Coupled Monge-Ampère equations, geometric structures, geophysical dynamics and all that...

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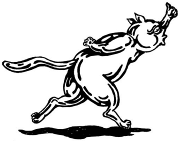
Plan

Bibliography:



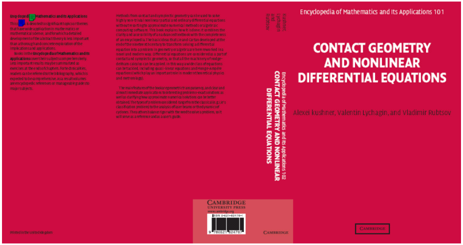


Figure: Cambridge University Press, 2007

Basic object

Figure: Monge and Ampère

$$\boxed{A\frac{\partial^2 f}{\partial q_1^2} + 2B\frac{\partial^2 f}{\partial q_1 \partial q_2} + C\frac{\partial^2 f}{\partial q_2^2} + D\Big(\frac{\partial^2 f}{\partial q_1^2}\cdot\frac{\partial^2 f}{\partial q_2^2} - \big(\frac{\partial f}{\partial q_1 \partial q_2}\big)^2\Big) + E = 0}$$

Monge-Ampère structure

Definition

A Monge-Ampère structure on a 2n-dimensional manifold X is a pair of differential form $(\Omega, \omega) \in \Omega^2(X) \times \Omega^n(X)$ such that Ω is symplectic and ω is Ω-effective i.e. $\Omega \wedge \omega = 0$.

Main idea

- Let $F : \mathbb{R}^n \to (i)\mathbb{R}^n$ be a vector-function and its graph is a subspace in $T^*(\mathbb{R}^n) = \mathbb{R}^n \oplus (i)\mathbb{R}^n$.

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- This graph is a Lagrangian subspace in $T^*(\mathbb{R}^n)$ iff $(dF)_x$ is a symmetric endomorphism . The matrix $|| \frac{\partial F_i}{\partial x_j} ||$ is symmetric $\forall x$ iff the differential form $\sum_i F_i dx_i \in \Lambda^1(\mathbb{R}^n)$ is closed or, equivalently, exact:

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- The projection of the graph of ∇f on $(\mathbb{R}^n)_x$ is given in coordinates by $\nabla^2(f) = \det || \frac{\partial^2 f_i}{\partial x_j^2} ||$.

Correspondence: Forms -Symplectic MAO

Let M be a smooth $n-$dimensional manifold and ω is a differential n-form on T^*M. A (symplectic) Monge-Ampère operator $\Delta_\omega : C^\infty(M) \to \Omega^n(M)$ is the differential operator defined by

$$\Delta_\omega(f) = (df)^*(\omega),$$

where $df : M \to T^*M$ is the natural section associated to f.

Examples

ω	$\boldsymbol{\Delta_\omega = 0}$
$dq_1 \wedge dp_2 - dq_2 \wedge dp_1$	$\Delta f = 0$
$dq_1 \wedge dp_2 + dq_2 \wedge dp_1$	$\Box f = 0$
$dp_1 \wedge dp_2 \wedge dp_3 - dq_1 \wedge dq_2 \wedge dq_3$	$Hess(f) = 1$
$dp_1 \wedge dq_2 \wedge dq_3 - dp_2 \wedge dq_1 \wedge dq_3$ $+dp_3 \wedge dq_1 \wedge dq_2 - dp_1 \wedge dp_2 \wedge dp_3$	$\Delta f - Hess(f) = 0$

Hodge-Lepage-Lychagin theorem

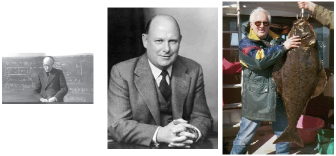

Figure: Lepage, Hodge and Lychagin

The next theorem stresses the fundamental role played by the effective forms in the theory of Monge-Ampère operators :

Theorem (Hodge-Lepage-Lychagin)

- *Every form $\omega \in \Lambda^k(V^*)$ can be uniquely decomposed into the finite sum*

$$\omega = \omega_0 + \top\omega_1 + \top^2\omega_2 + \dots,$$

where all ω_i are effective forms.

