Department of Mathematics, Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University in Prague, Trojanova 13, CZ-120 00 Prague, Czech Republic,
severin.posta@fjfi.cvut.cz

Differential invariants of realizations of Galilei algebras

Severin Pošta
joint work with M.Nesterenko

Full Galilei algebra

The full Galilei algebra $\mathrm{AG}_4(n)$ of $(n+1)$-dimensional Newton space-time is generated by the basis elements

P_i (operators of spatial translations, $i = 1, 2, \ldots, n$);
T (operator of temporal translation);
J_{ij} (operators of rotations, $i < j$, $i, j = 1, 2, \ldots, n$);
G_i (operators of pure Galilei transformations, $i = 1, 2, \ldots, n$);
D (dilatation operator);
S (projection operator);
Z (operator of scale transformation of spatial variables);
M (mass operator).

Commutation relations

$$[J_{ij}, J_{kl}] = \delta_{il} J_{jk} + \delta_{jk} J_{il} - \delta_{ik} J_{jl} - \delta_{jl} J_{ik}, \tag{1}$$
$$[P_i, J_{jk}] = \delta_{ij} P_k - \delta_{ik} P_j; \tag{2}$$
$$[T, G_i] = -P_i; \tag{3}$$
$$[G_i, J_{jk}] = \delta_{ij} G_k - \delta_{ik} G_j; \tag{4}$$
$$[D, G_i] = G_i; \tag{5}$$
$$[D, P_i] = -P_i; \tag{6}$$
$$[D, T] = -2T; \tag{7}$$
$$[S, P_i] = G_i; \tag{8}$$
$$[D, S] = 2S; \tag{9}$$
$$[T, S] = D; \tag{10}$$
$$[Z, G_i] = -G_i; \tag{11}$$
$$[Z, P_i] = -P_i; \tag{12}$$
$$[G_i, P_j] = \delta_{ij} M; \tag{13}$$
$$[Z, M] = -2M. \tag{14}$$

Galilei algebras

Galilei algebra	Notation	Basis elements	Dimension
Classical	$\mathrm{AG}_1(n)$	$\langle P_i, T, J_{ij}, G_i, M\rangle$	$\frac{1}{2}n(n-1)+2n+2$
Extended	$\mathrm{AG}_2(n)$	$\langle P_i, T, J_{ij}, G_i, D, M\rangle$	$\frac{1}{2}n(n-1)+2n+3$
Special	$\mathrm{AG}_3(n)$	$\langle P_i, T, J_{ij}, G_i, D, S, M\rangle$	$\frac{1}{2}n(n-1)+2n+4$
Full	$\mathrm{AG}_4(n)$	$\langle P_i, T, J_{ij}, G_i, D, S, Z, M\rangle$	$\frac{1}{2}n(n-1)+2n+5$
Reduced classical	$\mathrm{A\bar{G}}_1(n)$	$\langle P_i, T, J_{ij}, G_i\rangle$	$\frac{1}{2}n(n-1)+2n+1$
Reduced extended	$\mathrm{A\bar{G}}_2(n)$	$\langle P_i, T, J_{ij}, G_i, D\rangle$	$\frac{1}{2}n(n-1)+2n+2$
Reduced special	$\mathrm{A\bar{G}}_3(n)$	$\langle P_i, T, J_{ij}, G_i, D, S\rangle$	$\frac{1}{2}n(n-1)+2n+3$
Reduced full	$\mathrm{A\bar{G}}_4(n)$	$\langle P_i, T, J_{ij}, G_i, D, S, Z\rangle$	$\frac{1}{2}n(n-1)+2n+4$

Realizations

Notion of "realization" is widely explored in mathematical and physical literature and denotes a number of different objects, e.g.,

J. Draisma, Transitive Lie Algebras of Vector Fields: An Overview, *Qual. Theory Dyn. Syst.*, 11: 39–60, 2012.

G. Burdet, M. Perrin, Realizations of the central extension of the inhomogeneous symplectic algebra as time dependent invariance algebras of nonrelativistic quantum systems, *J. Math. Phys.*, 16 (8): 1692–1703, 1975.

