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Proca particle in Riemannian spacetimes

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Previous results: scalar and Dirac particles in Riemannian spacetimes

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Scalar particle in general inertial and gravitational fields and conformal invariance revisited

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1. **Exact Foldy-Wouthuysen transformation for a general static metric**
2. **Foldy-Wouthuysen Hamiltonian is conformally invariant for massless particles and conformally symmetric for massive ones**
3. **Proof of similarity of conformal transformations for scalar and Dirac particles**

$$\left(g^{\mu\nu}D_{\mu}D_{\nu} + m^2 - \lambda R\right)\psi = 0.$$

D_μ is the covariant derivative including the electromagnetic four-potential

$$D_{\mu}\psi = \left(\nabla_{\mu} + ieA_{\mu}\right)\psi = \left(\partial_{\mu} + ieA_{\mu}\right)\psi.$$

∇_μ is the covariant derivative (gravity)

Simple derivation leads to

$$\left[\frac{1}{\sqrt{-g}}(\partial_{\mu} + ieA_{\mu})\sqrt{-g}g^{\mu\nu}(\partial_{\nu} + ieA_{\nu}) + m^2 - \lambda R\right]\psi = 0.$$

After successive generalized Feshbach-Villars and Foldy-Wouthuysen transformations we obtain the Foldy-Wouthuysen Hamiltonian:

$$\mathcal{H}_{\rm FW} = \rho_3\epsilon - i\Upsilon' + eA_0$$

$$-\frac{1}{2\sqrt{\epsilon}}[\sqrt{\epsilon}, [\sqrt{\epsilon}, (i\partial_0 + i\Upsilon' - eA_0)]]\frac{1}{\sqrt{\epsilon}}$$

$$\varepsilon = \sqrt{D_i \frac{G^{ij}}{g^{00}} D_j + \frac{m^2}{g^{00}} + \Lambda},$$

$$G^{ij} = g^{ij} - \frac{g^{0i}g^{0j}}{g^{00}}, \qquad \Gamma^i = \sqrt{-g}\,g^{0i}.$$

$$\Upsilon' = \frac{1}{2f}\{\partial_i, \Gamma^i\}\frac{1}{f} = \frac{1}{2}\left\{D_i, \frac{g^{0i}}{g^{00}}\right\},$$

$$\Lambda = -\frac{f_{,0,0}}{f} - \left(\frac{g^{0i}}{g^{00}}\right)_{,i}\frac{f_{,0}}{f} - 2\frac{g^{0i}}{g^{00}}\frac{f_{,0,i}}{f} - \left(\frac{g^{0i}}{g^{00}}\right)_{,0}\frac{f_{,i}}{f}$$
$$-\frac{1}{2}\left(\frac{g^{0i}}{g^{00}}\right)_{,0,i} - \frac{1}{2f^2}\left(\frac{g^{0i}}{g^{00}}\right)_{,i}\Gamma^j_{,j} - \frac{g^{0i}}{2f^2 g^{00}}\Gamma^j_{,j,i}$$
$$+\frac{1}{4f^2}(\Gamma^i_{,i})^2 - \left(\frac{G^{ij}}{g^{00}}\right)_{,i}\frac{f_{,j}}{f} - \frac{G^{ij}}{g^{00}}\frac{f_{,i,j}}{f} - \frac{\lambda R}{g^{00}}.$$

Classical physics is very similar:

$$H = \sqrt{\frac{m^2 - G^{ij}p_i p_j}{g^{00}}} - \frac{g^{0i}p_i}{g^{00}}.$$

Covariant Dirac equation including the electromagnetic terms

$$(i\hbar\gamma^a D_a - mc)\psi = 0,$$

$$D_a = e_a^\mu D_\mu,$$

$$D_\mu = \partial_\mu + ieA_\mu + \frac{i}{4}\sigma^{ab}\Gamma_{\mu ab},$$

$$\Gamma_{\mu ab} = -\Gamma_{\mu ba} = e_\mu^c \Gamma_{cab},$$

$$\Gamma_{cab} = \frac{1}{2}(-C_{cab} + C_{abc} - C_{bca}),$$

$$C_{abc} = e_a^\mu e_b^\nu (e_{c\nu,\mu} - e_{c\mu,\nu}),$$

where e_μ^a is the tetrad and e_a^μ is the inverse tetrad.

