Evolution of a bouncing solutions in cosmological models with non-minimally coupled scalar fields

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Introduction

- Let us consider a cosmological model, described by the following action

$$S = \int d^4x\sqrt{-g}\left[U(\sigma)R - \frac{1}{2}g^{\mu\nu}\sigma_{,\mu}\sigma_{,\nu} - V(\sigma)\right], \qquad (1)$$

  where $U(\sigma)$ and $V(\sigma)$ are differentiable functions of the scalar field σ. If $U(\sigma)$ is a constant: $U(\sigma) = U_0$, then the scalar field is minimally coupled to gravity, otherwise we have a model with non-minimal coupling.

- Let us consider the FLRW metric. In this case the interval is

$$ds^2 = N(\tau)^2 d\tau^2 - a(\tau)^2\left(\frac{dr^2}{1-Kr^2} - r^2 d\theta^2 - r^2\sin^2(\theta)d\varphi^2\right), \quad (2)$$

  where $a(\tau)$ is the cosmological radius and $N(\tau)$ is the lapse function, K is a constant. As usual $K = 0$ describes a flat universe, $K = 1$ a closed universe, and $K = -1$ an open one.

- A bounce point is defined by two conditions: the Hubble parameter $H = 0$ and $\dot{H} > 0$.

Equations in flat Friedmann–Lemaître–Robertson–Walker (FLRW) universe

Firstly we consider the case of the spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) background

- $ds^2 = -dt^2 + a^2(t)\left(dx_1^2 + dx_2^2 + dx_3^2\right)$ the equations are:
- $$6UH^2 + 6\dot{U}H - \frac{1}{2}\dot{\sigma}^2 - V = 0, \tag{3}$$
- $$2U\left[2\dot{H} + 3H^2\right] + 2U'\left[\ddot{\sigma} + 2H\dot{\sigma}\right] = V - \left[2U'' + \frac{1}{2}\right]\dot{\sigma}^2, \tag{4}$$
- $$\ddot{\sigma} + 3H\dot{\sigma} - 6U'\left[\dot{H} + 2H^2\right] + V' = 0, \tag{5}$$

where a “dot” means a derivative with respect to the cosmic time t and a “prime” means a derivative with respect to the scalar field σ.

Bounce Solutions

- If a solution of Eqs. (3)–(5) has such a point t_b that

$$H(t_b) = \dot{a}(t_b)/a(t_b) = 0, \qquad \dot{H}(t_b) > 0,$$

  then it is a bounce solution.
- Let us find conditions that are necessary for the existence of a bounce solution.
  - Using Eq. (3), we get that from $H(t_b) = 0$ it follows $V(\varphi(t_b)) \leqslant 0$.
  - Subtracting equation (3) from equation (4), we obtain

$$4U\dot{H} = -\dot{\sigma}^2 - 2\ddot{U} + 2H\dot{U}. \qquad (6)$$

  - Therefore, if $U = const > 0$ a bounce solution **does not exist.** At the bounce point we get

$$2\left(U + 3U'^2\right)\dot{H}(t_b) = U'V' + \left[2U'' + 1\right]V. \qquad (7)$$

    Functions U and V and their derivatives are taken at the point $\varphi(t_b)$.
  - From here the condition $\dot{H}(t_b) > 0$ gives the restriction on functions U and V at the bounce point.

Equations (3)–(5) can be transformed into the following system of the first order equations which is useful for numerical calculations and analysis of stability:

$$\begin{cases} \dot{\sigma} = \psi, \\ \dot{\psi} = -3H\psi - \dfrac{\left[(6U''+1)\psi^2 - 4V\right]U' + 2UV'}{2\left(3U'^2+U\right)}, \\ \dot{H} = \dfrac{2U'H\psi}{3U'^2+U} - \dfrac{\left[2U''+1\right]\psi^2}{4\left(3U'^2+U\right)} - \dfrac{6U'^2H^2}{3U'^2+U} + \dfrac{U'V'}{2\left(3U'^2+U\right)}. \end{cases} \tag{8}$$

If Eq. (3) is satisfied in the initial moment of time, then from system (8) it follows that Eq. (3) is satisfied at any moment of time.