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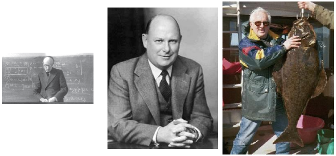

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$$\omega = \omega_0 + \top\omega_1 + \top^2\omega_2 + \dots,$$
where all ω_i *are effective forms.*
- *If two effective* k*-forms vanish on the same* k*-dimensional isotropic vector subspaces in* (V, Ω)*, they are proportional.*

Symplectic Monge-Ampère Equations: Solutions

- A generalised solution of a MAE $\Delta_\omega = 0$ is a lagrangian submanifold of (T^*M, Ω) which is an integral manifold for the MA differential form ω:

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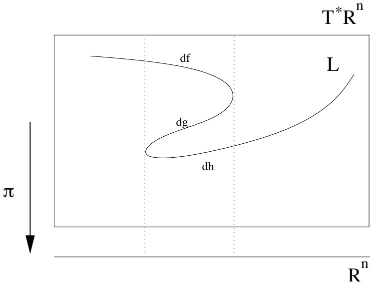
- A (generic) generalised solution locally is the graph of an 1-form df for a regular solution f.

Generalized solution



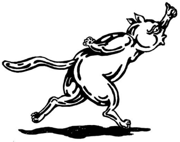
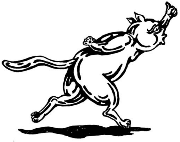


Figure: Generalised solution of a MAE

Generic types of singularities for Generalized solutions of MAE

Specific property of the graph-like Lagrangian submanifolds: their projection on the "configuration space" $\mathbb{R}^n$ is a diffeomorphism. Our generalised solutions are general Lagrangian immersions and they have Arnold's lagrangian singularities.



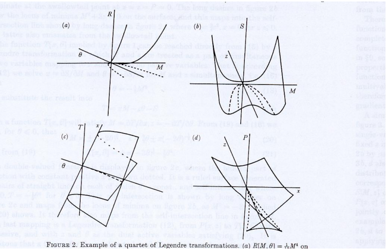


Figure: Lagrangian singularities (Wave fronts, foldings etc.)

This singularities describe the formation of atmospheric fronts (Chynoweth, Porter, Sewell 1988)

Symplectic Equivalence-1

- Two SMAE $\Delta_{\omega_1} = 0$ and $\Delta_{\omega_2} = 0$ are locally equivalent iff there is exist a local symplectomorphism $F : (T^*M, \Omega) \to (T^*M, \Omega)$ such that

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$$F^*\omega_1 = \omega_2.$$
- L is a generalised solution of $\Delta_{F^*\omega_1} = 0$ iff $F(L)$ is a generalised solution of $\Delta_\omega = 0$.

Legendre partial transformation

Figure: Legendre

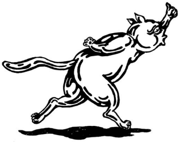
$$
\begin{tikzcd}
\boxed{u_{q_1q_1} + u_{q_2q_2} = 0} \arrow[r, dashed, leftrightarrow] \arrow[d] & \boxed{v_{q_1q_1}v_{q_2q_2} - v_{q_1q_2}^2 = 1} \arrow[d] \\
\boxed{\omega = dq_1 \wedge dp_2 - dq_2 \wedge dp_1} & \boxed{\tilde{\omega} = dp_1 \wedge dp_2 - dq_1 \wedge dq_2} \arrow[l, "\Phi^*"']
\end{tikzcd}
$$

Legendre partial transformation-2

$$
\begin{tikzcd}
\boxed{L_u = \left(q_1, q_2, u_{q_1}, u_{q_2}\right)} \arrow[r, "\Phi"] \arrow[d] & \boxed{\begin{array}{l} L_v = \left(\tilde{q}_1, \tilde{q}_2, v_{\tilde{q}_1}, v_{\tilde{q}_2}\right) \\ = \left(q_1, -u_{q_2}, u_{q_1}, q_2\right) \end{array}} \arrow[d] \\
\boxed{e^{q_1} \cos(q_2)} \arrow[r, leftrightarrow, dashed] & \boxed{\begin{array}{c} q_2 \arcsin\left(q_2 e^{-q_1}\right) \\ + \sqrt{e^{2q_1} - q_2^2} \end{array}}
\end{tikzcd}
$$

with $\Phi : T^*\mathbb{R}^2 \to T^*\mathbb{R}^2$, $(q_1, q_2, p_1, p_2) \mapsto (q_1, -p_2, p_1, q_2)$.

