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Hyper-Kähler and HKT manifolds through supersymmetric glasses

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A review.

Also [F. Delduc + S. Fedoruk + E. Ivanov + A.S., in progress]

DE RHAM COMPLEX
(1933)

p-forms

$$A^{(p)} \;=\; A_{M_1 \ldots M_p} dx^{M_1} \wedge \cdots \wedge dx^{M_p}$$

Exterior derivative

$$B^{(p+1)} \;=\; dA^{(p)} \;=\; \partial_N A_{M_1 \ldots M_p} dx^N \wedge dx^{M_1} \wedge \cdots \wedge dx^{M_p}$$

- nilpotent: $d^2 = 0$

Its conjugate

$d^\dagger = *d*, \;\; d^\dagger A^{(p)} = C^{(p-1)}, \;\; (d^\dagger)^2 = 0$

$(*A^{(p)} \;=\; B^{(N-p)}$ — duality)

Algebra

$$d^2 = (d^\dagger)^2 = 0, \qquad dd^\dagger + d^\dagger d = \triangle^{\mathrm{cov}}$$

SUPERSYMMETRIC INTERPRETATION
[Witten, 1982]

• p-forms $\longrightarrow$ wave functions $\Psi(x^M, \psi^M)$ expanded in ψ^M.

$d \rightarrow Q,\ d^\dagger \rightarrow \bar{Q},\quad \triangle^{\rm cov} \rightarrow H.$

Supercharges

$$\begin{aligned} Q &= -i\psi^M \partial_M \\ \bar{Q} &= \bar{\psi}^M \left[-i\partial_M + i(\partial_M g_{NP})\,\psi^N \bar{\psi}^P\right] \end{aligned}$$

with

$\partial_M \equiv \frac{\partial}{\partial x^M},\ \bar{\psi}^M \ = \ g^{MQ}\frac{\partial}{\partial \psi^Q}.$

Allows for superspace description

Real superfields (1,2,1)

$$X^M(t;\theta,\bar{\theta}) \; = x^M + \theta\bar{\psi}^M + \psi^M\bar{\theta} + F^M\theta\bar{\theta}$$

Action

$$S = \tfrac{1}{2}\int dt d\theta d\bar{\theta}\, g_{MN}(X)\, DX^M \bar{D}X^N$$

with $D = \frac{\partial}{\partial\theta} + i\bar{\theta}\frac{\partial}{\partial t}, \qquad \bar{D} = -\frac{\partial}{\partial\bar{\theta}} - i\theta\frac{\partial}{\partial t}$

In components:

$$L = \frac{1}{2} g_{MN}\, \dot{x}^M \dot{x}^N + \text{ fermionic terms.}$$

- Invariant under SUSY transformations,

$$\delta x^M = \; \bar{\epsilon}\bar{\psi}^M - \epsilon\psi^M, \qquad \delta\psi^M = \; \bar{\epsilon}(i\dot{x}^M + F^M)$$

COMPLEX MANIFOLDS

An almost complex manifold is an even-dimensional manifold involving a complex structure tensor I_{MN} with the properties

$$I_{MN} = -I_{NM}, \qquad I_N{}^P I_P{}^M = -\delta_N^M$$

• An almost complex manifold is complex iff

$$\nabla_{[M} I_{N]P} = I_M{}^Q I_N{}^S \nabla_{[Q} I_{S]P}$$

(vanishing of the Nijenhuis tensor)

• Then complex coordinates z^m can be globally defined.

$$ds^2 \ = \ h_{m\bar{n}} dz^m d\bar{z}^{\bar{n}}$$

with Hermitian $h_{m\bar{n}}$.

MORE COMPLICATED COMPLEXES $\leftrightarrow$ EXTENDED SUPERSYMMETRIES

Definition:

A *Kähler manifold* *is a complex manifold with*

$$\nabla_P I_{MN} = \partial_P I_{MN} - \Gamma^S_{PM} I_{SN} - \Gamma^S_{PN} I_{MS} \; = \; 0\,.$$

- The Kähler form $I = I_{MN} dx^M \wedge dx^N$ is closed, $dI = 0$.

de Rham-Kähler complex

- Introduce $\partial, \bar{\partial}$ - holomorphic and antiholomorphic exterior derivatives.

