Functional equations for Feynman integrals and Abel's addition theorem

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FE from recurrence relations

Functional equations (FE) for Feynman integrals were proposed in O.V.T. Phys.Lett. B670 (2008) 67.

Feynman integrals satisfy recurrence relations which can be written as

$$\sum_j Q_j I_{j,n} = \sum_{k,r<n} R_{k,r} I_{k,r}$$

where Q_j, R_k are polynomials in masses, scalar products of external momenta, space-time dimension d, and powers of propagators. $I_{k,r}$ - are integrals with r external lines. In recurrence relations some integrals are more complicated than the others: $I_{j,n}$ on the l.h.s have more arguments than $I_{k,r}$ on the r.h.s.

General method for deriving functional equations:

By choosing kinematic variables, masses, indices of propagators remove most complicated integrals, i.e. impose conditions :

$$Q_j = 0$$

keeping at least some other coefficients $R_k \neq 0$.

Introduction

Example: one-loop n-point integrals

$$I_n^{(d)} = \frac{1}{i\pi^{d/2}} \int \frac{d^d k_1}{[(k_1-p_1)^2 - m_1^2 + i\eta]\ldots(k_1-p_n)^2 - m_n^2 + i\eta]}$$

Integrals $I_n^{(d)}$ satisfy generalized recurrence relations O.T. in Phys.Rev.D54 (1996) p.6479

$$G_{n-1}\nu_j \mathbf{j}^+ I_n^{(d+2)} - (\partial_j \Delta_n) I_n^{(d)} = \sum_{k=1}^{n} (\partial_j \partial_k \Delta_n) \mathbf{k}^- I_n^{(d)},$$

Introduction

where $\mathbf{j}^{\pm}$ shifts indices $\nu_j \to \nu_j \pm 1$, $\partial_j \equiv \frac{\partial}{\partial m_j^2}$,

$$G_{n-1} = -2^n \begin{vmatrix} p_1 p_1 & p_1 p_2 & \dots & p_1 p_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ p_1 p_{n-1} & p_2 p_{n-1} & \dots & p_{n-1} p_{n-1} \end{vmatrix},$$

$$\Delta_n = \begin{vmatrix} Y_{11} & Y_{12} & \dots & Y_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \dots & Y_{nn} \end{vmatrix}, \quad Y_{ij} = m_i^2 + m_j^2 - p_{ij}, \qquad p_{ij} = (p_i - p_j)^2,$$

One-loop propagator type integral

At $n = 3,\ j = 1,\ m_3^2 = 0$, imposing conditions $G_2 = 0,\ \Delta_3 = 0$ we get

$$
\begin{aligned}
I_2^{(d)}(m_1^2, m_2^2, p_{12}) &= \frac{p_{12} + m_1^2 - m_2^2 - \alpha_{12}}{2p_{12}}\ I_2^{(d)}(m_1^2, 0, s_{13}) \\
&+ \frac{p_{12} - m_1^2 + m_2^2 + \alpha_{12}}{2p_{12}}\ I_2^{(d)}(0, m_2^2, s_{23})
\end{aligned}
$$

where

$$
\begin{aligned}
s_{13} &= \frac{\Delta_{12} + 2p_{12}m_1^2 - (p_{12} + m_1^2 - m_2^2)\alpha_{12}}{2p_{12}}, \\
s_{23} &= \frac{\Delta_{12} + 2p_{12}m_2^2 + (p_{12} - m_1^2 + m_2^2)\alpha_{12}}{2p_{12}}, \\
\alpha_{12} &= \pm\sqrt{\Delta_{12}}.
\end{aligned}
$$

$$
\Delta_{ij} = p_{ij}^2 + m_i^4 + m_j^4 - 2p_{ij}m_i^2 - 2p_{ij}m_j^2 - 2m_i^2 m_j^2.
$$

Integral with arbitrary masses and momentum can be expressed in terms of integrals with one propagator massless !!!

