned} \tag{99}$$

as well as, due to (99), the quantum Yang-Baxter equation (QYBE)

$$R_q^{12}R_q^{13}R_q^{23} = R_q^{23}R_q^{13}R_q^{12} \tag{100}$$

is called the *universal R-matrix*. Let $U_q(\mathfrak{b}_+)$ and $U_q(\mathfrak{b}_-)$ be quantum Borel subalgebras of $U_q(\mathfrak{g})$, generated by X_+, $q^{\pm X_0}$ and X_-, $q^{\pm X_0}$ respectively. We denote by $T_q(\mathfrak{b}_+ \otimes \mathfrak{b}_-)$ the Taylor extension of $U_q(\mathfrak{b}_+) \otimes U_q(\mathfrak{b}_-)$[4]. One can show that *there exists unique solution of equations (99) in the space $T_q(\mathfrak{b}_+ \otimes \mathfrak{b}_-)$ and such solution has the following form*

$$\begin{aligned} R_q(\mathfrak{g}) := R_q^{\succ} &= \exp_{q^{-2}}\left((q - q^{-1})X_+q^{-X_0} \otimes q^{X_0}X_-\right)q^{2X_0\otimes X_0} \\ &= q^{2X_0\otimes X_0}\exp_{q^{-2}}\left((q - q^{-1})X_+q^{X_0} \otimes q^{-X_0}X_-\right), \end{aligned} \tag{101}$$

where we use the standard definition of the q-exponential:

$$\begin{aligned} \exp_q(x) &:= \sum_{n\geq 0}\frac{x^n}{(n)_q!}, \quad (n)_q := \frac{(1-q^n)}{(1-q)}, \\ (n)_q! &:= (1)_q(2)_q\ldots(n)_q. \end{aligned} \tag{102}$$

[4] $T_q(\mathfrak{b}_+ \otimes \mathfrak{b}_-)$ is an associative algebra generated by formal Taylor series of the monomials $X_+^n \otimes X_-^m$ with coefficients which are rational functions of $q^{\pm X_0}$, $q^{\pm X_0\otimes X_0}$, provided that all values $|n-m|$ for each formal series are bounded, $|n-m| < N$.

Analogously, *there exits unique solution of equations (99) in the space* $T_q(\mathfrak{b}_- \otimes \mathfrak{b}_+) = \tau \circ T_q(\mathfrak{b}_+ \otimes \mathfrak{b}_-)$ *and such solution is given by the formula*

$$\begin{aligned} R_q(\mathfrak{g}) \;:=\; R_q^{\prec} \;&=\; \exp_{q^2}\left((q^{-1}-q)X_- q^{-X_0} \otimes q^{X_0} X_+\right) q^{-2X_0 \otimes X_0} \\ &=\; q^{-2X_0 \otimes X_0} \exp_{q^2}\left((q^{-1}-q)X_- q^{X_0} \otimes q^{-X_0} X_+\right), \end{aligned} \tag{103}$$

As formal Taylor series the solutions (101) and (103) are independent and they are related by the relation

$$R_q^{\prec} \;=\; \tau \circ R_{q^{-1}}^{\succ}. \tag{104}$$

It should be noted also that

$$(R_q^{\succ})^{-1} \;=\; R_{q^{-1}}^{\succ}, \quad (R_q^{\prec})^{-1} = R_{q^{-1}}^{\prec}. \tag{105}$$

From the explicite forms (101) and (103) we see that

$$(R_q^{\succ})^{*} = \tau \circ R_q^{\succ} = (R_q^{\prec})^{-1}, \quad (R_q^{\prec})^{*} = \tau \circ R_q^{\prec} = (R_q^{\succ})^{-1} \tag{106}$$

for $U_q(\mathfrak{su}(2))$, $U_q(\mathfrak{su}(1.1))$, and

$$(R_q^{\succ})^{\dagger} \;=\; (R_q^{\succ})^{-1}, \quad (R_q^{\prec})^{\dagger} \;=\; (R_q^{\prec})^{-1} \tag{107}$$

for $U_q(\mathfrak{sl}(2;\mathbb{R}))$, i.e. in the case $U_q(\mathfrak{sl}(2;\mathbb{R}))$ both R-matrices $R_q^{\succ}$, $R_q^{\prec}$ are unitary and in the case $U_q(\mathfrak{su}(1,1))$ they can be called "flip-Hermitian" or "τ-Hermitian".