Č. Burdík, A new class of realisations of the Lie algebra so$(q, 2n-q)$, *J. Phys. A: Math. Gen.*, 21: 284-295, 1988.

S. Krivonos, O. Lechtenfeld & A. Sorin, Minimal realization of l-conformal Galilei algebra, Pais-Uhlenbeck oscillators and their deformation, *J. High Energ. Phys.* (2016): 78, 2016.

We consider realizations of Lie algebras by **vector fields**, i.e., representations by first-order differential operators.

Application of realizations

- general theory of differential equations;
- integration of differential equations and their systems;
- group classification of ODEs and PDEs;
- classification of gravity fields of a general form with respect to motion groups;
- geometric control theory and in the theory of systems with superposition principles;
- difference schemes for numerical solutions of differential equations;
- solution of the Levine's problem in molecular dynamics, etc.

Definition of realization

Let V be an n-dimensional vector space over $\mathbb{R}$ (or $\mathbb{C}$).
$\mathfrak{g}$ is a Lie algebra on V spanned by $\{e_1, e_2, \ldots, e_n\}$ with the structure constants $C_{ij}^k \in \mathbb{R}$.
M is an open domain of $\mathbb{R}^m$ and $\mathrm{Vect}(M)$ is the Lie algebra of smooth vector fields on M.

Definition

A *realization of a Lie algebra* $\mathfrak{g}$ *in vector fields on* M is a homomorphism R: $\mathfrak{g} \to \mathrm{Vect}(M)$. The realization is called *faithful* if $\ker R = \{0\}$ and *unfaithful* otherwise.

In the paper M. Nesterenko, S. Pošta and O. Vaneeva, Realizations of Galilei algebras, *J. Phys. A: Math. Theor.*, V.49, 115203, 2016, all inequivalent realizations of the low-dimernsional Gellilei algebras were constructed.

Three-, four- and five-dimensional cases

- $\mathrm{A\bar{G}_1}(1) = \langle P,\ T,\ G\rangle$: $[G, T] = P \sim A_{3.1}$; (3+3)
- $AG_1(1) = \langle P,\ T,\ G,\ M\rangle$: $[G, T] = P$ and $[G, P] = M \sim A_{4.1}$; (8+8)
- $\mathrm{A\bar{G}_2}(1) = \langle P,\ T,\ G,\ D\rangle$: $[G, T] = P$, $[D, G] = G$, $[P, D] = P$ and $[T, D] = 2T \sim A_{4.8}^{b=-1/2}$; (7+8)
- $\mathrm{AG_2}(1) = \langle M,\ P,\ T,\ G,\ D\rangle$: $[T, G] = -P$, $[D, G] = G$, $[P, D] = P$, $[T, D] = 2T$, $[G, P] = M \sim \mathfrak{g}_{5.30}^{-2}$; (35)
- $\mathrm{A\bar{G}_3}(1) = \langle P,\ T,\ G,\ D,\ S\rangle$: $[T, G] = -P$, $[D, G] = G$, $[P, D] = P$, $[T, D] = 2T$, $[S, P] = G$, $[T, S] = D$, $[D, S] = 2S \sim \mathfrak{sl}(2, \mathbb{R}) \Subset 2A_1$.(19)

From realizations to invariant equations

To find differential equations invariant with respect to a realization of the Lie algebra it is necessary to construct it's appropriate differential invariants. What is explained by the fact that any differential equation invariant with respect to the Lie algebra $\mathfrak{g}$, can be written as a function of differential invariants of $\mathfrak{g}$ ('regular' part) or as a function of conditions of rank degeneration for the correspondingly prolonged generators ('singular' part). And it is essential that for any realization of a Lie algebra there exist differential invariants of some (probably rather high) order.
The natural question is if it is possible to choose a minimal set of differential invariants which allows one to obtain all differential invariants of a given order by a final number of defined operations. The answer to this question is positive.