They satisfy the algebra

$$\partial^2 = (\partial^\dagger)^2 = \bar{\partial}^2 = (\bar{\partial}^\dagger)^2 = 0,$$

$$\{\partial, \bar{\partial}^\dagger\} = \{\bar{\partial}, \partial^\dagger\} = 0, \; \{\partial, \partial^\dagger\} = \{\bar{\partial}, \bar{\partial}^\dagger\} = \triangle\,.$$

Supersymmetric interpretation

$$\partial, \partial^\dagger \;\to\; Q_1, Q_1^\dagger; \;\; \bar{\partial}, \bar{\partial}^\dagger \;\to\; Q_2, Q_2^\dagger; \;\; \triangle \to H\,.$$

$\mathcal{N} = 4$ supersymmetry

Superfield formulation

In terms of (1,2,1) superfields:

Extra “hidden” symmetry

$$\delta X^M = I^M{}_N(\epsilon\bar{D} - \bar{\epsilon}D)X^N$$

of the $\mathcal{N} = 2$ action

$S = \frac{1}{2}\int dt d\theta d\bar{\theta}\, g_{MN}(X)\, DX^M \bar{D}X^N$

In terms of (2,4,2) superfields:

- Introduce extended $\mathcal{N} = 4$ superspace $(t;\ \theta_\alpha, \bar{\theta}^\alpha)$.
- Introduce *chiral* superfields Z^m $(\bar{D}^\alpha Z^m = 0)$.

In chiral basis,

$$Z^m = z^m + \sqrt{2}\theta_\alpha\psi^{m\alpha} + \theta^2 F^m$$

The action:

$S = \int dt d^4\theta\, \mathcal{K}(\mathcal{Z}^m, \bar{\mathcal{Z}}^{\bar{m}})\,,$

where $\mathcal{K}$ — the Kähler potential.

- $h_{m\bar{n}} = \partial_m \bar{\partial}_{\bar{n}}\mathcal{K}$; Any Kähler metric can be obtained this way.

- Generic $\mathcal{N} = 4$ models may include gauge fields.

- Bi-Kähler manifolds. [Gates + Hull + Rocek, 1984; Fedoruk+A.S., 2016]

HYPER-KÄHLER MANIFOLDS

Definition: A HK manifold is a manifold admitting three different covariantly constant complex structures $I^{(1,2,3)} \equiv \{I, J, K\}$ satisfying the quaternionic algebra

$$I^a I^b = -\delta^{ab} + \epsilon^{abc} I^c \ .$$

• It has dimension $4n$. Locally, one can always choose coordinates where

$$I = \mathrm{diag}(\mathcal{I}, \ldots, \mathcal{I}), \quad J = \mathrm{diag}(\mathcal{J}, \ldots, \mathcal{J}),$$
$$K = \mathrm{diag}(\mathcal{K}, \ldots, \mathcal{K})$$

with self-dual

$$(\mathcal{I}, \mathcal{J}, \mathcal{K})_{MN} \ \equiv \ \eta^{a=1,2,3}_{MN}$$ ('t Hooft notation).

$$\mathcal{K} = \eta^3_{MN} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \text{etc.}$$

Theorem: [well known]

1. The holonomy group of an HK manifold of real dimension $4n$ is $Sp(2n) \subset SU(2n) \subset SO(4n)$.

2. A manifold with holonomy group $Sp(2n)$ is HK.

Explication:

• Choose $(I^a)_{MN} = \mathrm{diag}(\eta^a_{MN}, \ldots, \eta^a_{MN})$.

• Introduce $\Sigma_M = \mathrm{diag}(\sigma_\mu, \ldots, \sigma_\mu)$ and $\Sigma^\dagger_M = \mathrm{diag}(\sigma^\dagger_\mu, \ldots, \sigma^\dagger_\mu)$

with
$(\sigma_\mu)_{aj} = (\vec{\sigma}, i)_{aj}\ , \qquad (\sigma^\dagger_\mu)^{ja} = (\vec{\sigma}, -i)^{ja}.$

• Define

$$g_{aj,bk} = (\Sigma_M)_{aj} (\Sigma_N)_{bk}\, g^{MN}$$

Then in the flat case, $g_{aj,bk} = \Omega_{ab} \epsilon_{jk}$

($\Omega = \mathrm{diag}(i\sigma_2, \ldots, \sigma_2)$ — symplectic form)

• Indices are raised and lowered by $\Omega^{ab} = \Omega_{ab}$ and $\epsilon^{jk} = \epsilon_{jk}$.

