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NSVZ relation in supersymmetric theories regularized by higher derivatives

In $\mathcal{N} = 1$ supersymmetric theories the β-function is related to the anomalous dimension of the matter superfields by the equation

$$\beta(\alpha) = -\frac{\alpha^2\Big(3C_2 - T(R) + C(R)_i{}^j\gamma_j{}^i(\alpha)/r\Big)}{2\pi(1-C_2\alpha/2\pi)},\qquad \text{where}$$

$$\text{tr}\,(T^A T^B) \equiv T(R)\,\delta^{AB};\qquad (T^A)_i{}^k (T^A)_k{}^j \equiv C(R)_i{}^j;$$
$$f^{ACD} f^{BCD} \equiv C_2\delta^{AB};\qquad r \equiv \delta_{AA}.$$

V.Novikov, M.A.Shifman, A.Vainshtein, V.I.Zakharov, Nucl.Phys. **B 229** (1983) 381; Phys.Lett. **B 166** (1985) 329; M.A.Shifman, A.I.Vainshtein, Nucl.Phys. **B 277** (1986) 456; D.R.T.Jones, Phys.Lett. **B 123** (1983) 45.

The NSVZ β-function was obtained from different arguments: instantons, anomalies etc.

NSVZ β-function and calculations in the lowest loops

The NSVZ β-function can be compared with the results of calculations in the lowest orders of the perturbation theory. To make such calculations, a theory should be regularized.

The dimensional regularization breaks the supersymmetry and is not convenient for calculations in supersymmetric theories. That is why supersymmetric theories are mostly regularized by the dimensional reduction. However, the dimensional reduction is not self-consistent.

W.Siegel, Phys.Lett. **B 84** (1979) 193; **B 94** (1980) 37.

Using the dimensional reduction and $\overline{\text{DR}}$-scheme the β-function of $\mathcal{N}=1$ supersymmetric theories was calculated up to the four-loop approximation:

L.V.Avdeev, O.V.Tarasov, Phys.Lett. **B 112** (1982) 356; I.Jack, D.R.T.Jones, C.G.North, Phys.Lett. **B 386** (1996) 138; Nucl.Phys. **B 486** (1997) 479; R.V.Harlander, D.R.T.Jones, P.Kant, L.Mihaila, M.Steinhauser, JHEP **0612** (2006) 024.

The result coincides with the NSVZ β-function only in the one- and two-loop approximations. In the higher loops it is necessary to make a special tuning of the coupling constant.

We argue that the NSVZ relation is naturally obtained in the case of using the higher covariant derivative regularization

A.A.Slavnov, Nucl.Phys., **B31**, (1971), 301; Theor.Math.Phys. **13** (1972) 1064.

which is consistent and does not break supersymmetry:

V.K.Krivoshchekov, Theor.Math.Phys. **36** (1978) 745;
P.West, Nucl.Phys. B268, (1986), 113.

It can be also used for theories with $\mathcal{N}=2$ supersymmetry

V.K.Krivoshchekov, Phys.Lett. B **149** (1984) 128; I.L.Buchbinder, K.S., Nucl.Phys. **B883** (2014) 20; I.L.Buchbinder, N.G.Pletnev, K.S., Phys.Lett. **B751** (2015) 434.

For $\mathcal{N}=1$ SQED, regularized by higher derivatives, the NSVZ relation has been obtained by explicit summation of supergraphs

K.S., Nucl.Phys. **B 852** (2011) 71; JHEP **1408** (2014) 096.

Generalization of this result to the case of using the dimensional reduction is a complicated and so far unsolved problem

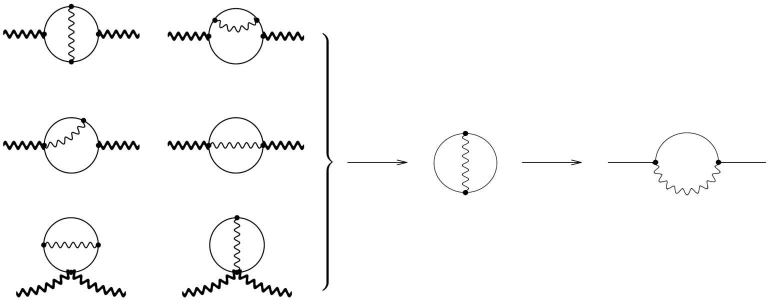
S.S.Aleshin, A.L.Kataev, K.S., JETP Lett. **103** (2016) 77.

Qualitative picture for $\mathcal{N}=1$ SQED

A.V.Smilga, A.I.Vainshtein, Nucl.Phys. **B 704** (2005) 445.

The NSVZ β-function for $\mathcal{N}=1$ SQED with N_f flavours has the form

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = \frac{N_f}{\pi}\Big(1-\gamma(\alpha_0)\Big).$$



Let us consider the $\mathcal{N}=1$ SYM theory described by the action

$$S = \frac{1}{2e_0^2}\,\mbox{Re}\,\mbox{tr}\int d^4x\,d^2\theta\,W^a W_a + \frac{1}{4}\int d^4x\,d^4\theta\,\phi^{*i}(e^{2V})_i{}^j\phi_j$$
$$+\Big\{\int d^4x\,d^2\theta\,\Big(\frac{1}{4}m_0^{ij}\phi_i\phi_j + \frac{1}{6}\lambda_0^{ijk}\phi_i\phi_j\phi_k\Big) + \mbox{c.c.}\Big\},$$

where the supersymmetric gauge field strength is defined as

$$W_a = \frac{1}{8}\bar D^2\left(e^{-2V}D_a e^{2V}\right).$$

We assume that the theory is invariant under the gauge transformations

$$\phi\to e^A\phi;\qquad e^{2V}\to e^{-A^+}e^{2V}e^{-A},$$

where the parameter $A = ie_0 A^B T^B$ is an arbitrary chiral superfield.