De Sitter Solutions

- We analyze the stability of de Sitter solutions with $H_{dS} > 0$ and $U(\sigma_{dS}) > 0$ only.
  For arbitrary differentiable functions V and $U > 0$, the model has a stable de Sitter solution with $H_{dS} > 0$ only if

$$V''_{eff}(\sigma_{dS}) > 0, \qquad V_{eff}(\sigma_{dS}) > 0, \quad V_{eff}(\sigma) = \frac{V(\sigma)}{4U(\sigma)^2}.$$

- The de Sitter point is a stable node (the scalar field decreases monotonically) at

$$\frac{3\left(U + 3U'^2\right)}{8U^2} \geqslant \frac{V''_{eff}}{V_{eff}}, \qquad (9)$$

  and a stable focus (the scalar field oscillations exist) at

$$\frac{3\left(U + 3U'^2\right)}{8U^2} < \frac{V''_{eff}}{V_{eff}}. \qquad (10)$$

Induce Gravity Model

Let us consider an induced gravity models with

$$U(\sigma)=\frac{\xi}{2}\sigma^2\,,$$

where $\xi>0$ is the non-minimal coupling constant.
In A.Yu. Kamenshchik, A. Tronconi, G. Venturi, and S.Yu. Vernov, Phys. Rev. D **87** (2013) 063503 (arXiv:1211.6272)
it has been found that the model with the following sixth degree polynomial potential:

$$V(\sigma)=\frac{(16\xi+3)(6\xi+1)\xi}{(8\xi+1)^2}C_2^2\sigma^6+\left[3\xi C_1^2+\frac{2(6\xi+1)(20\xi+3)\xi}{(8\xi+1)(4\xi+1)}C_0C_2\right]\sigma^4+$$

$$+\frac{6(6\xi+1)\xi}{8\xi+1}C_1C_2\sigma^5+\frac{6(6\xi+1)\xi}{4\xi+1}C_0C_1\sigma^3+\frac{(16\xi+3)(6\xi+1)\xi}{(4\xi+1)^2}C_0^2\sigma^2,$$

where C_i are constants, has exact solution with a non-monotonic Hubble parameter.

Exact Bounce Solution

In E.O. Pozdeeva and S.Yu. Vernov, AIP Conf. Proc. **1606** (2014) 48–58 (arXiv:1401.7550) exact bounce solution has been found.
The analytic form of this solution is

$$\sigma(t)=\frac{\sqrt{(8\xi+1)C_0}}{\sqrt{(8\xi+1)C_0e^{-\omega(t-t_0)}+(4\xi+1)C_2}}\,,$$

where $\omega=4\xi C_0/(4\xi+1)$, t_0 is an arbitrary integration constant.

$$H(t)=C_0+C_1\sigma(t)+C_2\sigma^2(t).$$

After the bounce point $H(t)$ is a monotonically increasing function.

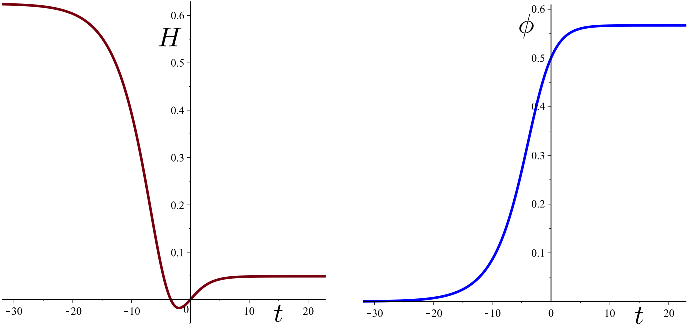


Figure: The functions $H(t)$ and $\sigma(t)$ that correspond to the exact bounce solution. The values of parameter are $\xi = 1,\ C_2 = 7/2,\ C_1 = -3,\ C_0 = 5/8$. Initial conditions is defined by $\sigma_0 = 0.5$, $\sigma_0 = 1/36 \simeq 0.0278$ ($t_0 = 2\ln(8/9)$).

This solution tends to unstable de Sitter solution.