The Lagrangian can be modified by nonminimal coupling terms

$$\frac{\mu'}{2c}F_{\alpha\beta}\bar{\Psi}\sigma^{\alpha\beta}\Psi+\frac{\delta'}{2}G_{\alpha\beta}\bar{\Psi}\sigma^{\alpha\beta}\Psi, \qquad G_{\alpha\beta}=\tfrac{1}{2}\eta_{\alpha\beta\mu\nu}F^{\mu\nu}$$

describing the general case of a fermion with an anomalous magnetic moment and an electric dipole moment.

Parameterization of the metric:

$$e_i^{\hat{0}}=V\delta_i^0, \qquad e_i^{\hat{a}}=W^{\hat{a}}{}_b(\delta_i^b-cK^b\delta_i^0), \qquad a=1,2,3.$$

The Foldy-Wouthuysen Hamiltonian is given by

$$\mathcal{H}_{\rm FW}=\mathcal{H}_{\rm FW}^{(1)}+\mathcal{H}_{\rm FW}^{(2)}+\mathcal{H}_{\rm FW}^{(3)}.$$

The three terms read, respectively,

$$\mathcal{H}_{\rm FW}^{(1)} = \beta\epsilon' + \frac{\hbar c^2}{16}\left\{\frac{1}{\epsilon'}, (2\epsilon^{cae}\Pi_e\{p_b, \mathcal{F}^d{}_c\partial_d\mathcal{F}^b{}_a\} + \Pi^a\{p_b, \mathcal{F}^b{}_a\tilde{\Upsilon}\})\right\} + \frac{\hbar mc^4}{4}\epsilon^{cae}\Pi_e\left\{\frac{1}{\mathcal{T}}, \{p_d, \mathcal{F}^d{}_c\mathcal{F}^b{}_a\partial_b V\}\right\},$$

$$\mathcal{H}_{\rm FW}^{(2)} = \frac{c}{2}(K^a p_a + p_a K^a) + \frac{\hbar c}{4}\Sigma_a\tilde{\Xi}^a + \frac{\hbar c^2}{16}\left\{\frac{1}{\mathcal{T}}, \left\{\Sigma_a\{p_e, \mathcal{F}^e{}_b\}, \left\{p_f, \left[\epsilon^{abc}\left(\frac{1}{c}\dot{\mathcal{F}}^f{}_c - \mathcal{F}^d{}_c\partial_d K^f + K^d\partial_d\mathcal{F}^f{}_c\right)\right.\right.\right.\right.$$
$$\left.\left.\left.\left. - \frac{1}{2}\mathcal{F}^f{}_d(\delta^{db}\tilde{\Xi}^a - \delta^{da}\tilde{\Xi}^b)\right]\right\}\right\}\right\},$$

$$\mathcal{H}_{\rm FW}^{(3)} = \frac{\hbar}{2}\Sigma^a\Omega_a^{(T)}.$$

Here the curly brackets {, } denote anticommutators and we introduced the operators

$$\Omega_a^{(T)} = -\frac{c}{2}V\delta_{ab}\check{T}^{\hat{b}} + \beta\frac{c^3}{8}\left\{\frac{1}{\epsilon'}, \{p_b, \mathcal{F}^b{}_a V\check{T}^{\hat{0}}\}\right\} + \frac{c^2}{16}\left\{\frac{1}{\mathcal{T}}, \left\{\{p_e, \mathcal{F}^e{}_b\}, \left\{p_f, \mathcal{F}^f{}_d V(\delta^{db}\check{T}^{\hat{a}} - \delta^{da}\check{T}^{\hat{b}})\right\}\right\}\right\},$$

$$\epsilon' = \sqrt{m^2c^4V^2 + \frac{c^2}{4}\delta^{ac}\{p_b, \mathcal{F}^b{}_a\}\{p_d, \mathcal{F}^d{}_c\}}, \qquad \mathcal{T} = 2\epsilon'^2 + \{\epsilon', mc^2V\}.$$

In the FW Hamiltonians for scalar and Dirac particles, terms independent of the spin are similar.