Quantum-background splitting is made by the substitution

$$e^{2V} \to e^{\mathbf{\Omega}^+} e^{2V} e^{\mathbf{\Omega}}.$$

The background superfield $\boldsymbol{V}$ is defined by $e^{2\boldsymbol{V}} = e^{\mathbf{\Omega}^+} e^{\mathbf{\Omega}}$.
We choose the following higher derivative term

$$S_\Lambda = \frac{1}{2e_0^2}\,\mathrm{Re}\,\mathrm{tr}\int d^4x\, d^2\theta\, e^{\Omega} e^{\mathbf{\Omega}} W^a e^{-\mathbf{\Omega}} e^{-\Omega} \Big[R\Big(-\frac{\bar\nabla^2\nabla^2}{16\Lambda^2}\Big) - 1\Big]_{Adj}$$
$$\times e^{\Omega} e^{\mathbf{\Omega}} W_a e^{-\mathbf{\Omega}} e^{-\Omega} + \frac{1}{4}\int d^4x\, d^4\theta\, \phi^+ e^{\mathbf{\Omega}^+} e^{\Omega^+}\Big[F\Big(-\frac{\bar\nabla^2\nabla^2}{16\Lambda^2}\Big) - 1\Big] e^{\Omega} e^{\mathbf{\Omega}}\phi,$$

and the gauge fixing term

$$S_{\mathrm{gf}} = \frac{1}{e_0^2}\mathrm{tr}\int d^4x\, d^4\theta\, \Big(16\xi_0\, f^+ \Big[e^{\mathbf{\Omega}^+} K^{-1}\Big(-\frac{\bar{\boldsymbol{\nabla}}^2\boldsymbol{\nabla}^2}{16\Lambda^2}\Big) e^{\mathbf{\Omega}}\Big]_{Adj} f$$
$$+ e^{\mathbf{\Omega}} f e^{-\mathbf{\Omega}} \boldsymbol{\nabla}^2 V + e^{-\mathbf{\Omega}^+} f^+ e^{\mathbf{\Omega}^+} \bar{\boldsymbol{\nabla}}^2 V\Big),$$

where the regulators R, F, and K have a rapid growth at infinity.

Actions for the Faddeev–Popov and Nielsen–Kallosh ghosts have the form

$$S_{\text{FP}} = \frac{1}{e_0^2}\text{tr}\int d^4x\, d^4\theta\, \Big(e^{\mathbf{\Omega}}\bar{c}e^{-\mathbf{\Omega}} + e^{-\mathbf{\Omega}^+}\bar{c}^+e^{\mathbf{\Omega}^+}\Big)$$
$$\times\Big\{\Big(\frac{V}{1-e^{2V}}\Big)_{Adj}\Big(e^{-\mathbf{\Omega}^+}c^+e^{\mathbf{\Omega}^+}\Big) + \Big(\frac{V}{1-e^{-2V}}\Big)_{Adj}\Big(e^{\mathbf{\Omega}}ce^{-\mathbf{\Omega}}\Big)\Big\};$$
$$S_{\text{NK}} = \frac{1}{2e_0^2}\text{tr}\int d^4x\, d^4\theta\, b^+\Big[e^{\mathbf{\Omega}^+}K\Big(-\frac{\bar{\mathbf{\nabla}}^2\mathbf{\nabla}^2}{16\Lambda^2}\Big)e^{\mathbf{\Omega}}\Big]_{Adj} b.$$

The remaining one-loop divergences are removed by inserting the Pauli–Villars determinants into the generating functional

A.A.Slavnov, Theor.Math.Phys. **33** (1977) 977.

$$Z[\mathbf{V},\text{Sources}] = \int DV\, D\phi\, Db\, D\bar{c}\, Dc \prod_I \text{Det}(PV, M_I)^{c_I}\,\text{Det}(PV, M_\varphi)^{-1}$$
$$\times\exp\Big(iS + iS_\Lambda + iS_{\text{gf}} + iS_{\text{FP}} + iS_{\text{NK}} + iS_{\text{source}}\Big),$$

In our notation the renormalization constants are defined by the equations

$$\frac{1}{\alpha_0} = \frac{Z_\alpha}{\alpha}; \qquad \frac{1}{\xi_0} = \frac{Z_\xi}{\xi}; \qquad \boldsymbol{V} = \boldsymbol{V}_R; \qquad V = Z_V Z_\alpha^{-1/2} V_R;$$

$$b = \sqrt{Z_b}\, b_R; \qquad \bar{c}c = Z_c Z_\alpha^{-1} \bar{c}_R c_R; \qquad \phi_i = (\sqrt{Z_\phi})_i{}^j (\phi_R)_j;$$

$$m^{ij} = m_0^{mn} (Z_m)_m{}^i (Z_m)_n{}^j; \qquad \lambda^{ijk} = \lambda_0^{mnp} (Z_\lambda)_m{}^i (Z_\lambda)_n{}^j (Z_\lambda)_p{}^k.$$

The subscript R denotes renormalized superfields, α, λ, and ξ are the renormalized coupling constant, the Yukawa couplings, and the gauge parameter, respectively; m denotes renormalized masses.
It is possible to impose the following constrains to these renormalization constants:

$$(Z_m)_i{}^j = (Z_\lambda)_i{}^j = (\sqrt{Z_\phi})_i{}^j; \qquad Z_\xi = Z_V^{-2}; \qquad Z_b = Z_\alpha^{-1}.$$

Non-renormalization of the vertices with two ghost legs and one leg of the quantum gauge superfield

The three-point vertices with two ghost legs and a single leg of the quantum gauge superfield are finite in all orders.