Sewell-Chynoweth SG- equation



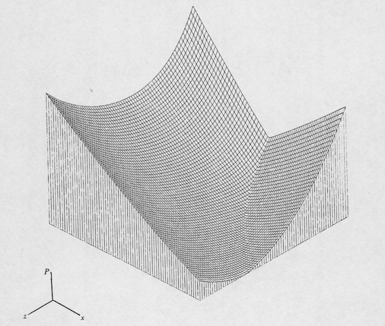


Figure: Numerical Solution of the semi-geostrophic $3D$ equation (Cullen, Sewell-Chynoweth...)

$$hess_{x,y}(u) + \frac{\partial^2 u}{\partial z^2} = hess(u) \tag{1}$$

Sewell-Chynoweth MAO form and its equivalence

- The effective form of (1):

$$\omega = dp \wedge dq \wedge dz + dx \wedge dy \wedge dr - \gamma dx \wedge dy \wedge dz,$$

$(x, y, z, p, q, r)-$ canonical coordinates system of $T^*\mathbb{R}^3$.

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- This form is a sum of two decomposable 3-forms:

$$\omega = dp \wedge dq \wedge dz + dx \wedge dy \wedge (dr - \gamma dz).$$

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- $\phi^*(\omega) = dp \wedge dq \wedge dr - dx \wedge dy \wedge dz$ where ϕ is the symplectomorphism

$$\phi(x, y, z, p, q, r) = (x, y, r, p, q, \gamma r - z).$$

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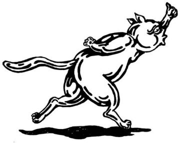
- The equation (1) is symplectically equivalent to the equation

$$hess(u) = 1. \qquad (2)$$

An exact solution of the SG 3D equation



$$f(x,y,z) = \int_a^{\sqrt{xy+yz+zx}} (b + 4\xi^3)^{1/3} d\xi$$

is a regular solution of (2). Therefore,

$$L = \Big\{(x, y, (x+y)\alpha, (y+z)\alpha, (z+x)\alpha, \gamma(x+y)\alpha - z)\Big\}$$

is a generalised solution of (1) with

$$\alpha = \frac{1}{2}\left(\frac{b}{(xy+yz+zx)^{\frac{3}{2}}} + 4\right)^{\frac{1}{3}}.$$

Hoskins geostrophic coordinate transformation

- The SG equations are used like a good approximation to the Boussinesq primitive equations when the rate of the flow momentum is smaller than the Coriolis force, or in other words, when the Rossby number $\mathrm{Ro} << 1$.

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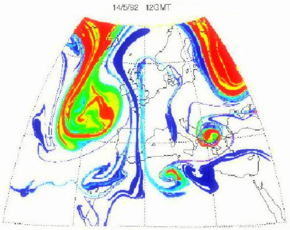
- Potential vorticity is a fundamental concept for understanding the generation of vorticity in cyclogenesis (the birth and development of a cyclone), especially along the polar front, and in analyzing flow in the ocean.
- B. Hoskins (1975) had proposed a remarkable coordinate transformation (a passage to geostrophic coordinates in $x - y$ directions) such that the geostrophic velocity and potential temperature may be represented in terms of one function both in the transformed coordinates as in physical ones

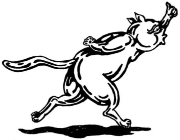


Distribution isentropique (320K) du poteniel tourbillon de Rossby-Ertel (PV) de 14 Mai 1992 , 12h 00 (GMT).

Au dessus de l'Europe, cette surface (320K) se situe à peu près à l'altitude de vol des avions long courriers z ~ 10 km.

Le petit tourbillon sur les Balkans est une rotation CYCLONIQUE dans le sens inverse des aiguilles d'une montre.