One-loop vertex type integral

At $n=4,\ j=1,\ m_4=0$, imposing conditions $G_3=0,\ \Delta_4=0$ we get

$$
\begin{aligned}
&I_3^{(d)}(m_1^2, m_2^2, m_3^2, s_{23}, s_{13}, s_{12}) = \\
&\frac{s_{13}+m_3^2-m_1^2+\alpha_{13}}{2s_{13}} \\
&\times I_3^{(d)}(m_2^2, m_3^2, 0, s_{34}^{(13)}, s_{24}(m_1^2, m_3^2, s_{23}, s_{13}, s_{12}), s_{23}) \\
&+\frac{s_{13}-m_3^2+m_1^2-\alpha_{13}}{2s_{13}} \\
&\times I_3^{(d)}(m_1^2, m_2^2, 0, s_{24}(m_1^2, m_3^2, s_{23}, s_{13}, s_{12}), s_{14}^{(13)}, s_{12})
\end{aligned}
$$

Again as it was for integral $I_2^{(d)}$ integral $I_3^{(d)}$ with arbitrary arguments can be expressed in terms of integrals with at least one propagator massless!!!

Question

We used recurrence relations to derive FE. All recurrence relations follow from the equation:

$$\int d^d k \frac{\partial}{\partial k_\mu} f(k, \{s_{ij}\}, \{m_r^2\}) = 0.$$

One can raise the question:
Functional equations hold for integrals or they can be obtained as a consequence of a relation between integrands?

Algebraic relations for propagators

Analyzing one-loop FE one can see that integrands are rather similar and differ only by one propagator:

Integrands for the one-loop propagator type integrals

$$\frac{1}{D_1 D_2}, \quad \frac{1}{D_0 D_2}, \frac{1}{D_1 D_0},$$

Integrands for the one-loop vertex type integrals

$$\frac{1}{D_1 D_2 D_3}, \quad \frac{1}{D_0 D_2 D_3}, \frac{1}{D_1 D_0 D_3}, \frac{1}{D_1 D_2 D_0}$$

where

$$D_0 = (k_1 - p_0)^2 - m_0^2 + i\eta, \quad D_1 = (k_1 - p_1)^2 - m_1^2 + i\eta,$$
$$D_2 = (k_1 - p_2)^2 - m_2^2 + i\eta, \quad D_3 = (k_1 - p_3)^2 - m_3^2 + i\eta,$$

Observation: since $G_n = 0$ vectors p_1,p_2,...are linearly dependent

Algebraic relations for propagators

Question: Would it be possible to find algebraic relations of the form:

$$\frac{1}{D_1 D_2} = \frac{x_1}{D_0 D_2} + \frac{x_2}{D_1 D_0}$$

where

$$p_0 = y_{01} p_1 + y_{02} p_2$$

and $x_1, x_2,\ y_{01}, y_{02}$ being independent of k_1.

The answer is - YES! Putting all terms over the common denominator and equating coefficients in front of different products of $(k_1^2)^a (k_1 p_1)^b (k_1 p_2)^c$ to zero we obtain system of equations:

$$\begin{aligned} & y_{02} - x_2 = 0, \quad y_{01} - x_1 = 0, \quad x_1 + x_2 = 1, \\ & p_1^2 (x_1 - y_{01}^2) + p_2^2 (x_2 - y_{02}^2) + y_{01} y_{02} (s_{12} - p_1^2 - p_2^2) \\ & \qquad\qquad - m_1^2 x_1 - m_2^2 x_2 + m_0^2 = 0. \end{aligned}$$

where $s_{12} = (p_1 - p_2)^2$

Algebraic relations for propagators

Solution of this system of equations is:

$$x_1 = y_{01} = \frac{m_2^2 - m_1^2 + s_{12}}{2s_{12}} - \frac{\sqrt{\Lambda_2 + 4s_{12}m_0^2}}{2s_{12}},$$

$$x_2 = y_{02} = \frac{m_1^2 - m_2^2 + s_{12}}{2s_{12}} + \frac{\sqrt{\Lambda_2 + 4s_{12}m_0^2}}{2s_{12}}.$$

and

$$\Lambda_2 = s_{12}^2 + m_1^4 + m_2^4 - 2s_{12}(m_1^2 + m_2^2) - 2m_1^2 m_2^2.$$