K.S., Nucl.Phys. **B909** (2016) 316.

There are 4 such vertices, $\bar{c}Vc$, $\bar{c}^+Vc$, $\bar{c}Vc^+$, and $\bar{c}^+Vc^+$.
They have the same renormalization constant $Z_\alpha^{-1/2} Z_c Z_V$. Therefore, the above statement can be rewritten as

$$\frac{d}{d\ln\Lambda}(Z_\alpha^{-1/2} Z_c Z_V) = 0.$$

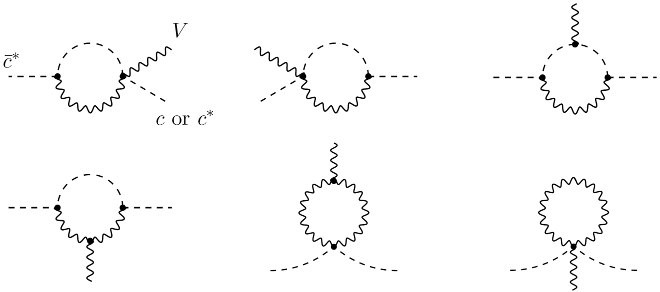
In the one-loop approximation this has first been noted in the paper

S.S.Aleshin, A.E.Kazantsev, M.B.Skopsov, K.S., JHEP **1605** (2016) 014.

Consequently, there is a subtraction scheme in which

$$-\frac{1}{2}\ln Z_\alpha + \ln Z_c + \ln Z_V = 0.$$

Important: Below we will demonstrate that Z_c is divergent. Therefore, The Green functions of the structure $\bar{c}V^n c$ are divergent for $n \neq 1$.



$$\frac{ie_0}{4} f^{ABC} \int d^4\theta\, \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \bar{c}^{*A}(\theta, p+q) \Big(f(p,q) \partial^2 \Pi_{1/2} V^B(\theta,-p)$$

$$+ F_\mu(p,q) (\gamma^\mu)_{\dot a}{}^b D_b \bar{D}^{\dot a} V^B(\theta,-p) + F(p,q) V^B(\theta,-p) \Big) c^C(\theta,-q);$$

$$\frac{ie_0}{4} f^{ABC} \int d^4\theta\, \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \bar{c}^{*A}(\theta, p+q) \widetilde{F}(p,q) V^B(\theta,-p) c^{*C}(\theta,-q).$$

Calculating these diagrams gives

$$F(p,q) = 1 + \frac{e_0^2 C_2}{4}\int \frac{d^4k}{(2\pi)^4}\Big\{ -\frac{(q+p)^2}{R_k k^2 (k+p)^2 (k-q)^2} - \frac{\xi_0\, p^2}{K_k k^2 (k+q)^2 (k+q+p)^2}$$
$$+\frac{\xi_0\, q^2}{K_k k^2 (k+p)^2 (k+q+p)^2} + \Big(\frac{\xi_0}{K_k} - \frac{1}{R_k}\Big)\Big(-\frac{2(q+p)^2}{k^4 (k+q+p)^2} + \frac{2}{k^2 (k+q+p)^2}$$
$$-\frac{1}{k^2 (k+q)^2} - \frac{1}{k^2 (k+p)^2}\Big)\Big\} + O(\alpha_0^2, \alpha_0\lambda_0^2).$$

$$\widetilde{F}(p,q) = 1 - \frac{e_0^2 C_2}{4}\int \frac{d^4k}{(2\pi)^4}\Big\{ \frac{p^2}{R_k k^2 (k+q)^2 (k+q+p)^2} + \frac{\xi_0\, (q+p)^2}{K_k k^2 (k-p)^2 (k+q)^2}$$
$$+\frac{\xi_0\, q^2}{K_k k^2 (k+p)^2 (k+q+p)^2} + \frac{2\xi_0}{K_k k^2 (k+p)^2} - \frac{2\xi_0}{K_k k^2 (k+q+p)^2} + \Big(\frac{\xi_0}{K_k} - \frac{1}{R_k}\Big)$$
$$\times\Big(\frac{2q^2}{k^4 (k+q)^2} + \frac{1}{k^2 (k+q+p)^2} - \frac{1}{k^2 (k+q)^2}\Big)\Big\} + O(\alpha_0^2, \alpha_0\lambda_0^2).$$

We see that these expressions are finite in the ultraviolet region.

RG functions defined in terms of the bare couplings

In terms of the bare coupling constants the RG functions are defined by

$$\beta(\alpha_0,\lambda_0) \equiv \frac{d\alpha_0}{d\ln\Lambda}; \qquad (\gamma_\phi)_i{}^j(\alpha_0,\lambda_0) \equiv -\frac{d\ln(Z_\phi)_i{}^j(\alpha,\lambda,\Lambda/\mu)}{d\ln\Lambda};$$

$$\gamma_V(\alpha_0,\lambda_0) \equiv -\frac{d\ln Z_V(\alpha,\lambda,\Lambda/\mu)}{d\ln\Lambda}; \qquad \gamma_c(\alpha_0,\lambda_0) \equiv -\frac{d\ln Z_c(\alpha,\lambda,\Lambda/\mu)}{d\ln\Lambda},$$

where the differentiation is made at fixed values of α and λ^{ijk}.
There renormalization group functions are

1. scheme independent at a fixed regularization;
2. depend on a regularization.