The covariant derivative in the Dirac equation includes the spin-dependent term and differs from that in the equation for a scalar particle:

$$D_\mu = \partial_\mu + \frac{ie}{\hbar} A_\mu - \frac{i}{4} \Gamma_\mu^{ab} \sigma_{ab}.$$

Previous results: a Proca particle interacting with electromagnetic fields in Minkowski spacetimes

Proca equations for a spin-1 particle improved by Corben and Schwinger have the form

$$U_{\mu\nu} = D_\mu U_\nu - D_\nu U_\mu, \quad \mu,\nu = 0,1,2,3,$$

$$D^\nu U_{\mu\nu} - m^2 U_\mu - ie\kappa U^\nu F_{\mu\nu} = 0, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

where $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative, $A_\mu = (\Phi, -\mathbf{A})$ is the four-potential, $F_{\mu\nu}$ is the electromagnetic field tensor, and $U_{\mu\nu} = -U_{\nu\mu}$. The Corben-Schwinger term is proportional to $\kappa = g - 1$, where $g = 2m\mu/(e\hbar s) = 2m\mu/(e\hbar)$ for spin-1 particles. Since the initial Proca equations correspond to $g = 1$, this term describes not only the AMM but also a part of the normal ($g = 2$) magnetic moment $\mu_0 = e\hbar/m$.

The equations for the ten-component wave function can be reduced to the equation for the six-component one (Sakata-Taketani transformation).

$$U_0 = \frac{1}{m^2}\left(D^i U_{0i} - ie\kappa U^i F_{0i}\right).$$

Next we introduce two vector functions, $\boldsymbol{\phi}$ and $\boldsymbol{U}$, whose components are given by iU_{i0}/m and U^i. Components of the vector $\boldsymbol{D}$ are equal to D_i. With these denotations,

$$U_0 = -\frac{i}{m}\boldsymbol{D}\cdot\boldsymbol{\phi} - \frac{ie\kappa}{m^2}\boldsymbol{E}\cdot\boldsymbol{U}.$$

It is convenient to define the spin-1 matrices as follows:

$$S^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S^{(2)} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad S^{(3)} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[S^{(i)}, S^{(j)}] = ie_{ijk}S^{(k)}, \quad S^{(i)}S^{(j)}S^{(k)} + S^{(k)}S^{(j)}S^{(i)} = \delta_{ij}S^{(k)} + \delta_{jk}S^{(i)},$$

$\boldsymbol{S}^2 = 2\mathscr{I}$, where $\mathscr{I}$ is the unit 3×3 matrix.

An exclusion of the components U_0 and U_{ij} results in

$$iD_0\boldsymbol{\phi} = m\boldsymbol{U} + \frac{1}{m}\boldsymbol{D}\times(\boldsymbol{D}\times\boldsymbol{U}) - \frac{ie\kappa}{m}\boldsymbol{B}\times\boldsymbol{U} + \frac{e\kappa}{m^2}\boldsymbol{E}(\boldsymbol{D}\cdot\boldsymbol{\phi}) + \frac{e^2\kappa^2}{m^3}\boldsymbol{E}(\boldsymbol{E}\cdot\boldsymbol{U}),$$

$$iD_0\boldsymbol{U} = m\boldsymbol{\phi} - \frac{1}{m}\boldsymbol{D}(\boldsymbol{D}\cdot\boldsymbol{\phi}) - \frac{e\kappa}{m^2}\boldsymbol{D}(\boldsymbol{E}\cdot\boldsymbol{U}).$$