Algebraic relations for propagators

Integrating obtained algebraic relation w.r.t. k_1 gives the following FE:

$$I_2^{(d)}(m_1^2, m_2^2, s_{12}) = \frac{s_{12} + m_1^2 - m_2^2 - \lambda}{2s_{12}} \; I_2^{(d)}(m_1^2, m_0^2, s_{13}(m_1^2, m_2^2, m_0^2, s_{12})) + \frac{s_{12} - m_1^2 + m_2^2 + \lambda}{2s_{12}} \; I_2^{(d)}(m_2^2, m_0^2, s_{23}(m_1^2, m_2^2, m_0^2, s_{12})).$$

where

$$s_{13} = \frac{\Lambda_2 + 2s_{12}(m_1^2 + m_0^2)}{2s_{12}} + \frac{m_1^2 - m_2^2 + s_{12}}{2s_{12}}\lambda$$

$$s_{23} = \frac{\Lambda_2 + 2s_{12}(m_2^2 + m_0^2)}{2s_{12}} + \frac{m_1^2 - m_2^2 - s_{12}}{2s_{12}}\lambda.$$

$$\lambda = \sqrt{\Lambda_2 + 4s_{12}m_0^2}$$

Parameter m_0 is arbitrary and can be taken at will. The same equation was obtained from recurrence relations by imposing conditions on Gram determinants.

Algebraic relations for propagators

Similar to the relation with two propagators one can find relation for three propagators:

$$\frac{1}{D_1 D_2 D_3} = \frac{x_1}{D_2 D_3 D_0} + \frac{x_2}{D_1 D_3 D_0} + \frac{x_3}{D_1 D_2 D_0}.$$

Here p_1, p_2 and p_3 are independent external momenta, k_1 will be integration momentum and

$$p_0 = y_{01} p_1 + y_{02} p_2 + y_{03} p_3.$$

Multiplying both sides of equation by the product $D_1 D_2 D_3 D_0$ and equating coefficients in front of $k_1^2, k_1 p_1, k_1 p_2, k_1 p_3$ and term independent of k_1 we obtain system of equations

$$y_{01} - x_1 = 0, \quad y_{02} - x_2 = 0, \quad y_{03} - x_3 = 0, \quad x_3 + x_2 + x_1 - 1 = 0,$$
$$[x_1 - y_{01}(y_{03} + y_{02} + y_{01})]p_1^2 + [x_2 - y_{02}(y_{03} + y_{02} + y_{01})]p_2^2$$
$$+[x_3 - y_{03}(y_{03} + y_{01} + y_{02})]p_3^2$$
$$+y_{02} y_{03} p_{23} + y_{01} y_{03} p_{13} + y_{01} y_{02} p_{12} - m_1^2 x_1 - m_2^2 x_2 - m_3^2 x_3 + m_0^2 = 0.$$

Algebraic relations for product of 3 propagators

This system has the following solution

$$x_1 = y_{01} = 1 - \alpha - y_{02}, \qquad x_2 = y_{02}, \qquad x_3 = y_{03} = \alpha,$$

where α is solution of the quadratic equation

$$\alpha^2 p_{13} + [m_3^2 - m_1^2 - p_{13} + y_{02}(p_{13} + p_{12} - p_{23})]\alpha$$
$$+m_1^2 - m_0^2 + (m_2^2 - m_1^2 - p_{12} + p_{12} y_{02}) y_{02} = 0.$$

Solution depends on 2 arbitrary parameters: m_0, y_{02}.
By integrating the obtained relation we get the same FE as it was given before.

Algebraic relations for propagators

The most general form of algebraic relation for obtaining FE can be written as:

$$\sum_k \prod_{j \in S_k} \frac{x_k}{[(k_1 - p_j)^2 - m_j^2 + i\eta]} + \frac{\partial f_\mu(k_1, p_1, \ldots)}{\partial k_{1\mu}} = 0,$$

where S_k are sets of integer numbers. This relation can be fulfilled by appropriate choice of x_k, m_k, p_k and f_μ.