We will assume that the RG functions defined in terms of the bare couplings satisfy the NSVZ relation in the case of using the higher covariant derivative regularization similarly to the Abelian case

$$\beta(\alpha_0) = -\frac{\alpha_0^2\Big(3C_2 - T(R) + C(R)_i{}^j\gamma_j{}^i(\alpha_0)/r\Big)}{2\pi(1-C_2\alpha_0/2\pi)}.$$

The NSVZ β-function can be equivalently rewritten in the form

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{3C_2 - T(R) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r}{2\pi} + \frac{C_2}{2\pi}\cdot\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0}.$$

Let us express the β-function in the right hand side in terms of the renormalization constant Z_α:

$$\beta(\alpha_0,\lambda_0) = \frac{d\alpha_0(\alpha,\lambda,\Lambda/\mu)}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}} = -\alpha_0\frac{d\ln Z_\alpha}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}}.$$

Then, using the identity $d(Z_\alpha^{-1/2}Z_V Z_c)/d\ln\Lambda = 0$ we obtain

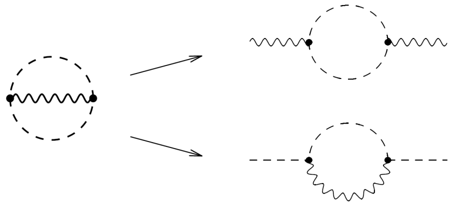
$$\beta(\alpha_0,\lambda_0) = -2\alpha_0\frac{d\ln(Z_c Z_V)}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}} = 2\alpha_0\Big(\gamma_c(\alpha_0,\lambda_0)+\gamma_V(\alpha_0,\lambda_0)\Big),$$

where γ_c and γ_V are anomalous dimensions of the Faddeev–Popov ghosts and of the quantum gauge superfield (defined in terms of the bare coupling constants), respectively.

Substituting this expression into the right hand side of the NSVZ relation we obtain

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R) - 2C_2\gamma_c(\alpha_0,\lambda_0)$$
$$-2C_2\gamma_V(\alpha_0,\lambda_0) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r\Big).$$

From this form of the NSVZ β-function we see that the matter superfields and ghosts similarly contribute to the right hand side.



Renormalization group functions defined in terms of the renormalized couplings

The RG functions are defined in terms of the renormalized couplings by the equations

$$\widetilde{\beta}(\alpha,\lambda) \equiv \frac{d\alpha}{d\ln\mu};$$

$$(\widetilde{\gamma}_\phi)_i{}^j(\alpha,\lambda) \equiv -\frac{d\ln(Z_\phi)_i{}^j(\alpha_0,\lambda_0,\Lambda/\mu)}{d\ln\mu};$$

$$\widetilde{\gamma}_V(\alpha,\lambda) \equiv -\frac{d\ln Z_V(\alpha_0,\lambda_0,\Lambda/\mu)}{d\ln\mu};$$

$$\widetilde{\gamma}_c(\alpha,\lambda) \equiv -\frac{d\ln Z_c(\alpha_0,\lambda_0,\Lambda/\mu)}{d\ln\mu},$$

where the differentiation is made at fixed values of α_0 and λ_0^{ijk}.
There renormalization group functions are
1. scheme and regularization dependent;
2. satisfy the NSVZ relation only for a special renormalization prescription, called the NSVZ scheme.

The RG functions defined in terms of the renormalized coupling constant are scheme dependent and satisfy the NSVZ relation only in a certain subtraction scheme. Similarly to

A.L.Kataev and K.S., Nucl.Phys. **B875** (2013) 459; Phys.Lett. **B730** (2014) 184.

we see that in the non-Abelian case the RG functions defined in terms of the bare coupling constant coincide with ones defined in terms of the renormalized coupling constants if the boundary conditions

$$Z_\alpha(\alpha,\lambda,x_0) = 1; \qquad (Z_\phi)_i{}^j(\alpha,\lambda,x_0) = \delta_i{}^j; \qquad Z_c(\alpha,\lambda,x_0) = 1,$$

where x_0 is a fixed value of $\ln\Lambda/\mu$, are imposed on the renormalization constants. (For $x_0 = 0$ we obtain minimal subtractions.) We also assume that the renormalization constants satisfy the equation

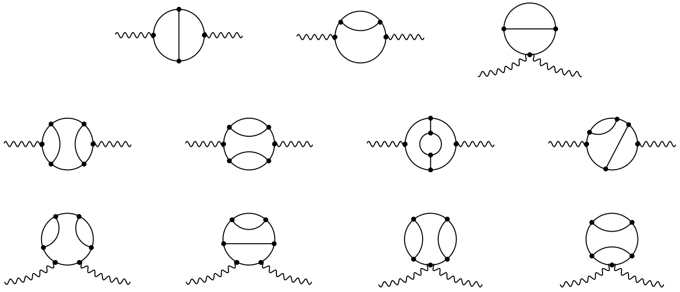
$$Z_V = Z_\alpha^{1/2} Z_c^{-1},$$

Possibly, these conditions give the NSVZ scheme with the higher covariant derivative regularization.

To verify the above results we consider the three-loop terms quartic in the Yukawa couplings. They correspond to the graphs

V.Yu.Shakhmanov, K.S., Nucl.Phys., **B920**, (2017), 345.

Attaching two external lines of the background gauge superfield we obtain the diagrams contributing to the β-function.