• One can show that the curvature 2-form of an HK manifold is $R_{aj,bk} = \epsilon_{jk}(\tilde{R})_{(ab)}$.

Ergo $(\tilde{R})_a{}^b \in sp(2n)$.

• Inverse is also true.

$\mathcal{N} = 8$ supersymmetry

• 3 complex structures → 3 extra "hidden" complex supersymmetries

$$\delta X^M = (I^a)^M{}_N(\epsilon_a \bar{D} - \bar{\epsilon}_a D)X^N \tag{1}$$

for the action

$$S = \frac{1}{2}\int dt d\theta d\bar{\theta}\, g_{MN}(X)\, DX^M \bar{D} X^N\,. \tag{2}$$

Theorem: [Alvarez-Gaumé + Freedman, 1981]
The action (2) enjoys extra symmetries (1) iff the three matrices I^a have all the properties of quaternionic covariantly constant complex structures and hence the manifold is hyper-Kähler.

Superfield description

• Ordinary $\mathcal{N} = 8$ superspace: $(t;\ \theta_{\alpha j}, \bar{\theta}^{\alpha j})$, $\alpha, j = 1, 2$.

• Harmonic $\mathcal{N} = 8$ superspace: $(t;\ \theta^+_\alpha, \bar{\theta}^{+\alpha}, \theta^-_\alpha, \bar{\theta}^{-\alpha})$, $\theta^\pm_\alpha = \theta_{\alpha j} u^{\pm j}$ (u_j are the harmonics; $u^{+j} u^-_j = 1$).

• Analytic harmonic superspace: $\zeta \equiv (t;\, \theta^+_\alpha, \bar\theta^{+\alpha})$.

• Introduce $2n$ G-analytic superfields of harmonic charge +1:

$$Q^{+a}(\zeta, u) \;=\; X^{+a}(t,u) + \ldots + (\theta^+)^2(\bar\theta^+)^2 P^{-3a}(t,u)$$

(harmonic charge is an eigenvalue of
$D^0 = u^{+j}\frac{\partial}{\partial u^{+j}} - u^{-j}\frac{\partial}{\partial u^{-j}}$)

• The lowest term in the u-expansion of X^{+a} is related to the coordinates on the manifold: $X^{+a} = x^{ja}u^+_j + \ldots$.

Superfield action

[Galperin, Ivanov, Ogievetsky, Sokatchev, 1986)]

$$S = \int dtd^4\theta^+ \, [\Omega^{ab} Q^+_a D^{++} Q^+_b - 2\mathcal{L}^{+4}(Q^+, u)]$$

with

$$D^{++} = u^{+j} \frac{\partial}{\partial u^{-j}} - 2i\theta^{+\alpha}\bar{\theta}^+_\alpha \frac{\partial}{\partial t} \, .$$

• $\mathcal{L}^{+4}$ is an *arbitrary* function of harmonic charge +4.

Equations of motion

$$D^{++}Q^{+a} = \Omega^{ab} \frac{\partial \mathcal{L}^{+4}}{\partial Q^{+b}} \, .$$

• Their solution allows one to exclude auxiliary fields and derive the metric.

Theorem [GIOS, 1988]:
Any HK metric is obtained from an appropriate $\mathcal{L}^{+4}$.

[proven by explicitly resolving the constraints $R_{AB} \in sp(2n)$].

• Consraints are complicated. Can be analytically resolved only in rare cases.

Example 1. TAUB-NUT : $\mathcal{L}^{+4} = (Q_1^+)^2(Q_2^+)^2$

• At the end of the (rather long) day one can derive

$$ds^2 \;=\; \left(1+\frac{a}{r}\right)d\vec{x}^2 + \frac{a^2}{1+\frac{a}{r}}(d\Psi + \cos\theta d\phi)^2$$

Example 2. Eguchi-Hanson: $\mathcal{L}^{+4} \;=\; (u_1^+)^2(u_2^+)^2/(Q^{+a}u_a^-)^2$

gives

$$ds^2 \;=\; \frac{dr^2}{\left[1-\left(\frac{a}{r}\right)^4\right]} + r^2\left\{\sigma_x^2+\sigma_y^2+\sigma_z^2\left[1-\left(\frac{a}{r}\right)^4\right]\right\}$$

BI-HKT AND HKT MODELS

• They may be expressed in terms of N=1 (1,1,0) superfields

$$\mathcal{X}^M \ = \ x^M + \theta\psi^M$$

with real θ and real ψ^M.