The following properties are valid for any operators $\boldsymbol{V}$ and $\boldsymbol{W}$ proportional to the unit matrix:

$$\boldsymbol{V}\times\boldsymbol{\phi} = -i(\boldsymbol{S}\cdot\boldsymbol{V})\boldsymbol{\phi}, \quad \boldsymbol{V}(\boldsymbol{W}\cdot\boldsymbol{\phi}) = \left[\boldsymbol{V}\cdot\boldsymbol{W} - S^{(i)}S^{(j)}V^{(j)}W^{(i)}\right]\boldsymbol{\phi},$$

$$V^{(j)}W^{(i)}\phi^{(j)} = [\boldsymbol{V}\cdot\boldsymbol{W} - (\boldsymbol{S}\cdot\boldsymbol{V})(\boldsymbol{S}\cdot\boldsymbol{W})]\,\phi^{(i)}.$$

In particular, $\boldsymbol{D}\times(\boldsymbol{D}\times\boldsymbol{U}) = -(\boldsymbol{S}\cdot\boldsymbol{D})^2$. Since $[D_i, D_j] = -iee_{ijk}B^{(k)}$,

$$iD_0\boldsymbol{\phi} = m\boldsymbol{U} - \frac{1}{m}(\boldsymbol{S}\cdot\boldsymbol{D})^2\boldsymbol{U} - \frac{e\kappa}{m}(\boldsymbol{S}\cdot\boldsymbol{B})\boldsymbol{U} - \frac{e\kappa}{m^2}\left[S^{(i)}S^{(j)}E^{(j)}D^{(i)} - \boldsymbol{E}\cdot\boldsymbol{D}\right]\boldsymbol{\phi} - \frac{e^2\kappa^2}{m^3}\left[(\boldsymbol{S}\cdot\boldsymbol{E})^2 - \boldsymbol{E}^2\right],$$

$$iD_0\boldsymbol{U} = m\boldsymbol{\phi} + \frac{1}{m}\left[(\boldsymbol{S}\cdot\boldsymbol{D})^2 - \boldsymbol{D}^2\right]\boldsymbol{\phi} - \frac{e}{m}(\boldsymbol{S}\cdot\boldsymbol{B})\boldsymbol{\phi} + \frac{e\kappa}{m^2}\left[S^{(i)}S^{(j)}D^{(j)}E^{(i)} - \boldsymbol{D}\cdot\boldsymbol{E}\right]\boldsymbol{U}.$$

The wave functions $\boldsymbol{\phi}$ and $\boldsymbol{\chi}$ form the six-component ST wave function

$$\Psi = \frac{1}{\sqrt{2}}\begin{pmatrix}\boldsymbol{\phi} + \boldsymbol{U}\\ \boldsymbol{\phi} - \boldsymbol{U}\end{pmatrix}.$$

The final equation in the Sakata-Taketani representation has the Hamiltonian form:

$$i\frac{\partial\Psi}{\partial t} = \mathscr{H}\Psi.$$

$$\mathscr{H} = e\Phi + \rho_3 m + i\rho_2 \frac{1}{m}(\boldsymbol{S}\cdot\boldsymbol{D})^2$$
$$-(\rho_3 + i\rho_2)\frac{1}{2m}(\boldsymbol{D}^2 + e\boldsymbol{S}\cdot\boldsymbol{B}) - (\rho_3 - i\rho_2)\frac{e\kappa}{2m}(\boldsymbol{S}\cdot\boldsymbol{B})$$
$$-\frac{e\kappa}{2m^2}(1+\rho_1)\left[(\boldsymbol{S}\cdot\boldsymbol{E})(\boldsymbol{S}\cdot\boldsymbol{D}) - i\boldsymbol{S}\cdot[\boldsymbol{E}\times\boldsymbol{D}] - \boldsymbol{E}\cdot\boldsymbol{D}\right]$$
$$+\frac{e\kappa}{2m^2}(1-\rho_1)\left[(\boldsymbol{S}\cdot\boldsymbol{D})(\boldsymbol{S}\cdot\boldsymbol{E}) - i\boldsymbol{S}\cdot[\boldsymbol{D}\times\boldsymbol{E}] - \boldsymbol{D}\cdot\boldsymbol{E}\right]$$
$$-\frac{e^2\kappa^2}{2m^3}(\rho_3 - i\rho_2)\left[(\boldsymbol{S}\cdot\boldsymbol{E})^2 - \boldsymbol{E}^2\right],$$

where ρ_i $(i = 1, 2, 3)$ are the 2×2 Pauli matrices.