D'après l'article d'Appenzeller et al., J. Geophys. Res. 101, 1435-1456 (1996), ``Fragmentation of stratospheric intrusions"



$$\begin{cases} X := x + \frac{v_g}{f} = x + \frac{1}{f^2}\frac{\partial \phi}{\partial x} \\ Y := y - \frac{u_g}{f} = y + \frac{1}{f^2}\frac{\partial \phi}{\partial y} \\ Z := z; \quad T := t. \end{cases}$$

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$$\frac{1}{f^2}(\Phi_{XX} + \Phi_{YY}) - \frac{1}{f^4}(\Phi_{XX}\Phi_{YY} - \Phi_{XY}^2) + \frac{1}{N^2}\Phi_{ZZ} = 1. \quad (3)$$

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- Here (and in what follows) f is the *Coriolis parameter* taking as a constant and N is the *Brunt - Väisälä* frequency:

$$N = \sqrt{\frac{q_g g}{f\theta_0}},$$

for the uniform potential vorticity q_g and the constant potential temperature θ_0.

Hoskins geostrophic MA effective form

- This is a 3D Monge-Ampére equation with the effective form

$$\omega = \frac{1}{f^2}(dp \wedge dy \wedge dz + dx \wedge dq \wedge dz) + \frac{1}{N^2} dx \wedge dy \wedge dr -$$

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- This form is the sum of two decomposable forms:

$$\omega = \frac{1}{N^2} dx \wedge dy \wedge dr - (dx - \frac{1}{f^2} dp) \wedge (dy - \frac{1}{f^2} dq) \wedge dz.$$

Hoskins geostrophic MA effective form : equivalence

- Consider the symplectomorphism

$$F(x, y, z, p, q, r) = (p, q, z, -x + f^2 p, -y + f^2 q, r). \qquad (4)$$

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- The new canonical coordinate system $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{p}, \tilde{q}, \tilde{r})$

$$\begin{cases} \tilde{p} := -x + f^2 p; \quad \tilde{x} := p; \\ \tilde{q} := -y + f^2 q; \quad \tilde{y} := q; \\ \tilde{r} := r; \quad \tilde{z} := z \end{cases}$$

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- The Hoskins SG (3) is equivalent to the (1):

$$\mathrm{hess}(u) = \frac{N^2}{f^4} = \frac{(q_g g)^2}{f^6(\theta_0)^2} \qquad (5)$$

by the symplectomorphism (4).

Geometric Structures on $T^*\mathbb{R}^2$.

Let (Ω, ω) be a Monge-Ampère structure on $X = \mathbb{R}^4$. The field of endomorphisms $A_\omega : X \to TX \otimes T^*X$ is defined by

$$\omega(\cdot, \cdot) = \Omega(A_\omega \cdot, \cdot).$$

REMARK The tensor

$$J_\omega = \frac{A_\omega}{\sqrt{|pf(\omega)|}}$$

gives

- an almost-complex structure on X if $pf(\omega) > 0$.

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- The tensor J_ω is integrable;
- the normalized form $\frac{\omega}{\sqrt{|pf(\omega)|}}$ is closed.

Table 1. Effective forms with constant coefficients in $2D$

$\boldsymbol{\Delta_\omega = 0}$	$\boldsymbol{\omega}$	$\boldsymbol{pf(\omega)}$
$\Delta f = 0$	$dq_1 \wedge dp_2 - dq_2 \wedge dp_1$	1
$\Box f = 0$	$dq_1 \wedge dp_2 + dq_2 \wedge dp_1$	-1
$\frac{\partial^2 f}{\partial q_1^2} = 0$	$dq_1 \wedge dp_2$	0

HyperKäler triple of MAE

The conservation law of the potential vorticity (the Ertel's theorem) obtains (using the Hamiltonian representation of the system):

$$\frac{d}{dt}\left(\frac{\partial(q_1, q_2)}{\partial(a, b)}\right) =$$

$$\frac{d}{dt}(1 + \phi_{q_1 q_1} + \phi_{q_2 q_2} + \det \mathrm{Hess}\, \phi\ ) = 0,$$