We can introduce the quantum Cartesian generators, for example for $U_q(\mathfrak{su}(1,1))$, by the formulas: $X''_{\pm} = \imath J''_1 \pm J''_3$, $q^{\pm X''_0} = q^{\pm \imath J''_2}$.[5] In terms of these generators the quantum algebra $U_q(\mathfrak{su}(1,1))$, which will be denoted by $U_{(r''_{st})}(\mathfrak{o}(2,1))$, can be reformulated as follows. The quantum deformation of $U(\mathfrak{o}(2,1))$, corresponding to the classical r-matrix r''_{st}, is an unital associative algebra $U_{(r''_{st})}(\mathfrak{o}(2,1))$ with the generators $\{J''_1,\ J''_3,\ q^{\pm \imath J''_2}\}$ and the defining relations ($k = 1, 3$):

$$\begin{aligned} q^{\imath J''_2} q^{-\imath J''_2} &= q^{-\imath J''_2} q^{\imath J''_2} = 1, \quad [J''_1,\, J''_3] = \frac{\imath(q^{2\imath J''_2} - q^{-2\imath J''_2})}{2(q - q^{-1})}, \\ q^{\pm \imath J''_2} J''_k &= \frac{1}{2}(q + q^{-1}) J''_k q^{\pm \imath J''_2} \pm \frac{\imath}{2}(q - q^{-1}) \varepsilon_{2kl} J''_l q^{\pm \imath J''_2} \end{aligned} \tag{108}$$

with the reality condition $(J''_1)^{\dagger} = J''_1$, $(J'_3)^{\dagger} = J''_3$, $(q^{\pm \imath J''_2})^{\dagger} = q^{\pm \imath J''_2}$, $q^* = q$ ($q := e^{\alpha}$, $\alpha \in \mathbb{R}$). The Hopf algebra structure on $U_{(r''_{st})}(\mathfrak{o}(2,1))$ is given as follows ($k = 1, 3$):

$$\begin{aligned} \Delta_q(J''_k) &= J''_k \otimes q^{\imath J''_2} + q^{-\imath J''_2} \otimes J''_k, \\ \Delta_q(q^{\pm \imath J''_2}) &= q^{\pm \imath J''_2} \otimes q^{\pm \imath J''_2}, \quad S_q(q^{\pm \imath J''_2}) = q^{\mp \imath J''_2}, \\ S_q(J''_k) &= -\frac{1}{2}(q + q^{-1}) J''_k + \frac{\imath}{2}(q - q^{-1}) \varepsilon_{k2l} J''_l, \\ \epsilon_q(q^{\pm \imath J''_2}) &= 1, \quad \epsilon_q(J''_k) = 0. \end{aligned} \tag{109}$$

[5] The generators $J''_i = (-1)^{i-1}(J''_i)^{\dagger}$ ($i = 1, 2, 3$) are q-analogs of the Cartesian basis for $U_q(\mathfrak{su}(1,1)) \simeq U_q(\mathfrak{o}(2,1))$ ($\lim_{q\to 1} J''_i \to I''_i$).

If one use the q-analog $U_q(\mathfrak{sl}(2;\mathbb{R}))$ we can also introduce another quantum Cartesian generators by the formulas: $X''_{\pm} = \imath J'''_1 \mp J'''_2$, $q^{\pm X''_0} = q^{\pm\imath J'''_3}$.[6] In terms of these generators the quantum algebra $U_q(\mathfrak{sl}(2;\mathbb{R})$, which will be denoted by $U_{(r'''_{st})}(\mathfrak{o}(2,1))$, can be reformulated as follows. The quantum deformation of $U(\mathfrak{o}(2,1))$, corresponding to the classical r-matrix r'''_{st}, is an unital associative algebra $U_{(r'''_{st})}(\mathfrak{o}(2,1))$ with the generators $\{J'''_1,\, J'''_2,\, q^{\pm\imath J'''_3}\}$ and the defining relations ($k=1,2$):

$$\begin{aligned} q^{\imath J'''_3} q^{-\imath J'''_3} &= q^{-\imath J'''_3} q^{\imath J'''_3} = 1, \quad [J'''_1, J'''_2] = -\frac{\imath(q^{2\imath J'''_3} - q^{-2\imath J'''_3})}{2(q-q^{-1})}, \\ q^{\pm\imath J'''_3} J'''_k &= \frac{1}{2}(q+q^{-1}) J'''_k q^{\pm\imath J'''_3} \pm \frac{\imath}{2}(q-q^{-1})\varepsilon_{3kl} J'''_l q^{\pm\imath J'''_3} \end{aligned} \tag{110}$$

with the reality conditions $J'''^{\dagger}_1 = J'''_1$, $J'''^{\dagger}_2 = -J'''_2$, $(q^{\imath J'''_3})^{\dagger} = q^{\imath J'''_3}$, $q^* = q^{-1}$ ($q := e^{\imath\alpha}$, $\alpha\in\mathbb{R}$). The Hopf structure on $U_{(r'''_{st})}(\mathfrak{o}(2,1))$ are provided by the formulae ($k=1,2$):

$$\begin{aligned} \Delta_q(J'''_k) &= J'''_k \otimes q^{\imath J'''_3} + q^{-\imath J'''_3} \otimes J'''_k, \\ \Delta_q(q^{\pm\imath J'''_3}) &= q^{\pm\imath J'''_3} \otimes q^{\pm\imath J'''_3}, \quad S_q(q^{\pm\imath J'''_3}) = q^{\mp\imath J'''_3}, \\ S_q(J'''_k) &= -\frac{1}{2}(q+q^{-1})J'''_k + \frac{\imath}{2}(q-q^{-1})\varepsilon_{k3l} J'''_l, \\ \epsilon_q(q^{\pm\imath J'''_3}) &= 1, \quad \epsilon_q(J'''_k) = 0, \end{aligned} \tag{111}$$