The Integrable Cosmological Model

- In the spatially flat FLRW metric $R = 6(\dot{H} + 2H^2)$.
- From (3)–(5) we get

$$2R\left(U + 3{U'}^2\right) + (6U'' + 1)\,\dot{\sigma}^2 = 4V + 6V'U'\,. \qquad (11)$$

- From the structure of Eq. (11) it is easy to see [1] that the simplest way to get a constant R is to choose such $U(\sigma)$ that

$$U + 3U'^2 = U_0, \qquad 6U'' + 1 = 0, \qquad U_0 R = 2V + 3V'U'.$$

- The solution to the first two equations is

$$U_c(\sigma) = U_0 - \frac{\sigma^2}{12} \qquad (12)$$

[1] E.O. Pozdeeva, M.A. Skugoreva, A.V. Toporensky, and S.Yu. Vernov, *JCAP* **1612** (2016) 006 (arXiv:1608.08214)

For $U = U_c$ Eq. (11) can be simplified:

$$2U_0 R = 4V(\sigma) - \sigma V'(\sigma). \tag{13}$$

and has the following solution:

$$V_{int} = \frac{\Lambda}{2U_0} + C_4\sigma^4, \qquad \Lambda = \frac{R}{4}. \tag{14}$$

where C_4 is an integration constant.
Thus, requiring that the Ricci scalar is a constant one can define both functions $U(\sigma) = U_c$ and $V(\sigma) = V_{int}$. To get a positive $G_{eff} = \frac{1}{16\pi U}$ for some values of σ we choose $U_0 > 0$.

- Using the explicit form of functions U_c and V_{int} we get that the condition $\dot{H}_b > 0$ is equivalent to $\Lambda > 0$, hence, from $V(\varphi_b) < 0$ it follows $C_4 < 0$.
  This integrable cosmological model has been considered in[2], where the behavior of bounce solutions has been studied in detail.
- Considering the equation

$$R = 6(\dot{H} + 2H^2) = 4\Lambda,$$

such as a differential equation for the Hubble parameter, we obtain two possible real solutions in dependence of the initial conditions:

$$H_1 = \sqrt{\frac{\Lambda}{3}}\tanh\left(\frac{2\sqrt{\Lambda}(t-t_0)}{\sqrt{3}}\right), \quad H_2 = \sqrt{\frac{\Lambda}{3}}\coth\left(\frac{2\sqrt{\Lambda}(t-t_0)}{\sqrt{3}}\right),$$

where t_0 is an integration constant.

[2] B. Boisseau, H. Giacomini, D. Polarski, and A.A. Starobinsky, J. Cosmol. Astropart. Phys. **1507** (2015) 002 (arXiv:1504.07927)

For future analysis let us introduce function P

$$P \equiv \frac{H}{\sqrt{U}} + \frac{U'\dot{\varphi}}{2U\sqrt{U}}, \qquad A \equiv \frac{U + 3U'^2}{4U^3} \tag{15}$$

have been introduced. If $U(\sigma) > 0$, then $A(\sigma) > 0$ as well.
In terms of these functions Eqs. (3) and (6) take the following form:

$$3P^2 = A\dot{\sigma}^2 + 2V_{eff}\,, \tag{16}$$

$$\dot{P} = \,-\,A\sqrt{U}\,\dot{\sigma}^2. \tag{17}$$

Equation (3) is a quadratic equation in $H(t)$ that has the following solutions:

$$H_{\pm} = \,-\,\frac{\dot{U}}{2U} \pm \frac{1}{6U}\sqrt{9U'^2\dot{\sigma}^2 + 3U\dot{\sigma}^2 + 6UV}. \tag{18}$$

The value of the function $P(\varphi)$ that correspond to $H_{\pm}$ is

$$P=\pm\frac{1}{6U}\sqrt{3\left[\frac{3U'^2}{U}\dot{\sigma}^2+\dot{\sigma}^2+2V\right]}=\pm\sqrt{\frac{A}{3}\dot{\sigma}^2+\frac{2}{3}V_{eff}}\,, \qquad (19)$$

So, a positive P corresponds to H_+ and a negative P corresponds to H_-. If the potential V is negative at some values of φ, whereas the function U is positive at these points, then one should restrict the domain of absolute values of $\dot{\sigma}$ from below to get a real Hubble parameter. In other words, there exists the unreachable domain on the phase plane. The boundary of this domain is defined by the condition $P=0$. If $V_{eff}>0$, then the sign of the function P cannot be changed. If in some domain $V_{eff}(\varphi)$ is negative, then the sign of P can be changed from plus to minus, but not vice verse. So, the sign of P cannot be changed twice.