Functional equations for Feynman integrals with integrands being rational functions strongly remind *addition formulae for Abelian integrals.*

Abelian integrals

Abelian integral is an integral in the complex plane of the form

$$\int_{z_0}^{z} R(x,y)dx,$$

where $R(x,y)$ is an arbitrary *rational* function of the two variables x and y. These variables are related by the equation

$$F(x,y) = 0,$$

where $F(x,y)$ is an irreducible polynomial in y,

$$F(x,y) \equiv \phi_n(x)y^n + ... + \phi_1(x)y + \phi_0(x),$$

whose coefficients $\phi_j(x)$, $j = 0, 1, ...n$ are rational functions of x. Abelian integrals are natural generalizations of elliptic integrals, which arise when

$$F(x,y) = y^2 - P(x),$$

where $P(x)$ is a polynomial of degree 3 and 4. If degree of the polynomial is greater than 4 then we have *hyperelliptic integral*.

Abel's theorem

Let C and C' be plane curves given by the equations

$$C: \quad F(x,y) = 0,$$
$$C': \quad \phi(x,y) = 0.$$

These curves have n points of intersections (x_1, y_1),... (x_n, y_n), where n is the product of degrees of C and C'. We will assume that $F(x,y)$ is fixed curve and $\phi(x,y)$ is a variable curve with variable coefficients, say, $a_1, \ldots$. Let $R(x,y)$ be a rational function of x and y where y is defined as a function of x by the relation $F(x,y) = 0$.
Consider the sum

$$I = \sum_{i=1}^{n} \int_{x_0,y_0}^{x_i,y_i} R(x,y) dx$$

Integrals being taken from a fixed point to the n points of intersections. If some of the coefficients a_1,a_2,... ,a_k of $\phi(x,y)$ are regarded as continuous variables, the points (x_i, y_i) will vary continuously and hence I will be a function, whose form is to be determined, of these variable coefficients.

Abel's theorem

Abel's theorem:
The partial derivatives of the sum I , with respect to any of the coefficients of the variable curve $\phi(x, y)$, is a *rational* function of the coefficients and hence I is equal to a *rational* function of the coefficients of $\phi(x, y)$, plus a finite number of logarithms or arc tangents of such rational functions.
Important: integrals themselves can be rather complicated transcendental functions but their sum can be simple.

Examples of Abel's theorem

The addition theorem for logarithm. Let:

$$y = \frac{1}{x}, \qquad \textit{the fixed curve},$$
$$y = x^2 + ax + b, \qquad \textit{the variable curve}$$

On eliminating y from these equations we have,

$$\phi(x, a, b) = x^3 + ax^2 + bx - 1 = 0,$$

According to Abel's theorem the sum

$$U(a, b) = \int_1^{x_1(a,b)} \frac{dx}{x} + \int_1^{x_2(a,b)} \frac{dx}{x} + \int_1^{x_3(a,b)} \frac{dx}{x}$$

must be rational function of a, b plus logarithms of ratios of such functions. The values of x_1, x_2, x_3 are roots of equation $\phi(x, a, b) = 0$ and they satisfy relations

$$x_1 + x_2 + x_3 = -a, \qquad x_1x_2 + x_1x_3 + x_2x_3 = b, \qquad x_1x_2x_3 = 1.$$

Addition theorem for logarithm

Differentiating expression for $U(a, b)$, and using equation $\phi(x, a, b) = 0$ one can show that

$$\frac{\partial U(a, b)}{\partial a} = \frac{\partial U(a, b)}{\partial b} = 0,$$

and therefore $U(a, b)$ is a constant that can be found from the boundary condition. From the boundary condition it follows that the constant is equal to zero. Performing integration we come to the addition formula

$$\log x_1 + \log x_2 = \log(x_1 x_2).$$

Addition theorem for logarithm

This formula can be obtained following our algorithm for finding FE for Feynman integrals. We consider the sum of “propagators”

$$\frac{y_1}{1+x_1 z}+\frac{y_2}{1+x_2 z}+\frac{y_3}{1+x_3 z}+\frac{df(z)}{dz}=0$$

There is a nontrivial solution of these equation for arbitrary z with $f(z)=-\ln(1+w_1 z)(1+w_2 z)(1+w_3 z)$. By integrating the above formula from 0 to 1 we arrive at the same relation for logarithm

$$\log x_1+\log x_2=\log(x_1 x_2).$$

The contribution from the term with derivative is zero.