This equation is a part of the HyperKähler triple of MAEs (R. and Roulstone 1997, 2001):

$$\begin{cases} \omega_I = \left[1 + a(p_{11} + p_{22}) + (a^2 - c^2)(p_{11}p_{22} - p_{12}^2)dq_1\right] \wedge dq_2 & , \\ \omega_J = \left[2cp_{12} + ac(p_{11}p_{22} - p_{12}^2)\right] dq_1 \wedge dq_2 & , \\ \omega_K = -c\Omega \end{cases}$$

$2D$ balanced model MAE

- The general family of (elliptic) MAE with constant coefficients carries all flat balanced models:

$$1 + \phi_{q_1 q_1} + a\phi_{q_2 q_2} + (a^2 - c^2)\det \mathrm{Hess}\,\phi = \zeta^{\mathbf{C}}/f\,, \qquad (6)$$

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- *The L_1 Salmon dynamics* with $a = c = 1$;
- *The $\sqrt{3}$ dynamics of McIntyre - Roulstone* for $a = 1, c = \sqrt{3}$ and $\zeta^{\mathbf{C}}/f < 3/2$;

  Our classification theorem in $2D$ gives a classification of all "almost-balanced"($0 < c < \sqrt{3}$) models with a uniform potential vorticity.

Courant Bracket

$T-$tangent bundle of M and T^*- cotangent bundle.

$$(X+\xi, Y+\eta) = \frac{1}{2}(\xi(Y)+\eta(X)),$$

-natural indefinite scalar product on $T \oplus T^*$.
The Courant bracket on sections of $T \oplus T^*$ is

$$[X+\xi, Y+\eta] = [X, Y] + L_X\eta - L_Y\xi - \frac{1}{2}d(\iota_X\eta - \iota_Y\xi).$$

Generalized Complex Geometry

Figure: Hitchin

DEFINITION [Hitchin]: An almost generalized complex structure is a bundle map $\mathbb{J} : T \oplus T^* \to T \oplus T^*$ with

$$\mathbb{J}^2 = -1, \ (\mathbb{J}\cdot, \cdot) = -(\cdot, \mathbb{J}\cdot).$$

An almost generalized complex structure is integrable if the spaces of sections of its two eigenspaces are closed under the Courant bracket.

$2D$ SMAE and Generalized Complex Geometry

- **DEFINITION** A Monge-Ampère equation $\Delta_\omega = 0$ has a divergent type if there existe a function μ such that the form $\omega' = \omega + \mu\Omega$ is closed.

$2D$ SMAE and Generalized Complex Geometry

- **DEFINITION** A Monge-Ampère equation $\Delta_\omega = 0$ has a divergent type if there existe a function μ such that the form $\omega' = \omega + \mu\Omega$ is closed.
- **THEOREM** (B.Banos)
  Let $\Delta_\omega = 0$ be a Monge-Ampère divergent type equation on $\mathbb{R}^2$ with closed ω (which might be non-effective). The generalized almost-complex structure defined by

$$\mathbb{J}_\omega = \begin{pmatrix} A_\omega & \Omega^{-1} \\ -\Omega(1 + A_\omega^2 \cdot, \cdot) & -A_\omega^* \end{pmatrix}$$

  is integrable.

Hitchin pairs (after M.Crainic)

A Hitchin pair is a pair of bivectors π and Π, $\Pi-$ non-degenerate, satisfying

$$\begin{cases} [\Pi, \Pi] = [\pi, \pi] \\ [\Pi, \pi] = 0. \end{cases} \tag{7}$$

PROPOSITION There is a 1-1 correspondence between Generalized complex structure

$$\mathbb{J} = \begin{pmatrix} A & \pi_A \\ \sigma & -A^* \end{pmatrix}$$

with σ non degenerate and Hitchin pairs of bivector (π, Π). In this correspondence

$$\begin{cases} \sigma = \Pi^{-1} \\ A = \pi \circ \Pi^{-1} \\ \pi_A = -(1 + A^2)\Pi \end{cases}$$

Hitchin pair of bivectors in $4D$

Π is non-degenerate $\Rightarrow$ two 2-forms ω and Ω, not necessarily closed and $\omega(\cdot,\cdot) = \Omega(A\cdot,\cdot)$.
A generalized lagrangian surface: closed under A, or equivalently, bilagrangian: $\omega|_L = \Omega|_L = 0$.
Locally, L is defined by two functions u and v satisfying a first order system:

Jacobi systems

$$\begin{cases} a + b\frac{\partial u}{\partial x} + c\frac{\partial u}{\partial y} + d\frac{\partial v}{\partial x} + e\frac{\partial v}{\partial y} + f \det J_{u,v} = 0 \\ A + B\frac{\partial u}{\partial x} + C\frac{\partial u}{\partial y} + D\frac{\partial v}{\partial x} + E\frac{\partial v}{\partial y} + E \det J_{u,v} = 0 \end{cases}$$

$$J_{u,v} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

Such a system generalizes both MAE and Cauchy-Riemann systems and is called a Jacobi system.

Dritschel-Viudez MAE

Recently a new approach to modelling stably-stratified geophysical flows was proposed by Dritschel and Viudez. This approach is based on the explicit conservation of potential vorticity and uses a change of variables from the usual primitive variables of velocity and density to the components of ageostrophic horizontal vorticity and a Monge-Ampère-like nonlinear equation with non-constant coefficients arises. The equation changes the type from elliptic to hyperbolic:

$$E\left(\Phi_{xx}\Phi_{zz}-\Phi_{xz}^2\right)+A\Phi_{xx}+2B\Phi_{xz}+C\Phi_{xz}+D=0 \qquad (8)$$

with

$$E=1\,,\quad A=1+\varphi_{xz}\,,\quad B=\frac{1}{2}\left(\varphi_{zz}-\varphi_{xx}\right)$$

$$C=1-\varphi_{xz}\,,\quad D=\varphi_{xx}\varphi_{zz}-\varphi_{xz}^2-\varpi$$

where φ is a given potential and the dimensionless PV anomaly ϖ may be also considered as a given quantity.

The corresponding Monge-Ampère structure

$$\begin{cases} \Omega = dx \wedge dp + dz \wedge dr \\ \omega = Edp \wedge dr + Adp \wedge dz + B(dx \wedge dp - dz \wedge dr) + \\ +Cdx \wedge dr + Ddx \wedge dz \end{cases}$$

The pfaffian is $\mathrm{pf}(\omega) = R$ with R the Rellich's parameter:

$$R = AC - ED - B^2 = 1 + \varpi - \left(\frac{\Delta\varphi}{2}\right)^2$$

A direct computation gives

$$d\omega = d\left(\frac{\Delta\varphi}{2}\right) \wedge \Omega \tag{9}$$

Integrability of the complex/product structure

- **THEOREM** (B.Banos, V.R.) $2D$ Dritschel-Viudez equation is locally equivalent to a Monge-Ampère equation with constant coefficients if and only if $\begin{cases} \Delta\varphi = 2c_1 \\ R = c_2 \end{cases}$

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- for $R > 0$, we see that

$$\omega + i\sqrt{R}\,\Omega = du \wedge dv$$

with

$$\begin{cases} u = x - (c_1 - ic_2)z - \varphi_z + p \\ v = -(c_1 + ic_2)x + \varphi_x + r \end{cases}$$

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$$\begin{cases} u = x - (c_1 - ic_2)z - \varphi_z + p \\ v = -(c_1 + ic_2)x + \varphi_x + r \end{cases}$$
- $\varphi = 2c_1$ and $R = c_2 > 0$ then $2D$ Dritschel-Viudez equation is equivalent to Laplace equation
$$\varphi_{xx} + \varphi_{zz} = 0$$
modulo the Legendre transform
$$F(x, z, p, r) = \frac{1}{\sqrt{R}}(x - c_1 z - \varphi_z, c_2 z, -c_2 x, -c_1 x + \varphi_x + r).$$

Hitchin hypersymplectic geometry-1

- if our 4-dimensional manifold M, endowed with the Monge-Ampère structure (Ω, ω) admits a lagrangian fibration (main example: M is the cotangent bundle of a smooth $2D-$manifold), then it exists a conformal split metric on M^4.