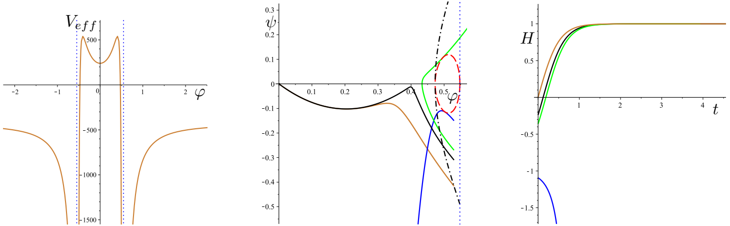


Figure: The effective potential (left picture), phase trajectories (middle picture) and the Hubble parameter $H(t)$ (right picture) for $V = C_4\sigma^4 + C_0$ and $U = U_0 - \sigma^2/12$. The parameters are $U_0 = 1/40$, $C_4 = -3$, $C_0 = 0.15$. The initial values are $\sigma_i = 0.53$, and $\psi_i = -0.4164479079$ (gold line), $\psi_i = -0.31$ (black line), $\psi_i = -0.27$ (green line), $\psi_i = -0.15$ (blue line). The black dash curve corresponds to $H = 0$. The blue point lines correspond to $U = 0$. The red dash curve corresponds to $P = 0$.

The red dash curve, defined by the equation $P = 0$, is the boundary of unreachable domain. Any point inside this curve corresponds to a non-real value of the Hubble parameter.

Such domain exists at any model with action (1) that has a bounce solution, because $V(\sigma_b) < 0$.

Solutions can touch the boundary of unreachable domain.

GENERALIZATION OF INTEGRABLE MODEL

There are two way of modification: modify U or modify V.

- Let us modify V and consider the model with

$$U_c(\varphi) = U_0 - \frac{\sigma^2}{12}, \quad \text{designation} \quad U_0 = \frac{1}{2K} \tag{20}$$

  and

$$V_c = C_4\sigma^4 + C_2\sigma^2 + C_0. \tag{21}$$

- Such as we consider only gravity regime ($G_{eff} > 0$) $U_c > 0$, we use the restriction $\sigma_b^2 < 6/K$.
-

$$V_{eff} = \frac{36(C_4\sigma^4 + C_2\sigma^2 + C_0)}{(K\sigma^2 - 6)^2}. \tag{22}$$

  The even potential V_{eff} has an extremum at $\sigma = 0$ and at points

$$\sigma_m = \pm\sqrt{\frac{-2(3C_2 + KC_0)}{12C_4 + KC_2}}. \tag{23}$$

Generalization of bouncing potential

- The model with $V_c = C_4\sigma^4 + C_2\sigma^2 + C_0$ has a bounce solution only if

$$C_4\sigma_b^4 + C_2\sigma_b^2 + C_0 < 0, \quad C_2\sigma_b^2 + 2C_0 > 0, \quad C_2 + 2C_4\sigma_b^2 < 0.$$

- We specify the case $C_4 < 0$. Supposing that ϕ_m are real we get

$$0 > C_2 + 2\sigma_b^2 C_4 > C_2 + \frac{12}{K} C_4.$$

- So, the model with a bounce solution has real σ_m only at

$$3C_2 + KC_0 > 0 \quad \text{and} \quad KC_2 + 12C_4 < 0. \tag{24}$$

- Using conditions to the model constants, we get

$$V''_{eff}(0) = \frac{\frac{C_0 K}{3} + C_2}{2} > 0,$$

$$V''_{eff}(\sigma_m) = -\frac{36(C_2 K + 12 C_4)^3 (C_0 K + 3 C_2)}{(C_0 K^2 + 6 C_2 K + 36 C_4)^3} < 0.$$

- Thus, the effective potential has a minimum at $\varphi = 0$ and maxima at $\sigma = \sigma_m$.