Abel's theorem for Elliptic integral

Example: Elliptic integral of the second type:

$$E(k,x) = \int_0^x \frac{(1-k^2x)dx}{\sqrt{x(1-x)(1-k^2x)}}.$$

Abel's theorem

We take as C and C'

$$
\begin{aligned}
C: &\quad y^2 = x(1-x)(1-k^2x),\\
C': &\quad y = ax + b.
\end{aligned}
$$

The elimination of y between two equations will give us as the abscissae x_1, x_2, x_3 of the points of intersection the three roots of the equation:

$$\phi(x) = k^2x^3 - (1 + k^2 + a^2)x^2 + (1 - 2ab)x - b^2 = 0.$$

The corresponding sum will be

$$I(a,b) = \int_0^{x_1} R(x)dx + \int_0^{x_2} R(x)dx + \int_{1/k^2}^{x_3} R(x)dx$$

where

$$R(x,y) = \frac{1 - k^2x}{y}.$$

Abel's theorem gives addition formula:

$$\int_0^{x_1} R(x)dx + \int_0^{x_2} R(x)dx + \int_{1/k^2}^{x_3} R(x)dx = -2a + \kappa,$$

where κ is an arbitrary constant to be found from boundary conditions. Finaly we get:

$$\int_0^{x_1} \frac{(1-k^2x)dx}{\sqrt{x(1-x)(1-k^2x)}} + \int_0^{x_2} \frac{(1-k^2x)dx}{\sqrt{x(1-x)(1-k^2x)}} \\ - \int_0^{x'} \frac{(1-k^2x)dx}{\sqrt{x(1-x)(1-k^2x)}} - 2k^2\sqrt{x_1x_2x'} = 0,$$

where

$$x' = \frac{1}{x_1x_2}\left(\frac{x_1y_2+x_2y_1}{1-k^2x_1x_2}\right)^2.$$

Deriving relations for propagators we used orthogonality condition

$$G_n = 0. \tag{1}$$

In fact it is not needed to assume such a relation. For example, to fix parameters in algebraic relations for products of two propagators

$$R_2(k_1, p_1, p_2, m_1^2, m_2^2, m_0^2) = \frac{1}{D_1 D_2} - \frac{x_1}{D_2 D_0} - \frac{x_2}{D_1 D_0} = 0, \tag{2}$$

instead of (1) we can impose conditions

$$\frac{\partial x_1}{\partial k_{1\mu}} = \frac{\partial x_2}{\partial k_{1\mu}} = 0. \tag{3}$$

Multiplying both sides of (2) by $D_0 D_1 D_2$ we get

$$\begin{aligned} & D_0 - x_1 D_1 - x_2 D_2 \\ & = (1 - x_1 - x_2) k_1^2 + 2x_1 k_1 p_1 + 2x_2 k_1 p_2 \\ & + x_1 m_1^2 + x_2 m_2^2 - x_1 p_1^2 - x_2 p_2^2 - 2k_1 p_3 - m_3^2 + p_3^2. \end{aligned} \tag{4}$$

Differentiating this relation, contracting with k_1, p_1, p_2, p_3 and taking into account (3) gives several equations:

$$
\begin{aligned}
&-2x_1(k_1^2 - k_1p_1) - 2x_2(k_1^2 - k_1p_2) + 2k_1^2 - 2k_1p_3 = 0,\\
&2(1 - x_1 - x_2)k_1p_1 + 2x_1p_1^2 + 2x_2p_1p_2 - 2p_1p_3 = 0,\\
&2(1 - x_1 - x_2)k_1p_2 + 2p_1p_2x_1 + 2x_2p_2^2 - 2p_2p_3 = 0,\\
&2(1 - x_1 - x_2)k_1p_3 + 2x_1p_1p_3 + 2x_2p_2p_3 - 2p_3^2 = 0.
\end{aligned}
\tag{5}
$$