Standard prolongation

Consider a realization

$$e_i = f_{ij}(x_1, \dots, x_m)\partial_{x_j}, \quad i = 1 \dots n, \quad j = 1 \dots m.$$

Let the set of independent variables $t = (t_1, t_2, \dots, t_{m_0})$ and the set of dependent variables $u = (u_1, u_2, \dots, u_{m-m_0})$ are fixed in the realization (10)

$$e_i = \xi_{ij}(t, u)\partial_{t_j} + \eta_{ik}(t, u)\partial_{u_k}, \qquad j = 1 \dots m_0, \quad k = 1, \dots, m - m_0.$$

Then the d-prolonged Lie algebra $\mathrm{pr}^{(d)}\mathfrak{g}$ is generated by the differential operators

$$e_i^{(d)} = \xi_{ij}(t, u)\partial_{t_j} + \eta_{ik}(t, u)\partial_{u_k} + \sum_{l=1}^{d} \eta_{ik}^{l}(t, u^{(l)})\partial_{u_k^{(l)}}, \text{where}$$

$$\eta_{ik}^{l+1}(t, u^{(l+1)}) = D\eta_{ik}^{l}(t, u^{(l)}) - u_k^{(l+1)} D\xi_{ij}(t, u), \quad \eta_{ik}^{0} = \eta_{ik}.$$

Differential invariants

Definition

A smooth function $I = I(t, u^{(d)})$ is called a *differential invariant* of order d of Lie algebra $\mathfrak{g}$, if $e_i^{(d)} I(t, u^{(d)}) = 0$ for the prolonged basis $\{e_i^{(d)}\}$ of the algebra $\mathrm{pr}^{(d)}\mathfrak{g}$.

Definition

A maximal set I_d of functionally independent differential invariants of the order not greater then d (i.e., invariants of a prolonged algebra $\mathrm{pr}^{(d)}\mathfrak{g}$) is called a *universal differential invariant* of the order d of an algebra $\mathfrak{g}$.

In the case of m_0 independent and $m - m_0$ dependent variables the number of functionally independent differential invariants of order d is

$$N_d = m + (m - m_0)\binom{m_0 + d}{m_0} - r_d.$$

Complete description of differential invariants

Consider the ranks

$$r_l = \text{rank}\{(\xi_i, \eta_i, \eta_i^1, \ldots, \eta_i^l),\ i = 1, \ldots, n\}.$$

There exists such a number $\nu = \min\{l \in \mathbb{Z}^+ \mid r_l = n\}$ that $r_\nu = r_{\nu+1} = \cdots = n$ holds true.
Complete description of differential invariants of a fixed realization of a Lie algebra consists of functional **basis of differential invariants** and **operators of invariant differentiation**.
Basis of differential invariants can be constructed as a universal differential invariant of the order $(\nu + 1)$ and operators of invariant differentiation are derived from the commutativity condition with formally infinitely prolonged elements of the Lie algebra.
In order to describe singular invariant system one should investigate degeneration condition for the rank of the correspondingly prolonged realizations of Lie algebras (Lie determinants).

The smallest Galilei algebra

The reduced classical Galilei algebra $\mathrm{A\bar{G}_1(1)}$ of one-dimensional space is generated by the operators P, G and T with the unique nonzero commutation relation $[G, T] = P$.

All its inequivalent realizations are the following.

Compl.set	Subalgebra	Realization
$\{P, T, G\}$	$\mathfrak{h}_0 = \langle 0 \rangle$	$R_{\mathfrak{h}_0}\colon P = \partial_1,\ G = -x_2\partial_1 + \partial_3,\ T = \partial_2$
$\{G, T\}$	$\mathfrak{h}_{1.1} = \langle P \rangle$	$R_{\mathfrak{h}_{1.1}}\colon P = 0,\ G = \partial_1,\ T = \partial_2$
$\{P, G\}$	$\mathfrak{h}^{\alpha}_{1.2} = \langle T + \alpha G \rangle$	$R_{\mathfrak{h}^{0}_{1.2}}\colon P = \partial_1,\ G = \partial_2,\ T = x_2\partial_1$
	$\mathfrak{h}^{x_3}_{1.2} = \langle T + \alpha G \rangle$	$R_{\mathfrak{h}^{x_3}_{1.2}}\colon P = \partial_1,\ G = \partial_2,\ T = x_2\partial_1 - x_3\partial_2$
$\{G\}$	$\mathfrak{h}^{\alpha}_{2} = \langle P,\ T + \alpha G \rangle$	$R_{\mathfrak{h}^{\alpha}_{2}}\colon P = 0,\ G = \partial_1,\ T = -\alpha\partial_1$
	$\mathfrak{h}^{x}_{2} = \langle P,\ T + \alpha G \rangle$	$R_{\mathfrak{h}^{x_2}_{2}}\colon P = 0,\ G = \partial_1,\ T = -x_2\partial_1$