J.A. Young and S.A. Bludman, Phys. Rev. 131, 2326 (1963).

It is important to mention that the three-component spin pseudovector *S* is defined in the particle rest frame which belongs to local Lorentz frames. Therefore, the spin matrices define tetrad components of the pseudovector *S* .

A possibility of dual transformations allows us to add the term defining the EDM:

$$D^{\nu}U_{\mu\nu} - m^2 U_{\mu} - ie\kappa U^{\nu} F_{\mu\nu} + ie\eta U^{\nu} G_{\mu\nu} = 0$$

$G_{\mu\nu} = (-\boldsymbol{B}, \boldsymbol{E})$ and $\eta = 2mcd/(e\hbar s) = 2mcd/(e\hbar)$.

$$
\begin{aligned}
\mathscr{H} &= e\Phi + \rho_3 m - i\rho_2 \frac{1}{m}(\boldsymbol{S}\cdot\boldsymbol{\pi})^2 \\
&+(\rho_3 + i\rho_2)\frac{1}{2m}(\boldsymbol{\pi}^2 - e\boldsymbol{S}\cdot\boldsymbol{B}) - (\rho_3 - i\rho_2)\frac{e\kappa}{2m}(\boldsymbol{S}\cdot\boldsymbol{B}) \\
&+\frac{e\kappa}{4m^2}\left(\boldsymbol{S}\cdot[\boldsymbol{\pi}\times\boldsymbol{E}] - \boldsymbol{S}\cdot[\boldsymbol{E}\times\boldsymbol{\pi}] + \{\boldsymbol{S}\cdot\nabla, \boldsymbol{S}\cdot\boldsymbol{E}\} - 2\nabla\cdot\boldsymbol{E}\right) \\
&+i\rho_1\frac{e\kappa}{2m^2}\left(\boldsymbol{\pi}\cdot\boldsymbol{E} + \boldsymbol{E}\cdot\boldsymbol{\pi} - \{\boldsymbol{S}\cdot\boldsymbol{\pi}, \boldsymbol{S}\cdot\boldsymbol{E}\} + \boldsymbol{S}\cdot(\nabla\times\boldsymbol{E})\right) \\
&-\frac{e^2\kappa^2}{2m^2}(\rho_3 - i\rho_2)\left[(\boldsymbol{S}\cdot\boldsymbol{E})^2 - \boldsymbol{E}^2\right] - (\rho_3 - i\rho_2)\frac{e\eta}{2m}(\boldsymbol{S}\cdot\boldsymbol{E}) \\
&-\frac{e\eta}{4m^2}\left(\boldsymbol{S}\cdot[\boldsymbol{\pi}\times\boldsymbol{B}] - \boldsymbol{S}\cdot[\boldsymbol{B}\times\boldsymbol{\pi}] + \{\boldsymbol{S}\cdot\nabla, \boldsymbol{S}\cdot\boldsymbol{B}\}\right) \\
&-i\rho_1\frac{e\eta}{2m^2}\left(\boldsymbol{\pi}\cdot\boldsymbol{B} + \boldsymbol{B}\cdot\boldsymbol{\pi} - \{\boldsymbol{S}\cdot\boldsymbol{\pi}, \boldsymbol{S}\cdot\boldsymbol{B}\} + \boldsymbol{S}\cdot(\nabla\times\boldsymbol{B})\right).
\end{aligned}
$$

$$\mathscr{H}_{FW} = \rho_3\epsilon' + e\Phi + \frac{e\hbar}{4m}\left[\left\{\left(\frac{g-2}{2} + \frac{m}{\epsilon'+m}\right)\frac{1}{\epsilon'}, (\boldsymbol{S}\cdot[\boldsymbol{\pi}\times\boldsymbol{E}] - \boldsymbol{S}\cdot[\boldsymbol{E}\times\boldsymbol{\pi}])\right\}\right.$$