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- When the corresponding Monge-Ampère equation is given by (8), this metric writes as

$$g = C(dx)^2 - 2Bdxdz + A(dz)^2 + E/2(dpdx + dqdz), \quad (10)$$

Using this metric, we get an additional 2-form $\hat{\omega}$ defined by

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- $$\hat{\omega}(\cdot\,,\,\cdot) = g(A_\omega \cdot\,,\,\cdot) \quad \text{with} \quad \omega(\cdot\,,\,\cdot) = \Omega(A_\omega \cdot\,,\,\cdot)$$

  In coordinates,

$$\hat{\omega} = \left(-2AC + 2B^2 + D\right) dx \wedge dz - Bdx \wedge dp - Cdx \wedge dr$$
$$+ Adz \wedge dp + Bdz \wedge dr - dp \wedge dr$$

Hitchin hypersymplectic geometry-2

- Introducing $\Theta = \frac{\Omega}{\sqrt{|R|}}$, we get an hypersymplectic triple $(\Theta, \omega, \hat{\omega})$ satisfying

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- Equivalently, we obtain 3 tensors I, S and T satisfying

$$I^2 = -1, \ S^2 = 1, \ T^2 = 1$$
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- Moreover we have

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we see that

$$\begin{cases} \omega = dU \wedge dV - dX \wedge dZ \\ \\ \hat{\omega} = -(dU \wedge dV + dX \wedge dZ) \\ \\ \Omega = \dfrac{1}{R}\left(dX \wedge dU - SdZ \wedge dU + RdZ \wedge dV\right) \quad \text{with } S = \displaystyle\int R_z dx \end{cases}$$

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- In the new coordinates:
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  we see that
  $$\begin{cases} \omega = dU \wedge dV - dX \wedge dZ \\ \\ \hat{\omega} = -(dU \wedge dV + dX \wedge dZ) \\ \\ \Omega = \dfrac{1}{R}(dX \wedge dU - S dZ \wedge dU + R dZ \wedge dV) \quad \text{with } S = \displaystyle\int R_z dx \end{cases}$$
- In other words, when φ is harmonic, a submanifold $L = \left\{(Z, \psi_Z, U, \psi_U),\ (Z, U) \in \mathbb{R}^2\right\}$ is a generalized solution of $2D$ - Dritschel Viudez equation if and only if the following Tricomi equation holds:
  $$\psi_{ZZ} + R\psi_{UU} = S.$$

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- The Hitchin pfaffian defined by

$$pf(\omega) = \frac{1}{6} tr A^2_\omega.$$

$3D$ Generalized Calabi-Yau structures

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    $$g(U, V) = \Omega(AU, V)$$
    for all tangent vectors U, V,
  - α and β are (eventually complex) decomposable 3-forms whose associated distributions are the distributions of A eigenvectors and such that
    $$\frac{\alpha \wedge \beta}{\Omega^3} \text{ is constant.}$$
- A generalized Calabi-Yau structure $(g, \Omega, K, \alpha, \beta)$ is integrable if α and β are closed.

Generalized CY and MA

Each nondegenerate Monge-Ampère structure (Ω, ω_0) defines a generalized almost Calabi-Yau structure $(q_\omega, \Omega, A_\omega, \alpha, \beta)$ with

$$\omega = \frac{\omega_0}{\sqrt[4]{|\lambda(\omega_0)|}}.$$

The generalized Calabi-Yau structure associated with the equation

$$\Delta(f) - \mathrm{hess}(f) = 0$$

is the canonical Calabi-Yau structure of $\mathbb{C}^3$

$$\begin{cases} g = -\sum_{j=1}^{3} dx_j.dx_j + dy_j.dy_j \\ A = \sum_{j=1}^{3} \frac{\partial}{\partial y_j} \otimes dx_j - \frac{\partial}{\partial x_j} \otimes dy_j \\ \Omega = \sum_{j=1}^{3} dx_j \wedge dy_j \\ \alpha = dz_1 \wedge dz_2 \wedge dz_3 \\ \beta = \overline{\alpha} \end{cases}$$