Deformations of $\mathrm{A\bar{G}_1(1)}$

$\mathrm{A\bar{G}_1(1)}$ can be deformed only to the following algebras.

- $\mathrm{A\bar{G}_1^{q}(1)}$: $[P,T]_q = qP,\ [G,T]_q = P + \alpha q G,\ q \in \mathbb{R}$, (for $\alpha = -1$ and $q \neq 0$ $\mathrm{A\bar{G}_1^{q}(1)}$ is isomorphic to the smallest Poincaré algebra $p(1,1) = \langle P_0,\ P_1,\ J_{01}\rangle$);
- $A\bar{G}_1^{q_1}(1)$: $[P,G]_{q_1} = -2q_1 G,\ [P,T]_{q_1} = 2q_1 T,$ $[G,T]_{q_1} = P,\ q_1 \in \mathbb{R}$;
- $A\bar{G}_1^{q_2}(1)$: $[P,G]_{q_2} = q_2^2 T,\ [P,T]_{q_2} = -q_2^2 G,$ $[G,T]_{q_2} = P,\ q_2 \in \mathbb{R}$;
- $\mathrm{A\bar{G}_1^{q_3}(1)}$: $[P,G]_{q_3} = q_3 P,\ [G,T]_{q_3} = P,\ q_3 \in \mathbb{R}$;
- $A\bar{G}_1^{q_4}(1)$: $[P,T]_{q_4} = q_4 P,\ [G,T]_{q_4} = P + q_4 G,\ q_4 \in \mathbb{R}$;
- $\mathrm{A\bar{G}_1^{q_5}(1)}$: $[P,T]_{q_5} = q_5 bP - q_5^2 G,$ $[G,T]_{q_5} = P + q_5 bG,\ q_5 \in \mathbb{R}$.

Realizations of the deformed Galilei algebras

Algebra	Compl. set	Realization
$\mathrm{A\bar{G}}_1^{q}(1)$	$\{P,G,T\}$	$P=\partial_1,\ G=\partial_2,\ T=(qx_1+x_2)\,\partial_1+\alpha q x_2\partial_2+\partial_3$
$\mathrm{A\bar{G}}_1^{q_1}(1)$	$\{G,P,T\}$	$P=2q_1x_1\partial_1+\partial_2,\ G=\partial_1,\ T=q_1x_1^2\partial_1+x_1\partial_2+e^{2q_1x_2}\partial_3$
$\mathrm{A\bar{G}}_1^{q_2}(1)$	$\{G,T,P\}$	$P=q_2\cos(q_2x_1)\tan(q_2x_2)\,\partial_1-q_2\sin(q_2x_1)\,\partial_2+\frac{\cos q_2x_1}{\cos q_2x_2}\partial_3,\ G=\partial_1,\ T=\sin(q_2x_1)\tan(q_2x_2)\,\partial_1+\cos(q_2x_1)\,\partial_2+\frac{\sin q_2x_1}{q_2\cos q_2x_2}\partial_3$
$\mathrm{A\bar{G}}_1^{q_3}(1)$	$\{P,T,G\}$	$P=\partial_1,\ G=(q_3x_1-x_2)\,\partial_1+\partial_3,\ T=\partial_2$
$\mathrm{A\bar{G}}_1^{q_4}(1)$	$\{P,G,T\}$	$P=\partial_1,\ G=\partial_2,\ T=(q_4x_1+x_2)\,\partial_1+q_4x_2\partial_2+\partial_3$
$\mathrm{A\bar{G}}_1^{q_5}(1)$	$\{P,G,T\}$	$P=\partial_1,\ G=\partial_2,\ T=(bq_5x_1+x_2)\,\partial_1+q_5\,(bx_2-q_5x_1)\,\partial_2+\partial_3$