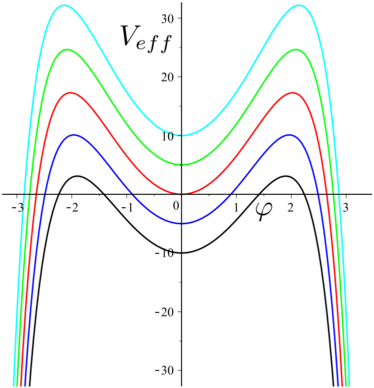


Figure: The effective potential V_{eff} at different values of parameters. In the picture we choose $K = 1/4$. The values of parameters are $C_4 = -1$, $C_2 = 7$ (left picture). The parameter $C_0 = -10$ (black curve), -5 (red curve), 0 (blue curve), 5 (green curve), and 10 (cyan curve).

$$0 < \sigma_m < \sigma_1^+ < \sigma_b < \sqrt{\frac{6}{K}}, \; \sigma_1^+ = \sqrt{\frac{1}{2}\left(\sqrt{\left(\frac{C_2}{C_4}\right)^2 - 4\frac{C_0}{C_4}} - \frac{C_2}{C_4}\right)}.$$

Analysis of numeric solutions

For $U = U_c$ and an arbitrary potential, system (8) has the following form:

$$\begin{aligned} \dot{\sigma} &= \psi, \\ \dot{\psi} &= -3H\psi - \frac{1}{6}\left(6 - K\sigma^2\right)V' + \frac{2}{3}K\sigma V, \\ \dot{H} &= -\frac{K}{6}\left[2\sigma^2H^2 + (4H\psi + V')\,\sigma + 2\psi^2\right]. \end{aligned} \tag{25}$$

We integrate this system with $V = V_c$ numerically.
We consider a positive σ_b such that $\sigma_1^+ < \sigma_b < \sqrt{6/K}$. The evolution of the scalar field starts at the bounce point with a negative velocity, defined by relation

$$\dot{\sigma}_b = -\sqrt{-2V(\sigma_b)}.$$

The field σ can come to zero passing the maximum of the potential. So, we keep in mind that the following subsequence of inequalities:

$$0 < \sigma_m < \sigma_1^+ < \sigma_b < \sqrt{\frac{6}{K}}.$$

Three possible evolutions

In the case $C_0 > 0$ there are three possible evolutions of the bounce solutions.

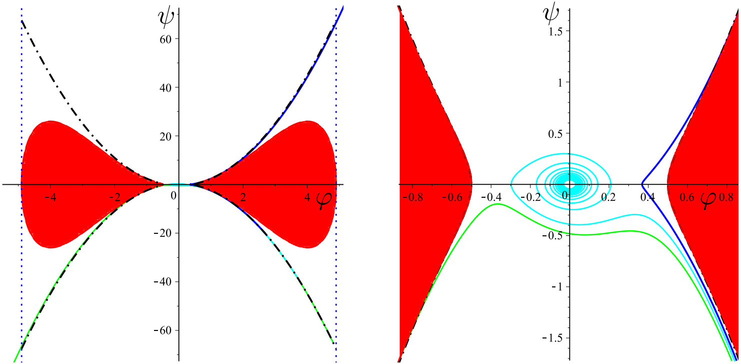


Figure: A phase trajectories for the models with U_c and V_c. The values of constants are $K = 1/4$, $C_4 = -4$, $C_2 = 1$, and $C_0 = 0$. The initial conditions are $\sigma_i = 2.7$, $\psi_i = -20.26259608$ (blue line), $\sigma_i = 3.7$, $\psi_i = -38.36598493$ (cyan line), and $\sigma_i = 4.8$, $\psi_i = -64.81244325$ (green line). The black curves are the lines of the points that correspond to $H = 0$. The unreachable domain, defined by $P < 0$, is in red. The blue point lines are $U = 0$.

For $C_0 > 0$ there exists the stable de Sitter solution $\sigma_{dS} = 0$ and $H_{dS} = \sqrt{\frac{C_0 K}{3}}$. It is a stable node at $KC_0 - 24C_2 \geqslant 0$ and a stable focus in the opposite case $C_0 K - 24C_2 < 0$.
The example of a stable node at $\sigma = 0$.

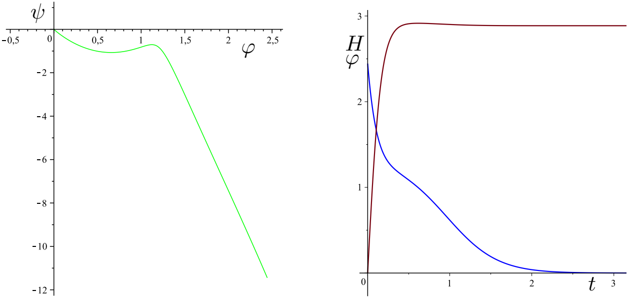


Figure: The field φ (blue line) and the Hubble parameter (red line) as functions of the cosmic time are presented in the right picture. The values of parameters are $K = 1$, $C_4 = -2.7$, $C_2 = 1$ and $C_0 = 25$. The initial conditions of the bounce solution are $\sigma_i = 2.445$, $\psi_i = -11.44650941$, and $H_i = 0$.