The corresponding contribution to the anomalous dimension is given by the diagrams

The calculation gives the following result:

$$\frac{\Delta\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{2\pi}{r}C(R)_i{}^j\frac{d}{d\ln\Lambda}\int\frac{d^4k}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\,\lambda_0^{imn}\lambda^*_{0jmn}\frac{\partial}{\partial q_\mu}\frac{\partial}{\partial q^\mu}$$
$$\times\Big(\frac{1}{k^2F_k\,q^2F_q\,(q+k)^2F_{q+k}}\Big) + \frac{4\pi}{r}C(R)_i{}^j\frac{d}{d\ln\Lambda}\int\frac{d^4k}{(2\pi)^4}\frac{d^4l}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}$$
$$\times\left(\lambda_0^{iab}\lambda^*_{0kab}\lambda_0^{kcd}\lambda^*_{0jcd}\Big(\frac{\partial}{\partial k_\mu}\frac{\partial}{\partial k^\mu}-\frac{\partial}{\partial q_\mu}\frac{\partial}{\partial q^\mu}\Big)+2\lambda_0^{iab}\lambda^*_{0jac}\lambda_0^{cde}\lambda^*_{0bde}\frac{\partial}{\partial q_\mu}\frac{\partial}{\partial q^\mu}\right)$$
$$\times\frac{1}{k^2F_k^2\,q^2F_q\,(q+k)^2F_{q+k}\,l^2F_l\,(l+k)^2F_{l+k}} = -\frac{1}{2\pi r}C(R)_i{}^j\Delta\gamma_\phi(\lambda_0)_j{}^i.$$

The simplest regulator has the form $F(k^2/\Lambda^2) = 1 + k^2/\Lambda^2$. In this case

$$\Delta\gamma_\phi(\alpha_0,\lambda_0)_j{}^i = \frac{1}{4\pi^2}\lambda_0^{iab}\lambda^*_{0jab} - \frac{1}{16\pi^4}\lambda_0^{iab}\lambda^*_{0jac}\lambda_0^{cde}\lambda^*_{0bde};$$

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R)\Big) - \frac{1}{2\pi r}C(R)_i{}^j\Delta\gamma_\phi(\lambda_0)_j{}^i + O(\alpha_0) + O(\lambda_0^6).$$

and the NSVZ relation is valid for the RG functions defined in terms of the bare couplings with the HD regularization.

$$(\ln Z_\phi)_j{}^i = -\frac{1}{4\pi^2}\lambda_0^{iab}\lambda^*_{0jab}\Big(\ln\frac{\Lambda}{\mu} + g_1\Big) + \frac{1}{32\pi^4}\lambda_0^{iab}\lambda^*_{0kab}\lambda_0^{kcd}\lambda^*_{0jcd}\Big(\ln^2\frac{\Lambda}{\mu}$$
$$+2g_1\ln\frac{\Lambda}{\mu} + 2g_1^2 - \widetilde{g}_2\Big) + \frac{1}{16\pi^4}\lambda_0^{iab}\lambda^*_{0jac}\lambda_0^{cde}\lambda^*_{0bde}\Big(\ln\frac{\Lambda}{\mu} + \ln^2\frac{\Lambda}{\mu} + 2g_1\ln\frac{\Lambda}{\mu}$$
$$+2g_1^2 - g_2\Big) + O(\alpha_0) + O(\lambda_0^6);$$

$$\frac{1}{\alpha_0} = \frac{1}{\alpha} + \frac{1}{2\pi}\Big(3C_2 - T(R)\Big)\Big(\ln\frac{\Lambda}{\mu} + b_1\Big) + \frac{1}{2\pi r}C(R)_i{}^j\Big[\frac{1}{4\pi^2}\lambda^{iab}\lambda^*_{jab}\Big(\ln\frac{\Lambda}{\mu} + b_2\Big)$$
$$+\frac{1}{32\pi^4}\lambda^{iab}\lambda^*_{kab}\lambda^{kcd}\lambda^*_{jcd}\Big(\ln^2\frac{\Lambda}{\mu} + 2g_1\ln\frac{\Lambda}{\mu} + \widetilde{b}_3\Big) + \frac{1}{16\pi^4}\lambda^{iab}\lambda^*_{jac}\lambda^{cde}\lambda^*_{bde}\Big(-\ln\frac{\Lambda}{\mu}$$
$$+\ln^2\frac{\Lambda}{\mu} + 2g_1\ln\frac{\Lambda}{\mu} + b_3\Big)\Big] + O(\alpha) + O(\lambda^6).$$

The RG functions defined in terms of the renormalized couplings are

$$\widetilde{\gamma}_\phi(\alpha,\lambda)_j{}^i = \frac{1}{4\pi^2}\lambda^{iab}\lambda^*_{jab} - \frac{1}{16\pi^4}\lambda^{iab}\lambda^*_{jac}\lambda^{cde}\lambda^*_{bde} + O(\alpha) + O(\lambda^6).$$

$$\frac{\widetilde{\beta}(\alpha,\lambda)}{\alpha^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R)\Big) + \frac{1}{2\pi r}C(R)_i{}^j\Big[-\frac{1}{4\pi^2}\lambda^{iab}\lambda^*_{jab} + \frac{1}{16\pi^4}$$
$$\times\lambda^{iab}\lambda^*_{kab}\lambda^{kcd}\lambda^*_{jcd}\Big(b_2 - g_1\Big) + \frac{1}{16\pi^4}\lambda^{iab}\lambda^*_{jac}\lambda^{cde}\lambda^*_{bde}\Big(1 + 2b_2 - 2g_1\Big)\Big]$$
$$+O(\alpha) + O(\lambda^6).$$