$$\left. -\rho_3\left\{\left(g-2+\frac{2m}{\epsilon'}\right), \boldsymbol{S}\cdot\boldsymbol{B}\right\} + \rho_3\frac{g-2}{4}\left\{\frac{1}{\epsilon'(\epsilon'+m)}, \{\boldsymbol{S}\cdot\boldsymbol{\pi}, (\boldsymbol{\pi}\cdot\boldsymbol{B}+\boldsymbol{B}\cdot\boldsymbol{\pi})\}\right\}\right]$$

$$+\frac{e\hbar\eta}{8m}\left[\left\{\frac{1}{\epsilon'}, (\boldsymbol{S}\cdot[\boldsymbol{B}\times\boldsymbol{\pi}] - \boldsymbol{S}\cdot[\boldsymbol{\pi}\times\boldsymbol{B}])\right\} - 4\rho_3\boldsymbol{S}\cdot\boldsymbol{E}\right.$$

$$\left. +\frac{\rho_3}{2}\left\{\frac{1}{\epsilon'(\epsilon'+m)}, \{\boldsymbol{S}\cdot\boldsymbol{\pi}, (\boldsymbol{\pi}\cdot\boldsymbol{E}+\boldsymbol{E}\cdot\boldsymbol{\pi})\}\right\}\right],$$

The Thomas-Bargmann-Mishel-Telegdi equation and its extension on particles with EDM are satisfied

A.J. Silenko, Quantum-mechanical description of spin-1 particles with electric dipole moments, Phys. Rev. D 87, 073015 (2013).

Quantum mechanics of a Proca particle in Riemannian spacetimes and its classical limit

The spin matrices define *tetrad* components of the pseudovector *S* . Similar situation exists for Dirac particles. This leaves place for the Cartan torsion. The initial equations remain unchanged:

$$U_{\mu\nu} = D_\mu U_\nu - D_\nu U_\mu, \quad \mu,\nu = 0,1,2,3,$$

$$D^\nu U_{\mu\nu} - m^2 U_\mu - ie\kappa U^\nu F_{\mu\nu} = 0, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

where $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative, $A_\mu = (\Phi, -\mathbf{A})$ is the four-potential, $F_{\mu\nu}$ is the electromagnetic field tensor, and $U_{\mu\nu} = -U_{\nu\mu}$. The Corben-Schwinger term is proportional to $\kappa = g - 1$, where $g = 2m\mu/(e\hbar s) = 2m\mu/(e\hbar)$ for spin-1 particles. Since the initial Proca equations correspond to $g = 1$, this term describes not only the AMM but also a part of the normal ($g = 2$) magnetic moment $\mu_0 = e\hbar/m$.

However, a derivation of Hamiltonians is much more cumbersome.

We confine ourselves to an exact derivation of leading terms in Hamiltonians. Any commutator of the momentum operator with potentials (including the metric tensor components) gives the additional factor $\hbar$. When only terms proportional to the zeroth power of $\hbar$ are taken into account, we omit all commutators of the momentum operator with potentials. In this approximation, it is convenient to pass to tetrad components

$$D_a = e^{\mu}_{a} D_{\mu}, \qquad \mathbf{D} = \left(D_{\hat{i}}\right).$$

The approximate form of the covariant Proca equations reads

$$U_{ab} = D_a U_b - D_b U_a, \qquad D^b U_{ab} - m^2 U_a = 0.$$

All terms proportional to the zeroth power of $\hbar$ are exact!

$$iD_0\phi = m\mathbf{U} - \frac{1}{m}\left(S\cdot\mathbf{D}\right)\mathbf{U},$$

$$iD_0\mathbf{U} = m\phi + \frac{1}{m}\left[\left(S\cdot\mathbf{D}\right)^2 - \mathbf{D}^2\right]\phi.$$

The wave functions $\boldsymbol{\phi}$ and $\boldsymbol{\chi}$ form the six-component ST wave function

$$\Psi = \frac{1}{\sqrt{2}}\begin{pmatrix}\boldsymbol{\phi}+\boldsymbol{U}\\ \boldsymbol{\phi}-\boldsymbol{U}\end{pmatrix}.$$

$$iD_0\Psi = \left[\rho_3 m + \frac{i\rho_2}{m}\left(S\cdot\mathbf{D}\right)^2 - \frac{1}{2m}\left(\rho_3 + i\rho_1\right)\mathbf{D}^2\right]\Psi.$$