Equations invariant w.r.t. the smallest Galilei algebra

In spite of the simplicity and small dimension of the Galilei algebra $\mathrm{A\bar{G}}_1(1)$ and its deformations a number of well-known differential equations are invariant with respect to it.

The Kawahara equation with time-dependent coefficient

$$u_t + uu_x + \lambda t^{\rho} u_{xxx} + \varepsilon t^{\frac{5\rho+2}{3}} u_{xxxxx} = 0$$

is invariant with respect to $\mathrm{A\bar{G}}_1^q(1)$.

Generalized KdV equation

$$u_t = u^n u_x + \varepsilon u_{xxx}, \quad n \neq 0, 1, \quad \varepsilon \neq 0$$

admits the generic realization of the deformed Galilei algebra $\mathrm{A\bar{G}}_1^q(1)$ for $q = 1$ and $\alpha = \frac{1}{3}$.

The 5-th order KdV-equation with time-dependent coefficient

$$u_t + uu_x + t^{\rho} u_{xxxxx} = 0, \quad \rho \neq 0$$

has the Lie symmetry isomorphic to the generic realization of the deformed Galilei algebra $\mathrm{A\bar{G}}_1^q(1)$ for $q = 1$ and $\alpha = \frac{\rho-4}{\rho+1}$.

The classical Kawahara equation

$$u_t + uu_x + \lambda u_{xxx} + \varepsilon u_{xxxxx} = 0, \quad \lambda\varepsilon \neq 0$$

admit the Galilei algebra $\mathrm{A\bar{G}_1(1)}$.
One more example is given by the reaction-diffusion equation

$$u_t = (u^n u_x)_x + \varepsilon u^m,$$

where n and m are arbitrary constants, invariant w.r.t the $\mathrm{A\bar{G}}_1^q(1)$ for $q = \frac{1-m+n}{2(1-m)}$.
Consider the generalized Ermakov system

$$\ddot{x} = \frac{1}{x^3} F\left(\frac{y}{x}\right), \quad \ddot{y} = \frac{1}{y^3} G\left(\frac{y}{x}\right),$$

which is invariant w.r.t the deformed Galilei algebra $\mathrm{A\bar{G}}_1^{q_1}(1)$.
Classical Kepler problem in polar coordinates

$$\ddot{r} - r\dot{\theta}^2 + \frac{\mu}{r^2} = 0, \quad r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$$

is invariant w.r.t the deformed Galilei algebra $\mathrm{A\bar{G}}_1^{q_3}(1)$.

Calculation of differential invariants

The above examples motivate complete study of differential invariants of realizations of low-dimensional Galilei algebras and their deformations.
Consider in detail the realization of the deformed Galilei algebra $A\bar{G}_1^{q_1}(1)$ (here we use x and y instead of u_1 and u_2)

$$e_1 = \partial_t, \quad e_2 = t\partial_t + x\partial_x, \quad e_3 = t^2\partial_t + 2tx\partial_x + x\partial_y.$$

Since $r_0 = 3$, there are no invariants of the zero order.
Consider the second prolongations of the basis operators:

$$e_1^{(2)} = \partial_t, \quad e_2^{(2)} = t\partial_t + x\partial_x - \dot{y}\partial_{\dot{y}} - \ddot{x}\partial_{\ddot{x}} - 2\ddot{y}\partial_{\ddot{y}},$$
$$e_3^{(2)} = t^2\partial_t + 2tx\partial_x + x\partial_y + 2x\partial_{\dot{x}} + (\dot{x} - 2t\dot{y})\partial_{\dot{y}} + \\ + (2\dot{x} - 2t\ddot{x})\partial_{\ddot{x}} + (\ddot{x} - 2\dot{y} - 4t\ddot{y})\partial_{\ddot{y}}.$$

We have $r_1 = r_2 = 3 = r_l,\ l \geq 3$, so $\nu = 0$ and basis of differential invariants is contained in I_1, moreover $N_1 = 5 - 3 = 2$ and $N_2 = 7 - 3 = 4$. So, there exist exactly two functionally independent invariants of the first order and four functionally independent invariants of the second order. At that, there is exactly two invariants of the first order among the second-order invariants.