We see that the considered part of this β-function is scheme-dependent. Imposing the boundary conditions

$$Z_\phi(\alpha,\lambda,x_0)_i{}^j = \delta_i{}^j; \qquad Z_\alpha(\alpha,\lambda,x_0) = \alpha/\alpha_0 = 1$$

we obtain $g_1 = b_1 = b_2 = -x_0$. Therefore, $b_2 - g_1 = 0.$

This implies that the NSVZ relation

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R) - 2C_2\gamma_c(\alpha_0,\lambda_0) -2C_2\gamma_V(\alpha_0,\lambda_0) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r\Big).$$

is really valid for the RG functions defined in terms of the renormalized couplings,

$$\frac{\widetilde\beta(\alpha,\lambda)}{\alpha^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R)\Big) + \frac{1}{2\pi r}C(R)_i{}^j\Big[-\frac{1}{4\pi^2}\lambda^{iab}\lambda^*_{jab} + \frac{1}{16\pi^4}\lambda^{iab}\lambda^*_{jac}\lambda^{cde}\lambda^*_{bde}\Big] + O(\alpha) + O(\lambda^6)$$
$$= -\frac{1}{2\pi}\Big(3C_2 - T(R)\Big) - \frac{1}{2\pi r}C(R)_i{}^j\widetilde\gamma_\phi(\alpha,\lambda)_i{}^j + O(\alpha) + O(\lambda^6).$$

Also we see that the NSVZ scheme is actually obtained with the higher covariant derivative regularization supplemented by minimal subtractions.

A NSVZ-like relation for the renormalization of the gaugino mass can be also written in theories with softly broken supersymmetry.

J.Hisano, M.A.Shifman, Phys.Rev., **D56**, (1997), 5475;
I.Jack, D.R.T.Jones, Phys.Lett. **B415**, (1997), 383;
L.V.Avdeev, D.I.Kazakov, I.N.Kondrashuk, Nucl.Phys. **B510** (1998) 289.

Here we will consider softly broken $\mathcal{N}=1$ SQED with N_f flavours, for which

$$\frac{\alpha m}{\widetilde\beta(\alpha)} = \mbox{RGI},$$

where m is the photino mass. In the NSVZ scheme

$$\widetilde\beta(\alpha) = \frac{\alpha^2 N_f}{\pi}\Big(1-\widetilde\gamma(\alpha)\Big).$$

Then the anomalous dimension of the photino mass can be related to the anomalous dimension of the matter superfields

$$\widetilde\gamma_m(\alpha) \equiv \frac{d\ln m}{d\ln\mu} = \frac{\alpha N_f}{\pi}\Big[1-\frac{d}{d\alpha}\Big(\alpha\widetilde\gamma(\alpha)\Big)\Big].$$

The NSVZ-like relation for the photino mass can be written in a simple form by the help of the θ-dependent coupling constant

$$A \equiv \alpha(1 + m\theta^2)$$

Then the equation $m\alpha/\widetilde{\beta}(\alpha) = \text{RGI}$ is encoded in the equation

$$\frac{dA}{d\ln\mu} = \widetilde{\beta}(A),$$

Using the NSVZ relation for the rigid theory we obtain

$$\frac{d}{d\ln\mu}\Big(\frac{m}{\alpha}\Big) = -\frac{m\alpha N_f}{\pi}\cdot\frac{d\widetilde{\gamma}(\alpha)}{d\alpha}.$$

This equation can be considered as θ^2-component of the NSVZ-like relation

$$\widetilde{\beta}(A) = \frac{A^2 N_f}{\pi}\Big(1 - \widetilde{\gamma}(A)\Big).$$

for the θ-dependent coupling constant A.

$$S=\frac{1}{4e_0^2}\mbox{Re}\int d^4x\,d^2\theta\,(1-2m_0\theta^2)W^aW_a+\frac{1}{4}\sum_{f=1}^{N_f}\int d^4x\,d^4\theta\,(1-\widetilde{m}^2_{\phi 0}\theta^4)$$
$$\times\Big(\phi_f^* e^{2V}\phi_f+\widetilde\phi_f^* e^{-2V}\widetilde\phi_f\Big)+\sum_{f=1}^{N_f}\Big(\frac{1}{2}\int d^4x\,d^2\theta\,m_{\phi 0}(1-\theta^2 b_0)\phi_f\widetilde\phi_f+\mbox{c.c.}\Big),$$

Here we are interested in the renormalization of the photino mass m in the limit when the parameters m_ϕ, $\widetilde{m}_\phi$, and b vanish. Then

$$S+S_\Lambda\to\frac{1}{4e_0^2}\mbox{Re}\int d^4x\,d^2\theta\,(1-2m_0\theta^2)W^a R(\partial^2/\Lambda^2)W_a$$
$$+\frac{1}{4}\sum_{f=1}^{N_f}\int d^4x\,d^4\theta\,\Big(\phi_f^* e^{2V}\phi_f+\widetilde\phi_f^* e^{-2V}\widetilde\phi_f\Big).$$

We choose the gauge fixing term

$$S_{\mbox{gf}}=-\frac{1}{32e_0^2}\int d^4x\,d^4\theta\,(1-m_0\theta^2-m_0\bar\theta^2)\,D^2V R(\partial^2/\Lambda^2)\bar D^2V.$$

The remaining one-loop divergences are removed by the Pauli–Villars determinants,

$$\mbox{Det}(PV,V,M_I)^{-1} \equiv \int D\Phi_I D\widetilde\Phi_I \exp\Big(\frac{i}{4}\int d^4x\, d^4\theta\, \Big(\Phi_I^* e^{2V}\Phi_I + \widetilde\Phi_I^* e^{-2V}\widetilde\Phi_I\Big) + \frac{i}{2}\int d^4x\, d^2\theta\, M_I\Phi_I\widetilde\Phi_I + \frac{i}{2}\int d^4x\, d^2\bar\theta\, M_I\Phi_I^*\widetilde\Phi_I^*\Big),$$

which are inserted into the generating functional

$$W = -i\ln\int DV\, D\phi\, D\widetilde\phi \prod_{I=1}^{m} \mbox{Det}(PV,V,M_I)^{N_f c_I} \times \exp\Big(iS + iS_\Lambda + iS_{\mbox{gf}} + iS_{\mbox{Sources}}\Big).$$

where $\sum_{I=0}^m c_I = 0$ with $c_0 \equiv -1$, and $M_I = a_I\Lambda$, $M_0 = 0$. The coefficients a_I are assumed to be independent of the coupling constant.