We parameterize the general metric and use the Schwinger gauge for the tetrad.

$$ds^2 = V^2c^2dt^2 - \delta_{\hat{a}\hat{b}}W^{\hat{a}}{}_c W^{\hat{b}}{}_d(dx^c - K^c cdt)(dx^d - K^d cdt)$$

$$e^0_{\hat{a}} = 0\,, \qquad e^{\hat{0}}_a = 0,\ a = 1,\ 2,\ 3.$$

$$e^{\hat{0}}_i = V\delta^0_i, \qquad e^{\hat{a}}_i = W^{\hat{a}}{}_b(\delta^b_i - cK^b\delta^0_i),$$

The inverse tetrad, such that $e^i_\alpha e^\alpha_j = \delta^i_j$,

$$e^i_{\hat{0}} = \frac{1}{V}(\delta^i{}_0 + \delta^i{}_a cK^a), \qquad e^i_{\hat{a}} = \delta^i{}_b W^b{}_{\hat{a}},$$

$$W^a{}_{\hat{c}}W^{\hat{c}}{}_b = \delta^a_b.$$

$$\mathcal{F}^a{}_b = VW^a{}_{\hat{b}}$$

The parameterization of the metric

$$g^{00} = \frac{1}{c^2V^2}, \qquad g^{0a} = \frac{K^a}{cV^2},$$

$$g^{ab} = \frac{1}{V^2}(K^aK^b - \mathcal{F}^a{}_c\mathcal{F}^b{}_d\delta^{cd}).$$

In the approximation used, the classical Hamiltonian takes the form $(\boldsymbol{\pi} = \mathbf{p} - e\mathbf{A})$

$$\mathcal{H} = \sqrt{m^2V^2 + \delta^{ij}\mathcal{F}_i^k\mathcal{F}_j^n\pi_k\pi_n} + \mathbf{K}\cdot\boldsymbol{\pi} + e\Phi.$$

The Hamiltonians for scalar and Dirac particles have the similar form (the square roots are multiplied by ρ_3 matrix). For a spin-1 particle, the Foldy-Wouthuysen transformation results in

$$iD_0\Psi_{FW} = \rho_3\sqrt{m^2 - \mathbf{D}^2}\Psi_{FW}.$$

$$D_0 = e^0_0 D_0 + e^i_0 D_i, \qquad \mathbf{D}^2 = \delta^{ij} W^k_{\hat{i}} W^n_j D_k D_n.$$

$$D_0 = \frac{1}{V} D_0 + \frac{K^i}{V} D_i, \qquad D_0 = V D_0 - K^i D_i,$$

$$V^2 \mathbf{D}^2 = \delta^{ij} \mathcal{F}^k_i \mathcal{F}^n_j D_k D_n.$$

The result is

$$\mathcal{H}\Psi_{FW} = \left(i D_0 + e\Phi\right) \Psi_{FW}$$

$$= \left(\rho_3 \sqrt{m^2 V^2 + \delta^{ij} \mathcal{F}^k_i \mathcal{F}^n_j \pi_k \pi_n} + \mathbf{K} \cdot \boldsymbol{\pi} + e\Phi\right) \Psi_{FW}.$$

The full agreement with the classical Hamiltonian and the corresponding Hamiltonians for scalar and Dirac particles!

Summary

- Previous results for scalar and Dirac particles in Riemannian spacetimes and for a Proca particle interacting with electromagnetic fields in Minkowski spacetimes are discussed
- Quantum mechanics of a Proca particle interacting with electromagnetic fields in Riemannian spacetimes is constructed and its classical limit is found. The corresponding Hamiltonian is obtained with allowance for zero order terms in the Planck constant
- It is proven that the Hamiltonian obtained for a Proca particle fully agrees with the corresponding classical Hamiltonian and the Hamiltonians for scalar and Dirac particles

Thank you for your attention