The example of a stable focus at $\varphi = 0$.

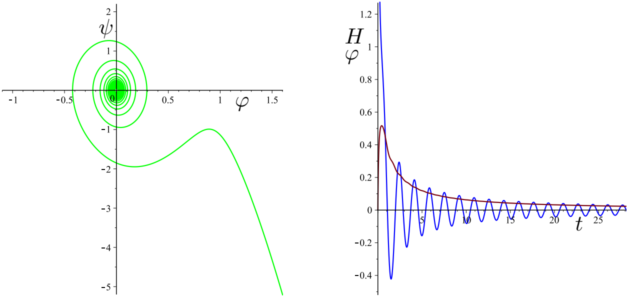


Figure: The values of constants are $K = 1/4$, $C_4 = -4$, $C_2 = 7$, $C_0 = 0$. The initial conditions are $\sigma_i = 3.4$ and $\psi_i = -30.12023904$. A zoom of the central part of phase plane is presented in the middle picture. The Hubble parameter (red) and the scalar field (blue) of functions of cosmic time are presented in the right picture.

We have presented solutions for models with positive value of C_0.
Let us consider now the phase trajectory at $C_0 = -0.1$.
We see that trajectories are similar at the beginning only. The scalar field tends to infinity and the system comes to antigravity domain with $U_c < 0$.

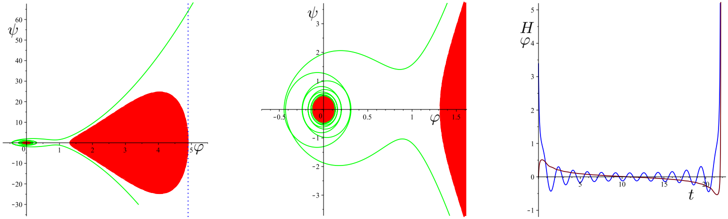
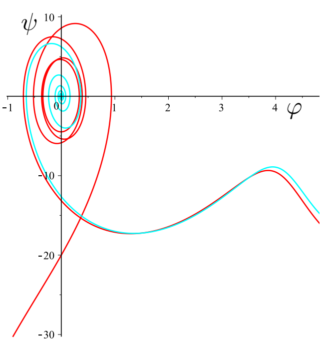


Figure: A phase trajectory for the models with U_c and V_c is presented in the left picture. The values of constants are $K = 1/4$, $C_4 = -4$, $C_2 = 7$, $C_0 = -0.1$. The initial conditions are $\sigma_i = 3.4$ and $\psi_i = -30.12355889$. A zoom of the central part of phase plane is presented in the middle picture. The Hubble parameter (red) and the scalar field (blue) of functions of cosmic time are presented in the right picture.

The difference between the solutions of system (25) with a positive and a negative C_0 is demonstrated. The cyan curves correspond to $C_0 = 10$, whereas the red curves correspond to $C_0 = -10$.
We see that the phase trajectories and the Hubble parameter are similar in the beginning, but stand essentially different in the future.



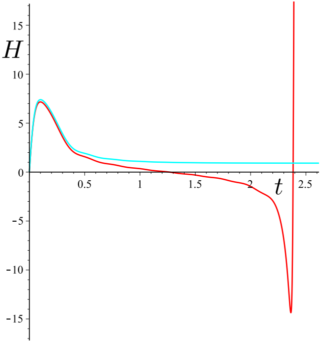


Figure: The phase trajectories (right picture) and the corresponding Hubble parameters (left picture) are presented. The values of parameters are $K = 1/4$, $C_4 = -4$, $C_2 = 90$. The initial conditions of bounce solution are $\sigma_i = 4.88$, $\psi_i = -15.17936692$ (cyan curve) and $\psi_i = -16.44424459$ (red curve).