The anomalous dimension of the photino mass is related to the two-point Green function of the gauge superfield,

$$\Gamma^{(2)}_{\boldsymbol{V}} - S_{\text{gf}} = -\frac{1}{16\pi}\int \frac{d^4p}{(2\pi)^4}\, d^4\theta\, \Big(\boldsymbol{V}(-p,\theta)\partial^2\Pi_{1/2}\boldsymbol{V}(p,\theta)\, d^{-1}(\alpha_0,\Lambda/p)$$
$$-\frac{m_0}{8}\Big(\theta^2 D^a\boldsymbol{V}(-p,\theta)\bar{D}^2 D_a\boldsymbol{V}(p,\theta) + \bar{\theta}^2\bar{D}^{\dot{a}}\boldsymbol{V}(-p,\theta)D^2\bar{D}_{\dot{a}}\boldsymbol{V}(p,\theta)\Big)$$
$$\times d_m^{-1}(\alpha_0,\Lambda/p)\Big) + O(m_0^2),$$

The derivative of the function $m_0 d_m^{-1}(\alpha_0,\Lambda/p)$ with respect to $\ln\Lambda$ in the limit $p\to 0$ can be extracted by the substitution

$$\boldsymbol{V} \to \bar{\theta}^{\dot{a}}\bar{\theta}_{\dot{a}}\theta^b\psi_b \equiv \bar{\theta}^2\theta^b\psi_b.$$

Also it is possible to set formally $m_0\bar{\theta}^2\to 0$.
Introducing the notation $\mathcal{V}_\psi \equiv \int d^4x\,\psi^a\psi_a \to \infty$, we obtain

$$\frac{d\Delta\Gamma^{(2)}_{\boldsymbol{V}}}{d\ln\Lambda}\Big|_{\boldsymbol{V}=\bar{\theta}^2\theta^b\psi_b} = -\frac{1}{8\pi}\mathcal{V}_\psi\cdot\frac{d}{d\ln\Lambda}\Big(\frac{m_0}{d_m(\alpha_0,\Lambda/p)} - \frac{m_0}{\alpha_0}\Big)\Big|_{p=0}.$$

The NSVZ-like relation for the RG functions defined in terms of the bare coupling constant: double total derivatives

The RG function defined in terms of the bare coupling constant satisfy the NSVZ-like relation in all loops in the case of using the higher derivative regularization independently of the subtraction scheme.

I.V.Nartsev, K.S., JHEP **1704**, (2017), 047.

This is obtained by the method similar to the one for the rigid theory, see

K.S., Nucl.Phys. **B852**, (2011), 71.

The main idea is to prove that the anomalous dimension of the photino mass is given by integrals of double total derivatives in all loops:

$$\frac{d\Delta\Gamma^{(2)}_{\boldsymbol{V}}}{d\ln\Lambda}\Big|_{\boldsymbol{V}=\bar\theta^2\theta^b\psi_b} = \frac{i}{8}N_f^2\Big\langle\Big(\mbox{Tr}\sum_{I=0}^{m} c_I\bar\theta^{\dot a}(\gamma^\mu)_{\dot a}{}^b\psi_b\,\widetilde{Q}\big[x_\mu,\ln(\star_I)\big]\Big)^2\Big\rangle$$

$$-\frac{iN_f}{8}\frac{d}{d\ln\Lambda}\sum_{I=0}^{m}c_I\,\mbox{Tr}\Big\langle\bar\theta^2\psi^2\big[x^\mu,\big[x_\mu,\ln(\star)\big]\big]\Big\rangle_I - \mbox{singularities} + O(m_0^2),$$

where the operator $\star$ encodes sequences of vertices and propagators.

The NSVZ-like relation for the RG functions defined in terms of the bare coupling constant: summing singularities

The higher order contributions to NSVZ-like relation for the photino mass come from the singularities,

$$\Big[x^\mu, \frac{\partial_\mu}{\partial^4}\Big] = \Big[-i\frac{\partial}{\partial q_\mu}, -\frac{iq_\mu}{q^4}\Big] = -2\pi^2\delta^4(q_E) = -2\pi^2 i\delta^4(q).$$

Summing the singularities we obtain

$$\frac{d}{d\ln\Lambda}\Big(\frac{m_0}{\alpha_0}\Big) = -\frac{N_f}{\pi}\Big[m_0\alpha_0\frac{d\gamma(\alpha_0)}{d\alpha_0} - m_0\alpha_0^3\frac{d}{d\alpha_0}\Big(\frac{\beta(\alpha_0)}{\alpha_0^2}\Big)\frac{\partial}{\partial\alpha_0}\ln G(\alpha_0,\Lambda/q)$$
$$+\frac{d}{d\ln\Lambda}\Big(\frac{m_0}{\alpha_0}\Big)\alpha_0^2\frac{\partial}{\partial\alpha_0}\ln G(\alpha_0,\Lambda/q)\Big|_{q=0}\Big].$$