They can be used to express $k_1p_3,\ p_1p_3,\ p_2p_3,\ x_1,\ x_2$ in terms of $k_1^2,\ k_1p_1,\ k_1p_2,\ p_1^2,\ p_1p_2,\ p_2^2$ considered to be independent variables.
For example, we get:

$$
k_1p_3 = x_1k_1p_1 - x_1k_1p_2 + k_1p_2 + \frac{x_1}{2}(m_1^2 - m_2^2 - p_1^2 + p_2^2)
+\frac{1}{2}(m_2^2 - p_2^2 - m_3^2 + p_3^2),
$$

and similar expressions for other scalar products of p_3. Solution for $x_1,\ x_2$ is the same as it was before and as a result we obtained the same relation between products of two propagators.

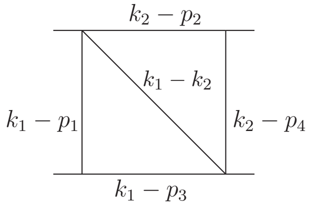
- Solution of the above system of equations is rather similar to finding intersections of two plane curves considered in Abel's theorem.
- Similar to usual algebraic integrals for one variable we can construct various integrands out of our different relationships for products of propagator. These integrands will be rational functions in independent variables.
- Integrations should be done over d dimensional space. Rational function must resemble integrands for Feynman integrands.

Similar to usual algebraic integrals for Feynman integrals we can construct various integrands out of obtained algebraic relations

For example, integrating product of different relationships between two propagators multiplied by $1/[(k_1 - k_2)^2 - m_5^2]^{\nu_5}$ with respect to k_1, k_2 leads to the FE for the integral

$$\int \frac{d^d k_1 d^d k_2}{[(k_1 - k_2)^2 - m_5^2]^{\nu_5}} R_2(k_1, p_1, p_2, m_1^2, m_2^2, m_0^2) R_2(k_2, p_3, p_4, m_3^2, m_4^2, \widetilde{m}_0^2) = 0$$

corresponding to the following diagram



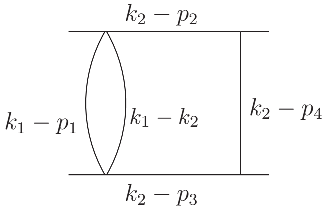
By integrating product of the relationship

$$R_3(k_1, p_1, p_2, p_3, m_1^2, m_2^2, m_3^2, m_0^2) = \frac{1}{D_1 D_2 D_3} - \frac{x_1}{D_2 D_3 D_0} - \frac{x_2}{D_1 D_3 D_0} - \frac{x_3}{D_1 D_2 D_0} = 0,$$

and one loop-propagator integral

$$\int\int \frac{d^d k_1 d^d k_2}{[k_1^2 - m_1^2]^{\nu_1}[(k_1 - k_2)^2 - m_5^2]^{\nu_5}} R_3(k_2, p_2, p_3, p_4, m_2^2, m_3^2, m_4^2) = 0,$$

we obtain the FE for the integral corresponding to the following diagram with arbitrary ν_1, ν_5, and arbitrary momenta and masses



Concluding remarks

- Two different methods for finding FE for Feynman integrals with any number of loops and external legs were discussed.
- FE reduce integrals with complicated kinematics to simpler integrals
- FE can be used for analytic continuation of Feynman integrals without knowing explicit analytic result.
- Application of these methods for some two- and three- loop integrals is in progress
- Systematic investigation of FE for Feynman integrals based on algebraic geometry and group theory is needed.
- Some improvements of these methods can be done by exploiting known methods for algebraic integrals.
- The methods can be extended for finding functional equations among hypergeometric as well as holonomic functions.