The generalized Calabi-Yau associated with the equation

$$\Box(f) + \mathrm{hess}(f) = 0$$

is the pseudo Calabi-Yau structure

$$\begin{cases} g = dx_1.dx_1 - dx_2.dx_2 + dx_3.dx_3 + dy_1.dy_1 - dy_2.dy_2 + dx_3.dx_3 \\ A = \frac{\partial}{\partial x_1} \otimes dy_1 - \frac{\partial}{\partial y_1} \otimes dx_1 + \frac{\partial}{\partial y_2} \otimes dx_2 - \frac{\partial}{\partial x_2} \otimes dy_2 - \frac{\partial}{\partial y_3} \otimes dx_3 \\ + \frac{\partial}{\partial x_3} \otimes dy_3 \\ \Omega = \sum_{j=1}^{3} dx_j \wedge dy_j \\ \alpha = dz_1 \wedge dz_2 \wedge dz_3 \\ \beta = \overline{\alpha} \end{cases}$$

The generalized Calabi-Yau structure associated with the equation

$$\mathrm{hess}(f) = 1$$

is the "real" Calabi-Yau structure

$$\begin{cases} g = \sum_{j=1}^{3} dx_j . dy_j \\ A = \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \otimes dx_j - \frac{\partial}{\partial y_j} \otimes dy_j \\ \Omega = \sum_{j=1}^{3} dx_j \wedge dy_j \\ \alpha = dx_1 \wedge dx_2 \wedge dx_3 \\ \beta = dy_1 \wedge dy_2 \wedge dy_3 \end{cases}$$

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- $$\begin{cases} \text{hess}(f) = 1 \\ \Delta f - \text{hess}(f) = 0 \\ \Box f + \text{hess}(f) = 0 \end{cases}$$
- iff the correspondingly defined generalized Calabi-Yau structure is integrable and locally flat.

	$\Delta_\omega = 0$	ω	$\varepsilon(q_\omega)$	$pf(\omega)$
1	$\nu\,\mathrm{hess}(f) = 1$	$-dq_1\wedge dq_2\wedge dq_3+\nu\cdot dp_1\wedge dp_2\wedge dp_3$	$(3,3)$	ν^2
2	$\Delta f - \nu\,\mathrm{hess}(f) = 0$	$dp_1 \wedge dq_2 \wedge dq_3 - dp_2 \wedge dq_1 \wedge dq_3$ $+dp_3\wedge dq_1\wedge dq_2-\nu\cdot dp_1\wedge dp_2\wedge dp_3$	$(0,6)$	$-\nu^2$
3	$\Box f + \nu\,\mathrm{hess}(f) = 0$	$dp_1 \wedge dq_2 \wedge dq_3 + dp_2 \wedge dq_1 \wedge dq_3$ $+dp_3\wedge dq_1\wedge dq_2+\nu\cdot dp_1\wedge dp_2\wedge dp_3$	$(4,2)$	$-\nu^2$
4	$\Delta f = 0$	$dp_1\wedge dq_2\wedge dq_3 - dp_2\wedge dq_1\wedge dq_3 +$ $dp_3 \wedge dq_1 \wedge dq_2$	$(0,3)$	0
5	$\Box f = 0$	$dp_1\wedge dq_2\wedge dq_3 + dp_2\wedge dq_1\wedge dq_3 +$ $dp_3 \wedge dq_1 \wedge dq_2$	$(2,1)$	0
6	$\Delta_{q_2,q_3} f = 0$	$dp_3 \wedge dq_1 \wedge dq_2 - dp_2 \wedge dq_1 \wedge dq_3$	$(0,1)$	0
7	$\Box_{q_2,q_3} f = 0$	$dp_3 \wedge dq_1 \wedge dq_2 + dp_2 \wedge dq_1 \wedge dq_3$	$(1,0)$	0
8	$\frac{\partial^2 f}{\partial q_1^2} = 0$	$dp_1 \wedge dq_2 \wedge dq_3$	$(0,0)$	0
9		0	$(0,0)$	0

Table: Classification of effective 3-formes in dimension 6

The subjects which I had no time to describe:

- Symmetries, conservation laws and Noether theorem for MAO and MAE;
- Self-similar solutions, shock waves and Hugoniot-Rankin conditions;
- Variational MAE, divergent MAE and Euler-Lagrange operators;
- $2D$ Euler equation as a reduction of the 2-nd Plebanski equation;
- Generalized Calabi-Yau 3D structures and Burgers vortices;
- Many-many other interesting things...

Thank you for your attention!