The universal differential invariant I_1 is generated by two functions $\omega_k = \omega_k(t, x, y, \dot{x}, \dot{y})$, $k = 1, 2$, that are defined by the conditions $e_i^{(1)}\omega_k = 0$, $i = 1, 2, 3$.

The set of functionally independent integrals of the corresponding characteristic system consists of two functions

$$\omega_1 = 2y - \dot{x},\ \omega_2 = 4x\dot{y} - \dot{x}^2,$$

and form our fundamental system of differential invariants.

Let us look for the operator of invariant differentiation in the form $X = \lambda(t, x, y)D_t$, where the function λ can be found in the implicit form $\varphi(\lambda, t, x, y)$ from the system $(e_i + \lambda D_t(\xi_i)\partial_\lambda)\varphi = 0$, $i = 1, 2, 3$.
For the given realization of the Lie algebra we have $\varphi(\lambda, t, x, y) = \varphi(\frac{\lambda}{x}) = 0$ and the operator of invariant differentiation has the form:

$$X = xD_t,$$

what gives us the complete description of all possible regular differential invariants of the chosen realization.
The obtained classification of invariants allow us to write explicitly any invariant system od PDEs or PDE of any fixed order invariant w.r.t this realization with one independent and two dependent variables.

Systems of two second-order ODEs

If we are interested by the regular systems of two second-order ODEs (e.g. Newtonian systems), which can be presented in the normal form and invariant w.r.t the three-dimensional Galilei algebra or it's deformations, then we obtain the following list of systems.

$\mathrm{A\bar{G}_1(1)}$:

$\omega_1 = y - \frac{1}{\dot{x}}$, $\omega_2 = \frac{\dot{y}}{\dot{x}}$,

$$\ddot{x} = \dot{x}^3 F(\omega_1, \omega_2), \quad \ddot{y} = \dot{x}^2 \dot{y} F(\omega_1, \omega_2) + \dot{x}^2 G(\omega_1, \omega_2).$$

$\omega_1 = y$, $\omega_2 = \frac{\dot{x}^2 - 2\dot{y}}{\dot{y}^2}$,

$$\ddot{x} = \dot{y}\left(\dot{x}^2 - \dot{y}\right) F(\omega_1, \omega_2) + \dot{x}\dot{y}^2 G(\omega_1, \omega_2),$$
$$\ddot{y} = \dot{x}\dot{y}^2 F(\omega_1, \omega_2) + \dot{y}^3 G(\omega_1, \omega_2).$$

$\omega_1 = y$, $\omega_2 = \frac{\dot{y}}{\dot{x}}$,

$$\ddot{x} = \dot{x}^3 F(\omega_1, \omega_2), \quad \ddot{y} = \dot{x}^2 \dot{y} F(\omega_1, \omega_2) + \dot{x}^2 G(\omega_1, \omega_2).$$

$A\bar{G}_1^q(1)$:
$\omega_1 = \dot{x}e^{(1-a)y}$, $\omega_2 = \dot{y}e^{y}$,

$$\ddot{x} = \dot{x}\dot{y}F\left(\omega_1, \omega_2\right), \quad \ddot{y} = \dot{y}^2 G\left(\omega_1, \omega_2\right).$$

$\omega_1 = y$, $\omega_2 = \dot{x}\dot{y}^{a-1}$,

$$\ddot{x} = \dot{x}\dot{y}F\left(\omega_1, \omega_2\right), \quad \ddot{y} = \dot{y}^2 G\left(\omega_1, \omega_2\right).$$