• Half as much fermionic variables!

Consider a supersymmetric action

$$S \ = \ -\frac{i}{2}\int dtd\theta \ g_{MN}(\mathcal{X})\dot{\mathcal{X}}^M D\mathcal{X}^N - \frac{1}{12}\int dtd\theta \ C_{SPM} D\mathcal{X}^S D\mathcal{X}^P D\mathcal{X}^M \,, \qquad (3)$$

with

$$D \ = \ \frac{\partial}{\partial\theta} + i\theta\frac{\partial}{\partial t}$$

It gives the component Lagrangian

$$L = \frac{1}{2} g_{MN} \dot{x}^M \dot{x}^N - \frac{i}{2} g_{MN} \left(\dot{\psi}^M + \hat{\Gamma}^M_{PS} \dot{x}^P \psi^S \right) \psi^N - \frac{1}{12} \partial_K C_{SPM} \psi^K \psi^S \psi^P \psi^M \,,$$

where

$$\hat{\Gamma}^M_{PS} = \Gamma^M_{PS} + \frac{1}{2} g^{MQ} C_{QPS} \,.$$

- C_{QPS} is the torsion.

- Let the dimension of the manifold be $4n$.

Theorem [Gibbons+ Papadopoulos + Stelle, 1997]

The action (3) is invariant with respect to three extra real supersymmetries

$$\delta \mathcal{X}^M = \epsilon_a (I^a)^M{}_N D\mathcal{X}^N$$

iff I^a *are integrable complex structures satisfying certain extra conditions. One of them is the Clifford algebra*

$$I^a I^b + I^b I^a = -2\delta_{ab} \, .$$

CKT is bi-HKT !

• Consider 3 complex structures I^a satisfying the Clifford algebra

$$I^a I^b + I^b I^a \;=\; -2\delta^{ab}\,.$$

• $I^a I^b \neq \frac{1}{2}\epsilon^{abc} I^b I^c$

Theorem: [known to mathematicians]

The closure of multiplication algebra based on I^a represents a direct sum of two quaternion algebras $\mathcal{H}_+ + \mathcal{H}_-$.

Proof:

• Consider

$$J^a = \frac{1}{2}\epsilon^{abc} I^b I^c, \qquad \Delta = -I^1 J^1 = -I^2 J^2 = -I^3 J^3$$

• Convince yourselves that the algebras generated by $\{\Delta_+, I^a_+\}$ and $\{\Delta_-, I^a_-\}$, where $I^a_\pm = (I^a \pm J^a)/2$ and $\Delta_\pm = (1 \pm \Delta)/2$ are closed and are quaternionic.

Theorem: [Fedoruk + A.S., 2016]

The bi-HKT complex structures can be locally chosen as

$$
\begin{aligned}
I^1 &= \mathrm{diag}(\underbrace{\mathcal{I},\ldots,\mathcal{I}}_{n^*},\underbrace{-\mathcal{I},\ldots,-\mathcal{I}}_{m^*})\,,\\
I^2 &= \mathrm{diag}(\underbrace{\mathcal{J},\ldots,\mathcal{J}}_{n^*},\underbrace{-\mathcal{J},\ldots,-\mathcal{J}}_{m^*})\,,\\
I^3 &= \mathrm{diag}(\underbrace{\mathcal{K},\ldots,\mathcal{K}}_{n^*},\underbrace{-\mathcal{K},\ldots,-\mathcal{K}}_{m^*})\,.
\end{aligned}
$$

with

$$
\mathcal{K} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \text{etc.}
$$

- Two sectors.
- *Possibility to include gauge fields*

A particular case - HKT manifolds
[Howe + Papadopoulos, 1996]

• HKT — "hyper-Kähler with torsion" (in fact, not HK and not even K)

Remark:

• For a generic complex manifold, it is still true that $\tilde{\nabla}_P I_{MN} = 0$, if $\tilde{\nabla}_P$ includes torsions,

$$\tilde{\Gamma}^M_{NK} \;=\; \Gamma^M_{NK} \;+\frac{1}{2}g^{ML}C_{LNK}\,,$$

$C_{LNK} = -C_{LKN}$. Different choices for $\tilde{\nabla}_P$ are possible.