General FLRW metric

On substituting FLRW metric into the action (1) and varying it with respect to N, a and σ, we obtain two Friedmann equations and a Klein-Gordon equation:

$$6U\left[\frac{\dot{a}^2}{a^2}+\frac{N^2}{a^2}K\right]+6U'\dot{\sigma}\frac{\dot{a}}{a}=\frac{1}{2}\dot{\sigma}^2+N^2V, \tag{26}$$

$$2U\left[2\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}-2\frac{\dot{a}\dot{N}}{aN}+\frac{N^2}{a^2}K\right]+2U'\left[\ddot{\sigma}+2\frac{\dot{a}}{a}\dot{\sigma}-\frac{\dot{N}}{N}\dot{\sigma}\right]=N^2V-\left[2U''+\frac{1}{2}\right]\dot{\sigma}^2, \tag{27}$$

$$\ddot{\sigma}+\left(3\frac{\dot{a}}{a}-\frac{\dot{N}}{N}\right)\dot{\sigma}-6U'\left[\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}-\frac{\dot{a}\dot{N}}{aN}+\frac{N^2}{a^2}K\right]+N^2V'=0, \tag{28}$$

where a "dot" means a derivative with respect to the parametric time τ and a "prime" means a derivative with respect to the scalar field.

- We consider model with $U(\sigma) = U_c$ and potential

$$V_c = \Lambda - c\sigma^4 \tag{29}$$

- Firstly in the case $K = 0$ the bounce for this model was obtained in B. Boisseau, H. Giacomini, D. Polarski and A. A. Starobinsky, JCAP 1507 (2015) 002, [arXiv:1504.07927 [gr-qc]].
- In our paper A. Y. Kamenshchik, E. O. Pozdeeva, A. Tronconi, G. Venturi and S. Y. Vernov, Class. Quant. Grav. **33**, no. 1, 015004 (2016) [arXiv:1509.00590 [gr-qc]] we look for bounce solutions with $K = \pm 1$

- The first Friedmann equation (26) for this model is

$$6\left(U_0-\frac{1}{12}\sigma^2\right)\frac{\dot{a}^2}{a^2}-\frac{\dot{a}\dot{\sigma}\sigma}{a}+6\left(U_0-\frac{1}{12}\sigma^2\right)\frac{KN^2}{a^2}=\frac{1}{2}\dot{\sigma}^2+N^2(\Lambda-c\sigma^4). \tag{30}$$

- The Klein-Gordon equation (28) is

$$\ddot{\sigma}+\left(3\frac{\dot{a}}{a}-\frac{\dot{N}}{N}\right)\dot{\sigma}+\sigma\left(\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}\right)-6\sigma\frac{\dot{a}\dot{N}}{aN}-4cN^2\sigma^3+\frac{N^2K\sigma}{a^2}=0. \tag{31}$$

- A substitution $\sigma = \frac{\chi}{a}$, in the Klein-Gordon equation (31), reduces it to

$$\ddot{\chi} + \frac{\dot{\chi}\dot{a}}{a} - \frac{\dot{\chi}\dot{N}}{N} - \frac{4c\chi^3}{a^2} + \frac{N^2 K\chi}{a^2} = 0. \tag{32}$$

- On choosing the lapse equation N as $N = a$, and multiplying obtained equation by the conformal time derivative of χ, $\frac{d\chi}{d\eta}$, we find its first integral

$$\frac{1}{2}\left(\frac{d\chi}{d\eta}\right)^2 - c\chi^4 + \frac{1}{2}K\chi^2 = A, \tag{33}$$

where A is a constant.

- Here let us now rewrite the expression (33) in terms of the initial scalar field σ and time derivatives in terms of the cosmic time:

$$\frac{1}{2}\dot{\sigma}^2 + \frac{1}{2}\sigma^2\dot{a}^2 + \sigma\dot{\sigma}\frac{\dot{a}}{a} - c\sigma^4 + \frac{K\sigma^2}{2a^2} = \frac{A}{a^4}. \tag{34}$$

- On comparing Eq. (34) with the Friedmann equation (30), we see that the latter reduces to a very simple form

$$6U_0\frac{\dot{a}^2}{a^2} + 6U_0\frac{K}{a^2} = \Lambda + \frac{A}{a^4}. \tag{35}$$

The equation for possible turning points (bounces or the points of maximal expansion is

$$a^4 - \frac{6U_0 K}{V_0} a^2 + \frac{A}{V_0} = 0\,. \qquad (36)$$

Let us give a list of the cosmological evolutions for different choices of the curvature K and the radiation constant A.