[6] The generators J'''_i ($i=1,2,3$) are also the q-analoq of the Cartesian basis for $\mathfrak{o}(2,1)$ ($\lim_{q\to 1} J'''_i \to I''_i$).

Now we consider the Jordanian quantizations that correspond to the Jordanian classical r-matrices for $\mathfrak{o}(3;\mathbb{C})\simeq\mathfrak{sl}(2;\mathbb{C})$ and $\mathfrak{o}(2,1)\simeq\mathfrak{sl}(2;\mathbb{R})$:

$$\mathfrak{sl}(2;\mathbb{C}):\quad r_J = -z(\imath I_1 - I_2)\wedge I_3 \;= -\imath z E_+\wedge H, \tag{112}$$

$$\mathfrak{sl}(2;\mathbb{R}):\quad r_J''' = -\alpha(\imath I_1'' - I_2'')\wedge I_3'' = \imath\alpha E_+'''\wedge H'''. \tag{113}$$

where $(H''')^\dagger = -H'''$, $(E_\pm''')^\dagger = -E_\pm'''$. The quantizations corresponding to the classical Jordanian r-matrices (112) and (113) are well known for a long time and they are defined by the twist F:

$$F \;=\; \exp(E_+\otimes\sigma),\qquad \sigma \;=\; \ln(1+zE_+), \tag{114}$$

where in the case of the real form $\mathfrak{sl}(2;\mathbb{R})$ we replace $E_\pm\to E_\pm'''$, $q^{E_0}\to q^{X_0'''}$, $z\to\imath\alpha$. The two-tensor F satisfies the 2-cocycle condition

$$F^{12}(\Delta\otimes\mathrm{id})(F) \;=\; F^{23}(\mathrm{id}\otimes\Delta)(F), \tag{115}$$

and the "unital"normalization

$$(\epsilon\otimes\mathrm{id})(F) \;=\; (\mathrm{id}\otimes\epsilon)(F) \;=\; 1. \tag{116}$$

In the case of $\mathfrak{o}(1,2)\simeq\mathfrak{sl}(2;\mathbb{R})$ the twist (114) is unitary

$$F^\dagger \;=\; F^{-1}. \tag{117}$$

The twisting element F defines a deformation of the universal enveloping algebra $U(\mathfrak{g})$ ($\mathfrak{g}=\mathfrak{sl}(2;\mathbb{C}),\mathfrak{sl}(2;\mathbb{R})$) considered as a Hopf algebra.

The new deformed coproduct and antipode are given as follows

$$\Delta^{(F)}(X) \;=\; F\Delta(X)F^{-1}, \qquad S^{(F)}(X) = uS(X)u^{-1} \tag{118}$$

for any $X \in U(\mathfrak{g})$, where $\Delta(X)$ and $S(X)$ are the coproduct and the antipode before twisting: $\Delta(X) = X \otimes 1 + 1 \otimes X$, $S(X) = -X$; and

$$u \;=\; m(\mathrm{id} \otimes S)(F) = \exp(-\imath\alpha H E_{+}). \tag{119}$$

It is easy to see that in the case of $\mathfrak{o}(1,2)$ we get the $\dagger$-Hopf algebra, i.e.

$$(\Delta^{(F)}(X))^{\dagger} \;=\; \Delta^{(F)}(X^{\dagger}), \quad (S^{(F)}(X))^{\dagger} = S^{(F)}(X^{\dagger}) \tag{120}$$

for any $X \in U(\mathfrak{o}(1,2))$.

Thus we described explicitly all quantizations of the complex $D = 3$ Euclidean $\mathfrak{o}(3;\mathbb{C})$ and its real forms: the compact Euclidean $\mathfrak{o}(3)$ and the non-compact Lorentz $\mathfrak{o}(2,1)$ symmetries.

Using these results we can describe all quantizations for all bialgebras of the complex $D = 4$ Euclidean algebra $\mathfrak{o}(4;\mathbb{C})$ and its real forms: Euclidian $\mathfrak{o}(4)$, quartenionic $\mathfrak{o}^{\star}(4)$, Kleinian $\mathfrak{o}(2,2)$ and Lorenzian $\mathfrak{o}(3,1)$ symmetries in explicit forms!!!.
It will be presented in the next talk by Pan Professor Jurek Lukierski.

THANK YOU FOR YOUR ATTENTION