Using the NSVZ relation for the rigid theory and solving the resulting equation, we obtain the NSVZ-like relation for the renormalization of the photino mass,

$$\frac{d}{d\ln\Lambda}\Big(\frac{m_0}{\alpha_0}\Big) = -\frac{m_0\alpha_0 N_f}{\pi}\cdot\frac{d\gamma(\alpha_0)}{d\alpha_0}.$$

Explicitly calculating two-loop diagrams defining the renormalization of the photino mass we have verified that the anomalous dimension of the photino mass is given by integrals of double total derivatives

$$\frac{d}{d\ln\Lambda}\Big(\frac{m_0}{d_m(\alpha_0,\Lambda/p)}-\frac{m_0}{\alpha_0}\Big)\Big|_{p=0;\alpha,m=\text{const}} = 16\pi^2 m\alpha N_f\sum_{I=0}^{m} c_I$$

$$\times\int\frac{d^4k}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d}{d\ln\Lambda}\frac{\partial}{\partial q^\mu}\frac{\partial}{\partial q_\mu}\frac{1}{R_k k^2\,(q^2+M_I^2)\,((k+q)^2+M_I^2)}+O(\alpha^2).$$

This integral does not vanish due to singularities of the integrand for $I=0$,

$$\int\frac{d^4q}{(2\pi)^4}\frac{q^\mu}{q^4}\frac{\partial}{\partial q^\mu}f(q^2) = -\frac{1}{8\pi^2}f(0).$$

The result confirms the exact equation presented earlier,

$$\frac{d}{d\ln\Lambda}\Big(\frac{m_0}{\alpha_0}\Big) = 8m_0\alpha_0 N_f\frac{d}{d\ln\Lambda}\int\frac{d^4k}{(2\pi)^4}\frac{1}{R_k k^4}+O(\alpha_0^2)$$

$$= -\frac{m_0\alpha_0 N_f}{\pi}\cdot\frac{d\gamma(\alpha_0)}{d\alpha_0}+O(\alpha_0^2) = \frac{m_0\alpha_0 N_f}{\pi^2}+O(\alpha_0^2).$$

Let us fix a value x_0 of $\ln\Lambda/\mu$ and impose the boundary conditions

$$Z_3(\alpha, x_0) = 1; \qquad Z(\alpha, x_0) = 1; \qquad Z_m(\alpha, x_0) = 1$$

similarly to the rigid case. Then both definitions of the RG functions give the same result,

$$\widetilde\beta(\alpha) = \beta(\alpha_0)\Big|_{\alpha_0=\alpha}; \qquad \widetilde\gamma(\alpha) = \gamma(\alpha_0)\Big|_{\alpha_0=\alpha}; \qquad \widetilde\gamma_m(\alpha) = \gamma_m(\alpha_0)\Big|_{\alpha_0=\alpha}.$$

For example,

$$\begin{aligned}\widetilde\gamma_m\left(\alpha(\alpha_0,x)\right) &= -\frac{d}{dx}\ln Z_m\left(\alpha(\alpha_0,x),x\right)\\ &= -\frac{\partial\ln Z_m(\alpha,x)}{\partial\alpha}\cdot\frac{\partial\alpha(\alpha_0,x)}{\partial x} - \frac{\partial\ln Z_m(\alpha,x)}{\partial x}.\end{aligned}$$

In the point $x=x_0$

$$\frac{\partial\ln Z_m(\alpha,x)}{\partial\alpha}\Big|_{x=x_0} = 0,$$

and the first term vanishes.

Therefore, taking into account that $\alpha(\alpha_0, x_0) = \alpha_0$ we obtain

$$\widetilde{\gamma}_m(\alpha) = \gamma_m(\alpha_0)\Big|_{\alpha_0=\alpha}.$$

Consequently, the NSVZ-like relation is valid for the RG functions defined in terms of the renormalized coupling constant in all loops,

$$\frac{d}{d\ln\mu}\Big(\frac{m}{\alpha}\Big) = -\frac{m\alpha N_f}{\pi}\cdot\frac{d\widetilde{\gamma}(\alpha)}{d\alpha}.$$

Equivalently,

$$\widetilde{\beta}(A) = \frac{A^2 N_f}{\pi}\Big(1-\widetilde{\gamma}(A)\Big).$$

Thus, the NSVZ-like scheme for the softly broken $\mathcal{N}=1$ SQED regularized by higher derivatives is given by the boundary conditions which are similar to the rigid case. In the case $x_0 = 0$ they correspond to the minimal subtractions.

I.V.Nartsev, K.S., JETP Lett. **105**, (2017), 69.

- ✓ Using finiteness of the three-point vertices with two ghost legs and a single leg of the quantum gauge superfield, the NSVZ β-function can be rewritten in term of the anomalous dimensions of the quantum gauge superfield, of the Faddeev–Popov ghosts, and of the matter superfields. The resulting expression for the NSVZ β-function has a simple qualitative interpretation.
- ✓ The NSVZ scheme seems to be obtained with the higher covariant derivative regularization and minimal subtractions.
- ✓ The non-trivial three-loop calculation for the terms quartic in the Yukawa couplings confirms this proposal for the NSVZ scheme.
- ✓ The NSVZ-like relation for the renomalization of the photino mass can be obtained in all loops for the softly broken $\mathcal{N}=1$ SQED in the case of using the higher derivative regularization for the RG functions defined in terms of the bare coupling constant independently of the subtraction scheme.
- ✓ The NSVZ-like scheme for the softly broken $\mathcal{N}=1$ SQED is also HD+MS.

Thank you for the attention!