• 1.• (a). $K = 0$, $A < 0$. We have a bounce at

$$a_B = \left(\frac{-A}{V_0}\right)^{1/4}.$$

This case was analyzed in detail in papers [3]

• (b) $K = 0$, $A = 0$. For this case we have an infinite de Sitter expansion or an infinite de Sitter contraction. The singularity is absent.

• (c) $K = 0$, $A > 0$. We have an infinite expansion which begins from the Big Bang singularity or an infinite contraction which ends in the Big Crunch singularity.

[3]B. Boisseau, H. Giacomini, D. Polarski and A. A. Starobinsky, JCAP 1507 (2015) 002.,
B. Boisseau, H. Giacomini and D. Polarski, JCAP **1510**, 033 (2015) [arXiv:1507.00792 [gr-qc]]

• 2. • (a) $K = 1$, $A < 0$. We have a bounce at

$$a_B = \left(\frac{3U_0}{\Lambda} + \sqrt{\frac{9U_0^2}{\Lambda^2} - \frac{A}{\Lambda}} \right)^{1/2} .$$

• (b) $K = 1$, $A = 0$. We have a closed de Sitter universe which contracts, has a bounce at

$$a_B = \left(\frac{6U_0}{\Lambda} \right)^{1/2} , \quad \text{and then expands infinitely.}$$

• (c) $K = 1$, $0 < A < \frac{9U_0^2}{\Lambda}$. For this case we have two types of possible evolutions. One begins at the Big Bang singularity, expands until the point of maximal expansion

$$a_M = \left(\frac{3U_0}{\Lambda} - \sqrt{\frac{9U_0^2}{\Lambda^2} - \frac{A}{\Lambda}} \right)^{1/2}$$

and then contracts until the encounter with the Big Crunch singularity. The second type of evolution is that with the bounce at

$$a_B = \left(\frac{3U_0}{\Lambda} + \sqrt{\frac{9U_0^2}{\Lambda^2} - \frac{A}{\Lambda}} \right)^{1/2} .$$

• (d) $K = 1$, $A = \frac{9U_0^2}{\Lambda}$. We have an Einstein static universe with radius

$$a_S = \frac{3U_0}{V}.$$

• (e) $K = 1$, $A > \frac{9U_0^2}{\Lambda}$. We have an infinitely expanding or an infinitely contracting universe.

•3. • (a) $K = -1$, $A < 0$. We have a bounce at

$$a_B = \left(-\frac{3U_0}{\Lambda} + \sqrt{\frac{9U_0^2}{\Lambda^2} - \frac{A}{\Lambda}}\right)^{1/2}.$$

• (b) $K = -1$, $A \geq 0$. We have an infinite expansion beginning from the Big Bang or an infinite contraction culminating in the Big Crunch.

For all these cases the Friedmann equation (35) can be integrated explicitly, because the Ricci scalar as an integral of motion. Combining Eqs. (26)–(28) it is easy to show that $R = R_0 = 2\Lambda/U_0$ at all values of K. We shall write down the explicit solution only for the case 2 (a). The expression for the cosmological radius is

$$a(t) = \left(\frac{3U_0}{\Lambda} + \sqrt{\frac{9U_0^2}{\Lambda^2} - \frac{A}{\Lambda}} \cosh \sqrt{\frac{\Lambda}{24U_0}}\, t \right)^{1/2} . \qquad (37)$$

The Hubble parameter is

$$h(t) = \frac{\Lambda}{96U_0} \frac{\sqrt{\frac{9U_0^2}{\Lambda^2} - \frac{A}{\Lambda}} \sinh \sqrt{\frac{\Lambda}{24U_0}}\, t}{\frac{3U_0}{\Lambda} + \sqrt{\frac{9U_0^2}{\Lambda^2} - \frac{A}{\Lambda}} \cosh \sqrt{\frac{\Lambda}{24U_0}}\, t} . \qquad (38)$$

Remarkably all the dependence of the cosmological evolution on the scalar field is encoded in the quantity A, i.e. it is nothing more than the evolution of the universe filled with the cosmological constant and the radiation-type fluid.

Conclusions

- The bounce solution with a non-monotonic Hubble parameter has been obtained.
- We shoe that the generalization of bounce potentials and effective gravity constant leads to the interesting behaviors of the Hubble parameter.
- The bounce are obtained in close and open FLRW universe.