• There is only one such connection with totally antisymmetric C_{LNK}. It is called the Bismut connection. Explicitly,

$$C^{\rm Bis}_{LNK} = I_L{}^P I_N{}^R I_K{}^T \left(\nabla_P I_{RT} + \nabla_R I_{TP} + \nabla_T I_{PR}\right)\,.$$

Definition:

A HKT manifold is a $4n$*-dimensional manifold with three quaternionic complex structures, with the latter being covariantly constant with respect to the Bismut connection (required to be one and the same for all three complex structures)*

• The simplest example: conformally flat 4-dim manifold

$$ds^2 = \frac{(dx^M)^2}{f^2}$$

(e.g. the Hopf manifold $S^3 \times S^1$).

Theorem:

A $\mathcal{N} = 4$ *supersymmetric bi-HKT manifold with the complex structures satisfying the quaternionic algebra is HKT.*

Superfield description
[Delduc + Ivanov, 2012]

• Consider $\mathcal{N} = 4$ superspace $(t;\, \theta_j, \bar\theta^j)$ and harmonic analytic superspace $\zeta = (t;\, \theta^+, \bar\theta^+)$.

• Take $2n$ ("short") superfields

$$q^{+a}(\zeta,u) \;=\; f^{+a}(t,u) \;+\; \theta^+\chi^a(t,u) \;+\; \bar\theta^+\bar\chi^a(t,u) \\ +\; \theta^+\bar\theta^+A^{-a}(t,u)\,.$$

• Impose nonlinear constraints

$$D^{++}q^{+a} \;=\; \mathcal{L}^{+3a}(q^+,u)$$

with arbitrary $\mathcal{L}^{+3a}$.

• Write the action

$$S \;=\; \int dt du d^4\theta\, \mathcal{L}(q^{+a}, q^{-b}, u^{\pm})\,,$$

where $q^{-a} \;=\; D^{--}q^{+a}$

Theorem:

This system defines an HKT supersymmetric σ model.

Proof : (i) By explicit calculations of the metric and connections [Delduc+Ivanov];

(ii) From the fact that complex structures are quaternionic.

Conjecture:

Any HKT metric can be derived with an appropriate choice of $\mathcal{L}^{+3a}$ and $\mathcal{L}$.

A particular case: HK

Theorem:

Let

$$\mathcal{L}^{+3a} = \Omega^{ab} \frac{\partial \mathcal{L}^{+4}}{\partial q^{+b}}, \quad \mathcal{L} = q^{+a} q^{-}_{a} \, .$$

This system defines a generic HK supersymmetric σ model.

A particular case:

$$\mathcal{L}^{+3a} = \Omega^{ab}\frac{\partial \mathcal{L}^{+4}}{\partial q^{+b}}, \qquad \text{arbitrary} \;\; \mathcal{L}\,.$$

• A family of HKT models derived from a "stem" HK model.

• For $\mathcal{L}^{+3a} = 0$, one obtains the HKT models described by linear $(4,4,0)$ multiplets.

[Ivanov+Lechtenfeld, 2003; Fedoruk + A.S., 2015]

The "stem" HK metric is flat in this case.

Nontrivial $\mathcal{L}^{+3a}$

- The only known explicit example:
[Delduc+ Valient, 1993]

$$\mathcal{L}_1^{+3} = (q_1^+)^2(q_2^+)(1 - i\rho),$$
$$\mathcal{L}_2^{+3} = -(q_1^+)(q_2^+)^2(1 + i\rho)\,.$$

- $\rho = 0 \quad \Rightarrow$ TAUB-NUT.

- Group manifolds? [$SU(3)$ etc.]

Obata connection

Definition:

$D_M^{Obata} (I^a)_{NP} = 0$; *no torsion*

• Coincides with the Levy-Civita connection for the HK manifolds.

• Generically, $D_M^{Obata} g_{NP} \neq 0$. Hence, vector lengths are not preserved during parallel transport. A noncompact holonomy group.

• The Obata connection depends only on $\mathcal{L}^{+3a}$ and is the same for all HKT manifolds belonging to one and the same HK family.

Conjecture:

Any HKT supersymmetric sigma model with Obata-flat metric is expressible in terms of linear multiplets.