$\omega_1 = y$, $\omega_2 = \frac{x\dot{y}}{\dot{x}}$,

$$\ddot{x} = \dot{x}^3 x^{\frac{2a-1}{1-a}} F\left(\omega_1, \omega_2\right), \quad \ddot{y} = \dot{x}^2 \dot{y} x^{\frac{2a-1}{1-a}} F\left(\omega_1, \omega_2\right) + \dot{y}^2 G\left(\omega_1, \omega_2\right).$$

$A\bar{G}_1^{q_1}(1)$:
$\omega_1 = 2y - \dot{x}$, $\omega_2 = 4x\dot{y} - \dot{x}^2$,

$$\ddot{x} = \frac{1}{x}F\left(\omega_1, \omega_2\right) + 2\dot{y}, \quad \ddot{y} = \frac{y}{x^2}F\left(\omega_1, \omega_2\right) + \frac{1}{x^2}G\left(\omega_1, \omega_2\right).$$

$\omega_1 = y$, $\omega_2 = \frac{x^2\dot{y}^2}{\dot{x}^2+1}$,

$$\ddot{x} = x^2\dot{y}^3F(\omega_1, \omega_2) - \frac{1+\dot{x}^2}{x}, \quad \ddot{y} = \dot{y}^2(\dot{x}F(\omega_1, \omega_2) + G(\omega_1, \omega_2)) - \frac{2\dot{x}\dot{y}}{x}.$$

$\omega_1 = y$, $\omega_2 = \frac{x^2\dot{y}^2}{\dot{x}^2-1}$,

$$\ddot{x} = x^2\dot{y}^3F(\omega_1, \omega_2) - \frac{\dot{x}^2-1}{x}, \quad \ddot{y} = \dot{y}^2(\dot{x}F(\omega_1, \omega_2) + G(\omega_1, \omega_2)) - \frac{2\dot{x}\dot{y}}{x}.$$

$\omega_1 = y$, $\omega_2 = x\dot{y}$,

$$\ddot{x} = \frac{1}{2x}F\left(\omega_1, \omega_2\right) + \frac{\dot{x}^2}{2x}, \quad \ddot{y} = \frac{1}{x^2}G\left(\omega_1, \omega_2\right) - \frac{\dot{x}\dot{y}}{x}.$$

$A\bar{G}_1^{q_2}(1)$:

$\omega_1 = y$, $\omega_2 = \frac{\dot{x}^2+\cos^2 x}{\dot{y}^2}$,

$$\ddot{x} = \frac{\dot{y}^3}{\cos x} F(\omega_1, \omega_2) - 2\dot{x}^2 \tan x - \sin x \cos x,$$

$$\ddot{y} = \frac{\dot{x}\dot{y}^2}{\cos x} F(\omega_1, \omega_2) + \dot{y}^2 G(\omega_1, \omega_2) - 2\dot{x}\dot{y} \tan x.$$

$\omega_1 = y - \arctan \frac{\dot{x}}{\cos x}$, $\omega_2 = \frac{\dot{x}^2+\cos^2 x}{(\dot{y}+\sin x)^2}$,

$$\ddot{x} = \frac{(\dot{x}^2 + \cos^2 x)}{\cos x} \left((\dot{y} + \sin x) F(\omega_1, \omega_2) + \dot{y}\right) - \dot{x}^2 \tan x,$$

$$\ddot{y} = \frac{\dot{x}(\dot{y} + \sin x)^2}{\cos x} F(\omega_1, \omega_2) + \frac{\dot{x}\dot{y}(\dot{y} + \sin x)}{\cos x} +$$
$$+ (\dot{y} + \sin x)^2 G(\omega_1, \omega_2) - \dot{x} \cos x - \dot{x}(\dot{y} + \sin x) \tan x.$$

$A\bar{G}_1^{q_3}(1)$

$\omega_1 = \dot{x}e^y$, $\omega_2 = \frac{\dot{y}}{\dot{x}}$,

$$\ddot{x} = \dot{x}^2 F(\omega_1, \omega_2), \quad \ddot{y} = \dot{y}^2 G(\omega_1, \omega_2).$$

$\omega_1 = y$, $\omega_2 = \frac{\dot{x}}{\dot{y}} + \ln \dot{y}$,

$$\ddot{x} = \dot{y}^2 F(\omega_1, \omega_2) + \dot{x}\dot{y} G(\omega_1, \omega_2), \quad \ddot{y} = \dot{y}^2 G(\omega_1, \omega_2).$$

$\omega_1 = y$, $\omega_2 = \frac{\dot{y}}{\dot{x}}$,

$$\ddot{x} = \dot{x}^2 F(\omega_1, \omega_2), \quad \ddot{y} = \dot{y}^2 G(\omega_1, \omega_2).$$

$\omega_1 = y$, $\omega_2 = \frac{\dot{y}x}{\dot{x}}$,

$$\ddot{x} = \dot{x}^2 \dot{y} F(\omega_1, \omega_2), \quad \ddot{y} = \dot{x}\dot{y}^2 F(\omega_1, \omega_2) + \dot{y}^2 G(\omega_1, \omega_2).$$

$A\bar{G}_1^{q_4}(1)$:

$\omega_1 = y - \frac{1}{\dot{x}}$, $\omega_2 = \frac{\dot{y}e^y}{\dot{x}}$,

$$\ddot{x} = \dot{x}^2\dot{y}F(\omega_1,\omega_2), \quad \ddot{y} = \dot{x}\dot{y}^2F(\omega_1,\omega_2) + \dot{y}^2G(\omega_1,\omega_2).$$

$\omega_1 = y$, $\omega_2 = \frac{\dot{y}e^{\frac{1}{\dot{x}}}}{\dot{x}}$,

$$\ddot{x} = \dot{x}^2\dot{y}F(\omega_1,\omega_2), \quad \ddot{y} = \dot{x}\dot{y}^2F(\omega_1,\omega_2) + \dot{y}^2G(\omega_1,\omega_2).$$

$\omega_1 = y$, $\omega_2 = \frac{\dot{y}}{\dot{x}}$,

$$\ddot{x} = \dot{x}^3e^{-x}F(\omega_1,\omega_2), \quad \ddot{y} = \dot{x}^2\dot{y}e^{-x}F(\omega_1,\omega_2) + \dot{x}^2G(\omega_1,\omega_2).$$

$A\bar{G}_1^{q_5}(1)$:

$\omega_1 = y + \arctan \dot{x}$, $\omega_2 = \frac{\dot{y}^2 e^{2by}}{1+\dot{x}^2}$,

$$\ddot{x} = \dot{y}\left(1+\dot{x}^2\right) F\left(\omega_1, \omega_2\right), \quad \ddot{y} = \dot{x}\dot{y}^2 F\left(\omega_1, \omega_2\right) + \dot{y}^2 G\left(\omega_1, \omega_2\right).$$

$\omega_1 = y$, $\omega_2 = \frac{1+\dot{x}^2}{\dot{y}^2} e^{2b \arctan \dot{x}}$,

$$\ddot{x} = \dot{y}\left(1+\dot{x}^2\right) F\left(\omega_1, \omega_2\right), \quad \ddot{y} = \dot{x}\dot{y}^2 F\left(\omega_1, \omega_2\right) + \dot{y}^2 G\left(\omega_1, \omega_2\right).$$

$\omega_1 = y$, $\omega_2 = \frac{\dot{y}\left(1+x^2\right)}{\dot{x}}$,

$$\ddot{x} = \dot{x}^3\left(1+x^2\right)^{-\frac{3}{2}} e^{-b \arctan x} F\left(\omega_1, \omega_2\right),$$

$$\ddot{y} = \frac{\dot{x}\dot{y}^2 e^{-b \arctan x}}{\sqrt{1+x^2}} F\left(\omega_1, \omega_2\right) + \dot{y}^2 G\left(\omega_1, \omega_2\right) - \frac{2x\dot{x}\dot{y}}{1+x^2}.$$